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Duality for Multiobjective Variational Problems

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Wolfe and Mond–Weir type duals for multiobjective variational problems are formulated. Under convexity assumptions on the functions involved weak, and converse duality theorems are proved to relate properly efficient solutions of the primal and dual problems. A close relationship between these variational problems and nonlinear multiobjective programming problems is also indicated. © 1992 Academic Press, Inc.

1. INTRODUCTION

Recently, programs with several conflicting objectives have been extensively studied in the literature. Introducing the concept of proper efficiency of solutions, Geoffrion [7] proved an equivalence between a multiobjective program with convex functions and a related parametric (scalar) objective program. Using this equivalence, Weir [13] formulated a dual program for a multiobjective program having differentiable convex functions. Subsequently, Egudo [6] and Weir [13] proved duality results for a differentiable multiobjective program with pseudoconvex/quasiconvex functions. Mond *et al.* [9] presented Mond–Weir [12] and Wolfe [14] type duals for a class of nondifferentiable multiobjective programs and proved weak, strong, and converse duality theorems for Mond–Weir type dual problems regarding Geoffrion's parameter as a variable.

Motivated by the results related to nonlinear multiobjective program-

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ming duality in the above-cited references, in this paper we propose studying Wolfe and Mond–Weir duality for multiobjective variational problems. In view of the remarks of Mond and Hanson [8], it is also shown here that our duality theorems can be considered as dynamic generalizations of the corresponding (static) multiobjective programming.

2. Notation and Preliminaries

Let $I=[a,b]$ be a real interval and $f: I\times R^n\times R^n\to R$ and $g: I\times R^n\times R^n\to R^m$ be continuously differentiable functions. In order to consider $f(t, x(t), \dot{x}(t))$, where $x: I\to R^n$ with dserivative $\dot{x}$, denote the partial derivative of f with respect to t, x, and $\dot{x}$, respectively, by f_t, f_x, and $f_{\dot{x}}$, such that

$$f_x=(\partial f/\partial x_1, \partial f/\partial x_2, ..., \partial f/\partial x_n), \qquad f_{\dot{x}}=(\partial f/\partial \dot{x}_1, \partial f/\partial \dot{x}_2, ..., \partial f/\partial \dot{x}_n).$$

Similarly, we write the partial derivatives of the vector function g using matrices with m rows instead of one. Let $C(I, R^n)$ denote the space of piecewise smooth functions x with norm $\|x\|=\|x\|_\infty+\|Dx\|_\infty$, where the differentiation operator D is given by

$$u=Dx \Leftrightarrow x(t)=\alpha+\int_a^t u(s)\, ds,$$

where α is a given boundary value. Therefore, $D=d/dt$ except at discontinuities.

Consider the following multiobjective variational primal problem as

(P) Minimize $\int_a^b f(t, x(t), \dot{x}(t))\, dt=(\int_a^b f^1(t, x(t), \dot{x}(t))\, dt, ..., \int_a^b f^p(t, x(t), \dot{x}(t))\, dt)$ subject to

$$x(a)=\alpha, \qquad x(b)=\beta, \tag{1}$$

$$g(t, x(t), \dot{x}(t))\leqslant 0, \qquad t\in I. \tag{2}$$

Let K the set of feasible solutions for (P) be given by

$$K=\{x\in C(I, R^n) \mid x(a)=\alpha,\ x(b)=\beta,\ g(t, x(t), \dot{x}(t))\leqslant 0,\ t\in I\}.$$

Definition 1. A point x^* in K is said to be an efficient solution of (P) if for all x in K:

$$\int_a^b f^i(t, x^*(t), \dot{x}^*(t))\, dt \geqslant \int_a^b f^i(t, x(t), \dot{x}(t))\, dt, \qquad \forall i\in\{1, 2, ..., p\},$$

$$\Rightarrow \int_a^b f^i(t, x^*(t), \dot{x}^*(t))\, dt = \int_a^b f^i(t, x(t), \dot{x}(t))\, dt, \qquad \forall i\in\{1, 2, ..., p\}.$$

DEFINITION 2 (Borwein [2]). A point x^* in K is said to be a weak minimum for (P) if there exists no other x in K for which

$$\int_a^b f(t, x^*(t), \dot{x}^*(t))\, dt > \int_a^b f(t, x(t), \dot{x}(t))\, dt.$$

From this it follows that if an x^* in K is efficient for (P) then it is also a weak minimum for (P).

DEFINITION 3 (Geoffrion [7]). A point x^* in K is said to be properly efficient solution of (P) if there exists a scalar $M>0$ such that, $\forall i \in \{1, 2, ..., p\}$,

$$\int_a^b f^i(t, x^*(t), \dot{x}^*(t))\, dt - \int_a^b f^i(t, x(t), \dot{x}(t))\, dt$$

$$\leqslant M\left(\int_a^b f^j(t, x(t), \dot{x}(t))\, dt - \int_a^b f^j(t, x^*(t), \dot{x}^*(t))\, dt\right)$$

for some j such that

$$\int_a^b f^j(t, x(t), \dot{x}(t))\, dt > \int_a^b f^j(t, x^*(t), \dot{x}^*(t))\, dt$$

whenever x is in K and

$$\int_a^b f^i(t, x(t), \dot{x}(t))\, dt < \int_a^b f^i(t, x^*(t), \dot{x}^*t))\, dt.$$

An efficient solution that is not properly efficient is said to be improperly efficient. Thus for x^* to be improperly efficient means that to every sufficiently large $M>0$, there is an x in K and an $i \in \{1, 2, ..., p\}$ such that

$$\int_a^b f^i(t, x(t), \dot{x}(t))\, dt < \int_a^b f^i(t, x^*(t), \dot{x}^*(t))\, dt$$

and

$$\int_a^b f^i(t, x^*(t), \dot{x}^*(t))\, dt - \int_a^b f^i(t, x(t), \dot{x}(t))\, dt$$

$$> M\left(\int_a^b f^j(t, x(t), \dot{x}(t))\, dt - \int_a^b f^j(t, x^*(t), \dot{x}^*(t))\, dt\right),$$

$\forall j \in \{1, 2, ..., p\}$, such that

$$\int_a^b f^j(t, x(t), \dot{x}(t))\, dt > \int_a^b f^j(t, x^*(t), \dot{x}^*(t))\, dt.$$

Now we consider the following Geoffrion type parametric variational problem for predeterminated Geoffrion's parameter $\lambda \in \Lambda^+$, where

$$\Lambda^+ = \{\lambda \in R^p \mid \lambda > 0, \lambda^T e = 1, e = (1, 1, ..., 1) \in R^p\}.$$

(P_λ) Minimize $\sum_{i=1}^p \lambda^i \int_a^b f^i(t, x(t), \dot{x}(t))\, dt = \int_a^b \lambda^{\mathrm{T}} f(t, x(t), \dot{x}(t))\, dt$ subject to (1) and (2).

Problems (P) and (P_λ) are equivalent in the sense of Geoffrion's [7] Theorems 1 and 2, which are valid when R^n is replaced by some normed space of functions, as the proofs of these theorems do not depend on the dimensionality of the space in which the feasible set of (P) lies. For our variational problems the feasible set K lies in the normed space $C(I, R^n)$. For completeness we shall merely state these theorems characterizing proper vector minima of (P) in terms of solutions of (P_λ).

THEOREM 1. *Let $\lambda \in R^p$ be fixed. If x^* is optimal for* (P_λ), *then x^* is properly efficient for* (P).

THEOREM 2. *Let f and g be convex in $(x, \dot{x})$ on K. Then x^* is properly efficient for* (P) *if and only if x^* is optimal for* (P_λ) *for some $\lambda \in \Lambda^+$.*

Before presenting two distinct duals to (P) we state the following necessary optimality conditions for (P_λ) and point out that they can be easily derived by invoking the results of Valentine [11] or those of Chandra, Craven, and Husain [3].

PROPOSITION 1. *If x is optimal for (P_λ) and is normal* [3, 8], *there exists a piecewise smooth $y: I \to R^m$ such that for $t \in I$*

$$\lambda^{\mathrm{T}} f_x(t, x(t), \dot{x}(t)) + y(t)^{\mathrm{T}} g_x(t, x(t), \dot{x}(t))$$
$$= D[\lambda^{\mathrm{T}} f_{\dot{x}}(t, x(t), \dot{x}(t)) + y(t)^{\mathrm{T}} g_{\dot{x}}(t, x(t), \dot{x}(t))] \tag{3}$$

$$y(t)^{\mathrm{T}} g(t, x(t), \dot{x}(t)) = 0 \tag{4}$$

$$y(t) \geqslant 0. \tag{5}$$

3. Wolfe Type Duality

A Wolfe type dual to (P_λ) as suggested in [7] is as follows:

(WD_λ) Maximize $\int_a^b [\lambda^{\mathrm{T}} f(t, u(t), \dot{u}(t)) + y(t)^{\mathrm{T}} g(t, u(t), \dot{u}(t))]\, dt$ subject to

$$x(a) = \alpha, \qquad x(b) = \beta,$$

$$\begin{aligned} &\lambda^{\mathrm{T}} f_x(t, u(t), \dot{u}(t)) + y(t)^{\mathrm{T}} g_x(t, u(t), \dot{u}(t)) \\ &\quad = D[\lambda^{\mathrm{T}} f_{\dot{x}}(t, u(t), \dot{u}(t)) + y(t)^{\mathrm{T}} g_{\dot{x}}(t, u(t), \dot{u}(t))], \qquad t \in I, \end{aligned}$$

$$y(t) \geqslant 0, \qquad t \in I.$$

In view of Theorems 1 and 2 above, we now define the following vector maximization variational problem as the Wolfe type dual (WD) of (P):

(WD) Maximize $(\int_a^b [f^1(t, u(t), \dot{u}(t)) + y(t)^{\mathrm{T}} g(t, u(t), \dot{u}(t))]\, dt, ..., \int_a^b [f^p(t, u(t), \dot{u}(t)) + y(t)^{\mathrm{T}} g(t, u(t), \dot{u}(t))]\, dt)$ subject to

$$x(a) = \alpha, \qquad x(b) = \beta, \tag{6}$$

$$\begin{aligned} &\lambda^{\mathrm{T}} f_x(t, u(t), \dot{u}(t)) + y(t)^{\mathrm{T}} g_x(t, u(t), \dot{u}(t)) \\ &\quad = D[\lambda^{\mathrm{T}} f_{\dot{x}}(t, u(t), \dot{u}(t)) + y(t)^{\mathrm{T}} g_{\dot{x}}(t, u(t), \dot{u}(t))], \qquad t \in I, \end{aligned} \tag{7}$$

$$y(t) \geqslant 0, \qquad t \in I, \tag{8}$$

$$\lambda \in \Lambda^+. \tag{9}$$

In problems (P_λ) and (WD_λ) the vector $0 < \lambda \in R^p$ is predetermined. Note that if $p = 1$ problems (P) and (WD) become the pair of Wolfe type dual variational problems studied by Mond and Hanson [8].

Let H denote the set of feasible solutions of (WD).

Theorem 3. *Let $x(t) \in K$ and $(u(t), \lambda, y(t)) \in H$. Let f and g be convex at $(u, \dot{u})$ over K. Then the following cannot hold*:

$$\begin{aligned} &\int_a^b f^i(t, x(t), \dot{x}(t))\, dt \\ &\quad \leqslant \int_a^b [f^i(t, u(t), \dot{u}(t)) + y(t)^{\mathrm{T}} g(t, u(t), \dot{u}(t))]\, dt, \qquad \forall i \in \{1, 2, ..., p\}, \\ &\int_a^b f^j(t, x(t), \dot{x}(t))\, dt \\ &\quad < \int_a^b [f^j(t, u(t), \dot{u}(t)) + y(t)^{\mathrm{T}} g(t, u(t), \dot{u}(t))]\, dt, \qquad \textit{for at least one } j. \end{aligned}$$

Proof. Convexity of f and g implies that $\lambda^{T}f + y(t)^{T}g$ is convex. This yields

$$\int_a^b (\lambda^{T}f(t, x, \dot{x}))\,dt$$

$$-\int_a^b [\lambda^{T}f(t, u(t), \dot{u}(t)) + y(t)^{T}g(t, u(t), \dot{u}(t))]\,dt$$

$$\geqslant \int_a^\beta \{((x(t)-u(t))^{T}[\lambda^{T}f_x(t, u(t), \dot{u}(t)) + y(t)^{T}g_x(t, u(t), \dot{u}(y))]$$

$$+ ((\dot{x}(t)-\dot{u}(t))^{T}[\lambda^{T}f_{\dot{x}}(t, u(t), \dot{u}(t)) + y(t)^{T}g_{\dot{x}}(t, u(t), \dot{u}(t))]\}\,dt$$

$$= \int_a^b \{((x(t)-u(t))^{T}D[\lambda^{T}f_{\dot{x}}(t, u(t), \dot{u}(t)) + y(t)^{T}g_{\dot{x}}(t, u(t), \dot{u}(t))]$$

$$+ ((\dot{x}(t)-\dot{u}(t))^{T}[\lambda^{T}f_{\dot{x}}(t, u(t), \dot{u}(t)) + y(t)^{T}g_{\dot{x}}(t, u(t), \dot{u}(t))]\}\,dt$$

Using integration by parts and boundary conditions (6), we obtain

$$\int_a^b (\lambda^{T}f(t, x, \dot{x}) + y(t)^{T}g(t, x, \dot{x}))\,dt$$

$$-\int_a^b [\lambda^{T}f(t, u(t), \dot{u}(t)) + y(t)^{T}g(t, u(t), \dot{u}(t))]\,dt \geqslant 0,$$

which in view of (2) and (8) yields

$$\int_a^b (\lambda^{T}f(t, x, \dot{x})\,dt \geqslant \int_a^b [\lambda^{T}f(t, u(t), \dot{u}(t)) + y(t)^{T}g(t, u(t), \dot{u}(t))]\,dt. \quad (10)$$

Thus, the following cannot hold:

$$\int_a^b f^i(t, x(t), \dot{x}(t))\,dt$$

$$\leqslant \int_a^b [f^i(t, u(t), \dot{u}(t)) + y(t)^{T}g(t, u(t), \dot{u}(t))]\,dt, \qquad \forall i \in \{1, 2, ..., p\},$$

$$\int_a^b f^j(t, x(t), \dot{x}(t))\,dt$$

$$< \int_a^b [f^j(t, u(t), \dot{u}(t)) + y(t)^{T}g(t, u(t), \dot{u}(t))]\,dt, \qquad \text{for at least one } j.$$

LEMMA 1. *Let $u^*(t) \in K$ and $(u^*(t), y^*(t)) \in H$. Let f and g be convex. If*

$$y^*(t)^{T}g(t, u^*(t), \dot{u}(t)) = 0, \qquad t \in I,$$

then $u^*(t)$ *is properly efficient for* (P) *and* $(u^*(t), \lambda^*, y^*(t))$ *is properly efficient for* (WD).

Proof. (i) From (10) together with $y^*(t)^T g(t, u^* \cdot (t), \dot{u}^*(t)) = 0$, $t \in I$, it follows that for all $x(t) \in K$,

$$\int_a^b \lambda^{*T} f(t, u^*(t), \dot{u}^*(t))$$

$$= \int_a^b [\lambda^{*T} f(t, u^*(t), \dot{u}^*(t)) + y^*(t)^T g(t, u^*(t), \dot{u}^*(t))]\, dt$$

$$\leqslant \int_a^b (\lambda^T f(t, x, \dot{x}))\, dt.$$

Thus $u(t)$ is an optimal solution of (P_λ). Hence Theorem 1 implies that $u^*(t)$ is a properly efficient solution of (P).

(ii) Assume that $(u^*(t), \lambda^*, y^*(t)$ is not efficient. This implies that we have $(\tilde{u}(t), \tilde{\lambda}, \tilde{y}(t)) \in H$ such that

$$\int_a^b [f^i(t, \tilde{u}(t), \tilde{u}(t)) + \tilde{y}(t)^T g(t, \tilde{u}(t), \tilde{u}(t))]\, dt$$

$$\geqslant \int_a^b [f^i(t, u^*(t), \dot{u}^*(t)) + y^*(t)^T g(t, u^*(t), \dot{u}(t))]\, dt,$$

$$\forall i \in \{1, 2, ..., p\},$$

and

$$\int_a^b [f^j(t, \tilde{u}(t), \tilde{u}(t)) + \tilde{y}(t)^T g(t, \tilde{u}(t), \tilde{u}(t))]\, dt$$

$$> \int_a^b [f^j(t, u^*(t), \dot{u}^*(t)) + y^*(t)^T g(t, u^*(t), \dot{u}^*(t))]\, dt,$$

for at least one $j \in \{1, 2, ..., p\}$.

Using $y^*(t)^T g(t, u^*(t), \dot{u}^*(t)) = 0$ in the above we obtain

$$\int_a^b [f^i(t, \tilde{u}(t), \tilde{u}(t)) + \tilde{y}(t)^T g(t, \tilde{u}(t), \tilde{u}(t))]\, dt$$

$$\geqslant \int_a^b f^i(t, u^*(t), \dot{u}^*(t)),\, dt, \qquad \forall i \in \{1, 2, ..., p\},$$

and

$$\int_a^b [f^j(t, \tilde{u}(t), \dot{\tilde{u}}(t)) + \tilde{y}(t)^{\mathrm{T}} g(t, \tilde{u}(t), \dot{\tilde{u}}(t))]\, dt$$

$$> \int_a^b f^j(t, u^*(t), \dot{u}^*(t))\, dt, \qquad \text{for at least one } j \in \{1, 2, ..., p\},$$

which contradicts Theorem 3. Hence $(u^*(t), \lambda^*, y^*(t))$ is efficient.

Now we assume that $(u^*(t), \lambda^*, y^*(t))$ is improperly efficient; i.e., there exists $(\tilde{u}(t), \tilde{\lambda}, \tilde{y}(t)) \in H$ such that for some i and all $M > 0$,

$$\int_a^b [f^i(t, \tilde{u}(t), \dot{\tilde{u}}(t)) + \tilde{y}(t)^{\mathrm{T}} g(t, \tilde{u}(t), \dot{\tilde{u}}(t))]\, dt$$

$$- \int_a^b [f^i(t, u^*(t), \dot{u}^*(t)) + y^*(t)^{\mathrm{T}} g(t, u^*(t), \dot{u}^*(t))]\, dt$$

$$> M \left\{ \int_a^b [f^j(t, u^*(t), \dot{u}^*(t)) + y^*(t)^{\mathrm{T}} g(t, u^*(t), \dot{u}^*(t))]\, dt \right.$$

$$\left. - \int_a^b [f^j(t, \tilde{u}(t), \dot{\tilde{u}}(t)) + \tilde{y}(t)^{\mathrm{T}} g(t, \tilde{u}(t), \dot{\tilde{u}}(t))]\, dt \right\}$$

and $\forall j \in \{1, 2, ..., p\}$, such that

$$\int_a^b [f^j(t, u^*(t), \dot{u}^*(t)) + y^*(t)^{\mathrm{T}} g(t, u^*(t), \dot{u}^*(t))]\, dt$$

$$> \int_a^b [f^j(t, \tilde{u}(t), \dot{\tilde{u}}(t)) + \tilde{y}(t)^{\mathrm{T}} g(t, \tilde{u}(t), \dot{\tilde{u}}(t))]\, dt.$$

Since $\tilde{\lambda} \in \Lambda^+$,

$$\int_a^b [\tilde{\lambda}^{\mathrm{T}} f(t, u^*(t), \dot{u}^*(t) + y^*(t)^{\mathrm{T}} g(t, u^*(t), \dot{u}^*(t))]\, dt$$

$$> \int_a^b [\tilde{\lambda}^{\mathrm{T}} f(t, \tilde{u}(t), \dot{\tilde{u}}(t) + \tilde{y}(t)^{\mathrm{T}} g(t, \tilde{u}(t), \dot{\tilde{u}}(t))]\, dt,$$

which along with $y^*(t)^{\mathrm{T}} g(t, u^*(t), \dot{u}^*(t)) = 0$ yields

$$\int_a^b \tilde{\lambda}^{\mathrm{T}} f(t, u^*(t), \dot{u}^*(t))\, dt$$

$$> \int_a^b [\tilde{\lambda}^{\mathrm{T}} f(t, \tilde{u}(t), \dot{\tilde{u}}(t)) + \tilde{y}(t)^{\mathrm{T}} g(t, \tilde{u}(t), \dot{\tilde{u}}(t))]\, dt.$$

This contradicts (10). Thus $(u^*(t), \lambda^*, y^*(t))$ is properly efficient.

THEOREM 4. *Let f and g be convex at $(u, \dot{u})$ over K. Let x^* be normal* [8] *and a properly efficient solution for* (P). *Then for some $\bar{\lambda} \in \Lambda^+$, there exists a piecewise smooth $y^*: I \to R^m$ such that $(x^*, \bar{\lambda}, y^*)$ is a properly efficient solution of* (WD) *and*

$$\int_a^b [f(t, x^*(t), \dot{x}^*(t))]\, dt$$

$$= \int_a^b [f(t, x^*(t), \dot{x}^*(t)) + y^*(t)^{\mathrm{T}} g(t, x^*(t), \dot{x}^*(t))]\, dt.$$

Proof. Since f and g are convex and x^* is a properly efficient solution of (P), by Theorem 2, x^* is optimal for $(P_{\bar{\lambda}})$ for some $\bar{\lambda} \in \Lambda^+$. Therefore, by Theorem 3, there exists a piecewise smooth $y^*: I \to R^m$ such that for $t \in I$

$$\begin{aligned} &\bar{\lambda}^{\mathrm{T}} f_x(t, x^*(t), \dot{x}^*(t)) + y^*(t)^{\mathrm{T}} g_x(t, x^*(t), \dot{x}^*(t)) \\ &\quad = D[\bar{\lambda}^{\mathrm{T}} f_{\dot{x}}(t, x^*(t), \dot{x}^*(t)) + y(t)^{\mathrm{T}} g_{\dot{x}}(t, x^*(t), \dot{x}^*(t))] \end{aligned} \tag{11}$$

$$y^*(t)^{\mathrm{T}} g(t, x^*(t), \dot{x}(t)) = 0 \tag{12}$$

$$y^*(t) \geqslant 0. \tag{13}$$

From (11) and (13) it follows that $(x^*, \bar{\lambda}, y^*) \in H$. Lemma 1 and (12) imply that $(x^*, \bar{\lambda}, y^*)$ is a properly efficient solution of (WD). Using (12) we have

$$\int_a^b [f^i(t, x^*(t), \dot{x}^*(t))]\, dt$$

$$= \int_a^b [f(t, x^*(t), \dot{x}^*(t)) + y^*(t)^{\mathrm{T}} g(t, x^*(t), \dot{x}^*(t))]\, dt.$$

For validating the converse duality theorem (Theorem 5), we make the assumption that X_2 denotes the space of piecewise differentiable function $x: I \to R^n$ for which $x(a) = 0 = x(b)$ equipped with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2 x\|_\infty$, defining D as before. The problem (WD) may be rewritten in the form

Minimize $-\phi(u, \lambda, y) = (-\phi^1(u, \lambda, y), -\phi^2(u, \lambda, y), ..., -\phi^p(u, \lambda, y))$ subject to

$$u(a) = \alpha, \qquad u(b) = \beta,$$

$$\Theta(t, u(t), \dot{u}(t), \ddot{u}(t), \lambda, y(t), \dot{y}(t)) = 0, \qquad t \in I,$$

$$y(t) \geqslant 0, \qquad t \in I,$$

where

$$\phi^i(u, \lambda, y) = \int_a^b [f^i(t, u(t), \dot{u}(t)) + y(t)^{\mathrm{T}} g(t, u(t), \dot{u}(t))]\, dt, \qquad i = 1, 2, ..., p,$$

and

$$\begin{aligned} \Theta &\equiv \Theta(t, u(t), \dot{u}(t), \ddot{u}(t), \lambda, y(t), \dot{y}(t)) \\ &\lambda^{\mathrm{T}} f_x(t, u(t), \dot{u}(t)) + y(t)^{\mathrm{T}} g_x(t, u(t), \dot{u}(t)) \\ &\quad - D[\lambda^{\mathrm{T}} f_{\dot{x}}(t, u(t), \dot{u}(t)) + y(t)^{\mathrm{T}} g_{\dot{x}}(t, u(t), \dot{u}(t))], \qquad t \in I, \end{aligned}$$

with $\ddot{u}(t) = D^2 u(t)$.

Consider $\Theta(\cdot, u(\cdot), \dot{u}(\cdot), \ddot{u}(\cdot), \lambda, y(\cdot), \dot{y}(\cdot))$ as defining a map $\Psi: X_2 \times Y \times \Lambda^+ \to A$, where Y is the space of piecewise differentiable function $y: I \to R^m$ and A is a Banach space. A Fritz John Theorem [4, 5] for infinite dimensional multiobjective programming problem may be applied to problem (WD) along with the analysis outlined in [8] or [3] for the derivation of optimality conditions. However, some restrictions are required as in [3] on the equality constraint $\Theta(\cdot) = 0$, since infinite dimensional space is involved here. It suffices to assume that the Fréchet derivative $\Psi' = (\Psi_x, \Psi_y, \Psi_\lambda)$ has a (weak*) closed range.

THEOREM 5. *Let f and g be convex at $(u, \dot{u})$ over K. Let (u^*, λ^*, y^*) with $u^* \in X_2$, $\lambda \in \Lambda^+$, and $y^* \in Y$ be a properly efficient solution of* (WD). *Let,*

- (I) *Ψ' have a (weak*) closed range,*
- (II) *f and g be twice continuously differentiable,*
- (III) *$f^i_x - Df^i_{\dot{x}}$, $i = 1, 2, ..., p$, be linearly independent, and*
- (IV) $(\beta(t)^{\mathrm{T}} \Theta_x - D\beta(t)^{\mathrm{T}} \Theta_{\dot{x}} + D^2 \beta(t)^{\mathrm{T}} \Theta_{\ddot{x}})\, \beta(t) = 0 \Rightarrow \beta(t) = 0$, $t \in I$.

Then, the objective functions of (P) *and* (WD) *are equal and u^* is a properly efficient solution of* (P).

Proof. Since (u^*, λ^*, y^*), with $u^* \in X_2$ and Ψ' having a (weak*) closed range, is properly efficient, it is a weak maximum. Hence, there exists $\alpha \in R^p$, $\mu \in R^p$, and piecewise smooth $\beta: I \to R^n$ and $\delta: I \to R^m$ satisfying the following Fritz John conditions [4, 5], which are derived by means of the analysis of [3],

$$\begin{aligned} &\alpha^{\mathrm{T}}(f_x + y(t)^{\mathrm{T}} g_x) - D\alpha^{\mathrm{T}}(f_{\dot{x}} + y(t)^{\mathrm{T}} g_{\dot{x}}) \\ &\quad - (\beta(t)^{\mathrm{T}} \Theta_x - D\beta(t)^{\mathrm{T}} \Theta_{\dot{x}} + D^2 \beta(t)^{\mathrm{T}} \Theta_{\ddot{x}}) = 0, \qquad t \in I, \qquad (14) \end{aligned}$$

$$(\alpha^{\mathrm{T}}e)\, g-(\beta(t)^{\mathrm{T}}\Theta_y - D\beta(t)^{\mathrm{T}}\Theta_{\dot{y}})+\delta(t)=0, \qquad t\in I, \tag{15}$$

$$\beta(t)^{\mathrm{T}}(f_x - Df_{\dot{x}})=\mu, \qquad t\in I, \tag{16}$$

$$\delta(t)^{\mathrm{T}}y(t)=0, \qquad t\in I, \tag{17}$$

$$\mu^{\mathrm{T}}\lambda=0 \tag{18}$$

$$(\alpha,\ \beta(t),\ \delta(t),\ \mu)\geqslant 0, \qquad t\in I, \tag{19}$$

$$(\alpha,\ \beta(t),\ \delta(t),\ \mu)\neq 0, \qquad t\in I, \tag{20}$$

where $f\equiv f(t,u^*(t),\dot{u}^*(t))$, $g\equiv g(t,u^*(t),\dot{u}^*(t))$, $f_x\equiv f_x(t,u^*(t),\dot{u}^*(t))$, etc., with all derivatives evaluated at $u=u^*$. Note that the term $D^2(\beta(t)^{\mathrm{T}}\Theta_{\dot{x}})$ in (14) is obtained by using integration by parts with boundary conditions $\beta(a)=0=\beta(b)$, adjoined to the differential equations (14) and (15) so that the integrated parts occurring in their derivation vanish. Since f and g are twice continuously differentiable, Θ also is twice continuously differentiable.

Equation (14) along with (7) yields

$$\begin{aligned}&(\alpha-(\alpha^{\mathrm{T}}e)\,\lambda^*)^{\mathrm{T}}(f_x-Df_{\dot{x}})\\ &\quad -(\beta(t)^{\mathrm{T}}\Theta_x - D\beta(t)^{\mathrm{T}}\Theta_{\dot{x}}+D^2\beta(t)^{\mathrm{T}}\Theta_{\dot{x}})=0, \qquad t\in I.\end{aligned} \tag{21}$$

Since $\lambda^*\in\Lambda^+$ and $\mu\geqslant 0$, (18) implies $\mu=0$. Thus (14) implies

$$\beta(t)^{\mathrm{T}}(f_x-Df_{\dot{x}})=0, \qquad t\in I. \tag{22}$$

Equation (22) along with (21) yields

$$(\beta(t)^{\mathrm{T}}\Theta_x - D\beta(t)^{\mathrm{T}}\Theta_{\dot{x}}+D^2\beta(t)^{\mathrm{T}}\Theta_{\dot{x}})\,\beta(t)=0, \qquad t\in I, \tag{23}$$

which in view of hypothesis (IV) of the theorem gives

$$\beta(t)=0, \qquad t\in I. \tag{24}$$

Equations (21) and (24) now yield

$$(\alpha-(\alpha^{\mathrm{T}}e)\,\lambda^*)^{\mathrm{T}}(f_x-Df_{\dot{x}})=0,$$

which along with hypothesis (III) yields

$$\alpha=(\alpha^{\mathrm{T}}e)\,\lambda^*. \tag{25}$$

We now claim that $\alpha>0$.

If $\alpha=0$, then from (15) we have $\delta(t)=0$, $t\in I$. Thus we have $(\alpha,\beta(t),\delta(t),\mu)=0$ for $t\in I$. This contradicts condition (20). Hence we conclude that $\alpha > 0$, which means $\alpha^{\mathrm{T}}e > 0$. From (25) it follows that $\alpha > 0$. Therefore, from (15) we obtain

$$g(t, u^*(t), \dot{u}^*(t)) = -\delta(t)/(\alpha^{\mathrm{T}}e) \leqslant 0, \qquad t \in I. \tag{26}$$

Relation (26) yields that $u^*(t) \in K$. Again, (26) together with (17) gives

$$y(t)^{\mathrm{T}}g(t, u^*(t), \dot{u}^*(t)) = 0, \qquad t \in I. \tag{27}$$

From (27) we have

$$\begin{aligned}\int_a^b &[f(t, u^*(t), \dot{u}^*(t) + y^*(t)^{\mathrm{T}}g(t, u^*(t), \dot{u}(t))]\, dt \\ &= \int_a^b f(t, u^*(t), \dot{u}^*(t))\, dt \end{aligned} \tag{28}$$

and, by Lemma 1, $u^*(t)$ is a properly efficient solution of (P).

4. Related Problems

As in [3, 8] the foregoing duality results can be extended to the corresponding problems (P_0) and (WD_0) given below. We obtain (P_0) by omitting the boundary conditions (1) and (WD_0) by adjoining the "natural boundary conditions."

(P_0) Minimize $\int_a^b f(t, x(t), \dot{x}(t))\, dt = (\int_a^b f^1(t, x(t), \dot{x}(t))\, dt, ..., \int_a^b f^p(t, x(t), \dot{x}(t))\, dt)$

subject to

$$g(t, x(t), \dot{x}(t)) \leqslant 0, \qquad t \in I.$$

(WD_0) Maximize $(\int_a^b [f^1(t, u(t), \dot{u}(t)) + y(t)^{\mathrm{T}}g(t, u(t), \dot{u}(t))]\, dt, ..., \int_a^b [f^p(t, u(t), \dot{u}(t)) + y(t)^{\mathrm{T}}g(t, u(t), \dot{u}(t))]\, dt)$

subject to

$$\begin{gathered}\lambda^{\mathrm{T}}f_x(t, u(t), \dot{u}(t)) + y(t)^{\mathrm{T}}g_x(t, u(t), \dot{u}(t)) \\ = D[\lambda^{\mathrm{T}}f_{\dot{x}}(t, u(t), \dot{u}(t)) + y(t)^{\mathrm{T}}g_{\dot{x}}(t, u(t), \dot{u}(t))], \qquad t \in I, \\ y(t) \geqslant 0, \qquad t \in I, \\ \lambda^{\mathrm{T}}f_{\dot{x}}(t, u(t), \dot{u}(t)) + y(t))^{\mathrm{T}}g_{\dot{x}}(t, u(t), \dot{u}(t) = 0, \qquad \text{when } t = a,\ t = b, \\ \lambda \in \Lambda^+.\end{gathered}$$

In particular if (P_0) and (WD_0) are independent of t; i.e., if f and g do not depend explicitly on t, then these problems essentially reduce to the static case $(\bar{\mathrm{P}})$ and $(\bar{\mathrm{D}})$ of multiobjective nonlinear programs studied by several authors, notably by Weir [13].

$(\bar{\mathrm{P}})$ Minimize $f(x)=(f^1(x), f^2(x), ..., f^p(x))$

subject to

$$g(x) \leqslant 0.$$

$(\bar{\mathrm{D}})$ Maximize $(f^1(u)+y^{\mathrm{T}}g(u), f^2(u)+y^{\mathrm{T}}g(u), ..., f^p(u) + y^{\mathrm{T}}g(u))$

subject to

$$\lambda^{\mathrm{T}}f_x(u)+y^{\mathrm{T}}g_x(u)=0$$

$$y \geqslant 0$$

$$\lambda \in \Lambda^+.$$

5. Mond–Weir Type Duality

The Mond–Weir type dual for (P_λ) is as follows:
(MD_λ) Maximize $\int_a^b \lambda^{\mathrm{T}} f(t, u(t), \dot{u}(t))\, dt$

subject to

$$u(\alpha)=\alpha, \qquad u(b)=\beta,$$

$$\lambda^{\mathrm{T}}f_x(t, u(t), \dot{u}(t))+y(t)^{\mathrm{T}}g_x(t, u(t), \dot{u}(t))$$
$$=D[\lambda^{\mathrm{T}}f_{\dot{x}}(t, u(t), \dot{u}(t))+y(t)^{\mathrm{T}}g_{\dot{x}}(t, u(t), \dot{u}(t))], \qquad t \in I,$$

$$\int_a^b y^{\mathrm{T}}f(t, u(t), \dot{u}(t))\, dt \geqslant 0,$$

$$y(t) \geqslant 0, \qquad t \in I,$$

where the vector $\lambda > 0$, $\lambda \in R^p$ is predetermined.

We white the following vector maximization variational problem as the Mond–Weir type dual (MD) of (P).

(MD) Maximize $\int_a^b f(t, u(t), \dot{u}(t))\, dt = (\int_a^b f^1(t, u(t), \dot{u}(t))\, dt, ..., \int_a^b f^p(t, u(t), \dot{u}(t))\, dt)$

subject to

$$u(a)=\alpha, \qquad u(b)=\beta,$$

$$\begin{aligned}\lambda^{\mathrm{T}} f_x(t, u(t), \dot{u}(t)) &+ y(t)^{\mathrm{T}} g_x(t, u(t), \dot{u}(t)) \\ &= D[\lambda^{\mathrm{T}} f_{\dot{x}}(t, u(t), \dot{u}(t)) + y(t)^{\mathrm{T}} g_{\dot{x}}(t, u(t), \dot{u}(t))], \qquad t \in I,\end{aligned}$$

$$\int_a^b y^{\mathrm{T}} f(t, u(t), \dot{u}(t))\, dt \geqslant 0,$$

$$y(t) \geqslant 0, \qquad t \in I,$$

$$\lambda \in \Lambda^+.$$

In the above it is easily seen that for $p=1$, problems (P) and (MD) become the nonsymmetric dual variational problems studied by Bector, Chandra, and Husain [1].

Denoting by G the set of feasible solutions of (MD) we state the following duality Theorems 6–8, that can be proved as in Wolfe's duality.

THEOREM 6. *Let $x(t) \in K$ and $(u(t), \lambda, y(t)) \in G$. Let f and g be convex at $(u, \dot{u})$ over K. Then the following cannot hold:*

$$\int_a^b f^i(t, x(t), \dot{x}(t))\, dt \leqslant \int_a^b f^i(t, u(t), \dot{u}(t))\, dt, \qquad \forall i \in \{1, 2, ..., p\},$$

$$\int_a^b f^j(t, x(t), \dot{x}(t))\, dt < \int_a^b f^j(t, u(t), \dot{u}(t))\, dt, \qquad \textit{for at least one } j.$$

THEOREM 7. *Let f and g be convex at $(u, \dot{u})$ over K. Let x^* be normal and a properly efficient solution for* (P). *Then for some $\bar{\lambda} \in \Lambda^+$, there exists a piecewise smooth $y^*: I \to R^m$ such that $(x^*, \bar{\lambda}, y^*)$ is a properly efficient solution of* (MD) *and*

$$\int_a^b [f(t, x^*(t), \dot{x}^*(t))\, dt = \int_a^b f(t, x^*(t), \dot{x}^*(t))\, dt.$$

THEOREM 8. *Let f and g be convex at $(u, \dot{u})$ over K. Let (u^*, λ^*, y^*) with $u^* \in X_2$, $\lambda \in \Lambda^+$, and $y^* \in Y$ be properly efficient solution of* (MD). *Let*

(I) *Ψ' have a (weak*) closed range,*

(II) *f and g twice continuously differentiable,*

(III) *$f^i_x - Df^i_{\dot{x}}$, $i=1, 2, ..., p$, be linearly independent, and*

(IV) $(\beta(t)^{\mathrm{T}}\Theta_x - D\beta(t)^{\mathrm{T}}\Theta_{\dot{x}} + D^2\beta(t)^{\mathrm{T}}\Theta_{\dot{x}})\,\beta(t)=0 \Rightarrow \beta(t)=0,\ t \in I.$

Then, the objective functionals of (P) *and* (MD) *are equal and* u^* *is a properly efficient solution of* (P).

The duality results similar to those contained in the above section can also be established for the following pair of problems with "natural boundary values."

(P_0) Minimize $\int_a^b f(t, x(t), \dot{x}(t))\,dt = (\int_a^b f^1(t, x(t), \dot{x}(t))\,dt, ..., \int_a^b f^p(t, x(t), \dot{x}(t)\,dt)$
subject to

$$g(t, x(t), \dot{x}(t)) \leqslant 0, \qquad t \in I.$$

(MD_0) Maximize $(\int_a^b f^1(t, u(t), \dot{u}(t)), dt, ..., \int_a^b f^p(t, u(t), \dot{u}(t))\,dt)$
subject to

$$\lambda^T f_x(t, u(t), \dot{u}(t)) + y(t)^T g_x(t, u(t), \dot{u}(t))$$
$$= D[\lambda^T f_{\dot{x}}(t, u(t), \dot{u}(t)) + y(t)^T g_{\dot{x}}(t, u(t), \dot{u}(t))], \qquad t \in I,$$
$$\int_a^b y(t)^T g(t, u(t), \dot{u}(t))\,dt \geqslant 0,$$
$$y(t) \geqslant 0, \qquad t \in I,$$
$$\lambda^T f_{\dot{x}}(t, u(t), \dot{u}(t)) + y(t)^T g_{\dot{x}}(t, u(t), \dot{u}(t)) = 0, \qquad \text{when } t = a,\ t = b,$$
$$\lambda \in \Lambda^+.$$

If the problems (P_0) and (MD_0) are independent of t; i.e., if f and g do not depend explicitly on t, then these problems essentially reduce to be the static case ($\bar{P}$) and ($\bar{D}$) of multiobjective nonlinear programs studied by several authors, notably by Weir [13].

($\bar{P}$) Minimize $f(x) = (f^1(x), f^2(x), ..., f^p(x))$
subject to

$$g(x) \leqslant 0.$$

($\bar{D}$) Maximize $(f^1(u), f^2(u), ..., f^p(u))$
subject to

$$\lambda^T f_x(u) + y^T g_x(u) = 0$$
$$y^T g(u) \geqslant 0$$
$$y \geqslant 0, \qquad \lambda \in \Lambda^+.$$

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REFERENCES

1. C. R. BECTOR, S. CHANDRA, AND I. HUSAIN, Generalized concavity and duality in continuous programming, *Utilitas Math.* **25** (1984), 171–190.
2. J. M. BORWEIN, "Optimization with Respect to Partial Ordering," Ph.D. Thesis, Oxford University, 1974.
3. S. CHANDRA, B. D. CRAVEN, AND I. HUSAIN, A class of nondifferentiable continuous programming problems, *J. Math. Anal. Appl.* **107** (1985), 122–131.
4. B. D. CRAVEN, Lagrangian conditions and quasi duality, *Bull. Austral. Math. Soc.* **16** (1977), 325–339.
5. B. D. CRAVEN, "Lagrangian Conditions, Vector Minimization and Local Duality," Research Report 37, Department of Mathematics, University of Melbourne, Australia, 1980.
6. R. R. EGUDO, Proper efficiency and multiobjective duality in nonlinear programming, *J. Inform. Optim. Sci.* **8** (1987), 155–166.
7. A. M. GEOFFRION, Proper efficiency and the theory of vector maximization, *J. Math. Anal. Appl.* **22** (1968), 618–630.
8. B. MOND AND M. A. HANSON, Duality for variational problem, *J. Math. Anal. Appl.* **18** (1967), 355–364.
9. B. MOND, I. HUSAIN, AND M. V. DURGAPARSAD, Duality for a class of nondifferential multiobjective programming problems, *J. Inform. Optim. Sci.* **9** (1988), 331–341.
10. B. MOND, I. HUSAIN, AND M. V. DURGAPARSAD, "Duality for a Class of Nondifferentiable Multiobjective Programming Problems," Mathematics Research Paper 89-11, Department of Mathematics, La Trobe University, Melbourne, Australia, April 1989.
11. F. A. VALENTINE, "The Problem of Lagrange with Differential Inequality as Added Side Conditions, in Contribution to the Calculus of Variations, 1933–1937," pp. 407–448, Univ. of Chicago Press, Chicago, 1937.
12. B. MOND AND T. WEIR, Generalized concavity and duality, *in* "Generalized Concavity in Optimization and Economics" (S. Schaible and W. T. Ziemba, Eds.), pp. 263–279, Academic Press, New York, 1981.
13. T. WEIR, Proper efficiency and duality for vector valued optimization problems, *J. Austral. Math. Soc. Ser. A* **43** (1987), 21–35.
14. P. WOLFE, A duality theorem for nonlinear programming, *Quart. Appl. Math.* **19** (1961), 239–244.