Continuous spectrum and square-integrable solutions of differential operators with intermediate deficiency index

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Abstract

We explore the connection between square-integrable solutions for real-values of the spectral parameter $\lambda$ and the continuous spectrum of self-adjoint ordinary differential operators with arbitrary deficiency index $d$. We show that if, for all $\lambda$ in an open interval $I$, there are $d$ of linearly independent square-integrable solutions, then for every extension of $D_{\text{min}}$ the point spectrum is nowhere dense in $I$, and there is a self-adjoint extension of $S_{\text{min}}$ which has no continuous spectrum in $I$. This analysis is based on our construction of limit-point (LP) and limit-circle (LC) solutions obtained recently in an earlier paper.

Keywords: Differential operators; Intermediate deficiency indices; Continuous spectrum; Square-integrable solutions; Singular boundary conditions

1. Introduction

This paper is a continuation of our study of self-adjoint ordinary differential operators in [17] and [16]. In [17] we constructed solutions of 'limit-circle (LC) type' for all values of the deficiency index $d$ between the minimal and maximal values, and used these to characterize the
self-adjoint domains; in [16] we constructed such domains with singular separated, coupled, and mixed boundary conditions and classified the various types of boundary conditions. In this paper we exploit this characterization and construction, especially the construction of separated singular boundary conditions, to obtain information about the spectrum, particularly the continuous spectrum.

The spectrum of a self-adjoint operator in Hilbert space is real. For singular differential operators the spectrum consists, in general, of eigenvalues and of essential spectrum. (In some of the literature ‘essential spectrum’ and ‘continuous spectrum’ are used interchangeably, we will use Weidmann’s definitions in [19] which differentiate between these terms, see Definitions 4–6.) Which real numbers $\lambda$ are in the spectrum of an operator? Given a self-adjoint differential operator $S$, a real number $\lambda$ is an eigenvalue of $S$ if the corresponding differential equation has a nontrivial solution which happens to satisfy the boundary condition of $S$. This happens ‘coincidentally.’ On the other hand, the essential spectrum is independent of the boundary conditions and thus depends only on the coefficients, including the weight function $w$, of the equation. This dependence is implicit and highly complicated. The coefficients and the weight function determine which solutions are in the Hilbert space $H = L^2(J, w)$. In this paper we study the relationship between the solutions in $H$ for real-values of $\lambda$ and the continuous spectrum.

Our main theorem extends a result in Weidmann’s monograph [19] from the minimal deficiency index case to an arbitrary deficiency index. In the minimal case there is no singular boundary condition, whereas for any deficiency greater than the minimal value there are singular boundary conditions. Although our proof uses the general approach in [19], there is a major difference due to the presence of singular boundary conditions. To overcome the formidable obstacles posed by these, we use the construction of separated singular boundary conditions from [16], which is based on the characterization of self-adjoint domains in [17].

2. Statement of the main result

We study spectral properties of self-adjoint realizations of the equation

$$My = \lambda wy \quad \text{on } J = (a, b), \ -\infty < a < b \leq \infty,$$

in the Hilbert space $H = L^2(J, w)$, where $M$ is a general symmetric quasi-differential expression of order $n = 2k$, $k > 1$, with real-valued coefficients (the case $k = 1$ is discussed in the book [22]), $w \in L_{\text{loc}}(J), w > 0$ on $J$, the endpoint $a$ is regular and the endpoint $b$ is singular. (Our results below also hold for the case when $b$ is regular but it is convenient to state them for the general singular case.)

For sufficiently smooth real-valued coefficients, the most general symmetric (formally self-adjoint) differential expressions of order $n = 2k$, $k > 1$, have the form [1,10],

$$My = \sum_{j=0}^{k} (p_j y^{(j)})^{(j)}. \quad \quad (2.2)$$

We are interested in using much weaker conditions, i.e., local Lebesgue integrability, on the coefficients. For this purpose Eq. (2.2) is modified by using quasi-derivatives $y^{[j]}$ as follows.

For $J = (a, b)$ an interval with $-\infty < a < b \leq \infty$ and $n = 2k$, $k > 1$, let
\[ Z_n(J, R) := \{ Q = (q_{rs})_{r,s=1}^n, \ q_{rs} \text{ real-valued,} \]
\[ q_{r,r+1} \neq 0 \text{ a.e. on } J, \ q_{r,r+1}^{-1} \in L_{\text{loc}}(J), \ 1 \leq r \leq n - 1, \]
\[ q_{rs} = 0 \text{ a.e. on } J, \ 2 \leq r < s \leq n; \]
\[ q_{rs} \in L_{\text{loc}}(J), \ s \neq r + 1, \ 1 \leq r \leq n - 1 \}. \quad (2.3) \]

For \( Q \in Z_n(J, \mathbb{R}) \) we define
\[ V_0 := \{ y : J \to \mathbb{C}, \ y \text{ is measurable} \} \]
\[ y^{[0]} := y \quad (y \in V_0). \quad (2.4) \]

Inductively, for \( r = 1, \ldots, n \), we define
\[ V_r = \{ y \in V_{r-1} : y^{[r-1]} \in \left( AC_{\text{loc}}(J) \right) \}, \]
\[ y^{[r]} = q_{r,r+1}^{-1} \left\{ y^{[r-1]} - \sum_{s=1}^{r} q_{rs} y^{[s-1]} \right\} \quad (y \in V_r), \quad (2.5) \]

where \( q_{n,n+1} := 1 \), and \( AC_{\text{loc}}(J) \) denotes the set of complex-valued functions which are absolutely continuous on all compact subintervals of \( J \). Finally we set
\[ M y = M_Q y := (-1)^k y^{[n]} \quad (y \in V_n). \quad (2.6) \]

The expression \( M = M_Q \) is called the quasi-differential expression associated with \( Q \). For \( V_n \) we also use the notations \( V(M) \) and \( D(Q) \). The vector function \( y^{[r]} (0 \leq r \leq n) \) is called the \( r \)th quasi-derivative of \( y \). Since the quasi-derivative depends on \( Q \), we sometimes write \( y^{[r]}_Q \) instead of \( y^{[r]} \). We now define symmetric quasi-differential expressions \( Q \), these generate symmetric and self-adjoint operators in the Hilbert space \( H \).

**Definition 1.** Let \( Q \in Z_n(J, \mathbb{R}) \) and let \( M = M_Q \) be defined as above. Assume that
\[ Q = -E^{-1} Q^* E, \quad \text{where} \ E = \left( (-1)^{\delta_{r,n+1-s}} \right)_{r,s=1}^n. \quad (2.7) \]

Then \( M = M_Q \) is called a symmetric differential expression.

A simple example of a symmetric differential expression is
\[ M y = (-1)^k y^{(2k)} + q y, \quad q \in L_{\text{loc}}(J, \mathbb{R}). \quad (2.8) \]

For a discussion of symmetric quasi-differential expressions see [2–4,9,15,21].

**Definition 2.** Let \( Q \in Z_n(J, \mathbb{R}), \ J = (a, b) \). The expression \( M = M_Q \) is said to be regular at \( a \) if for some \( c, a < c < b \), we have
\[ q_{r,r+1}^{-1} \in L(a, c), \quad r = 1, \ldots, n - 1; \]
\[ q_{rs} \in L(a, c), \quad 1 \leq r, s \leq n, \ s \neq r + 1. \]
Note that, from (2.3) it follows that if the above hold for some \( c \in J \) then they hold for any \( c \in J \).

**Definition 3.** Let \( Q \in Z_n(J, \mathbb{R}) \), and assume that \( M = MQ \) is symmetric and regular at \( a \). The deficiency index \( d = d(M, w) \) is the number of linearly independent solutions of (2.1) with \( \lambda = i \) which lie in \( H \).

It is well known [10,19] that \( d \) is independent of \( \lambda \) for all \( \lambda \in \mathbb{C} \) with \( \text{Im}(\lambda) \neq 0 \), satisfies the inequality
\[
k \leq d \leq 2k
\] (2.9) and that all values of \( d \) in this range are realized.

For real \( \lambda \) the number of linearly independent solutions of (2.1) lying in \( H \) may be less than \( d \) but cannot be greater than \( d \), see [17]. The minimal deficiency case \( d = k \) is called the limit-point (LP) case and the maximal deficiency case \( d = 2k \) is called the limit-circle (LC) case in analogy with the celebrated Weyl terminology when \( k = 1 \). We refer to the cases when \( k < d < 2k \) as the ‘intermediate’ cases; these have no analogue when \( n = 2 \) and are much more difficult to study. As in [17], by a self-adjoint realization of Eq. (2.1) we mean any operator \( S \) in \( H \) which satisfies
\[
S_{\min} \subset S = S^* \subset S_{\max},
\] (2.10) see [17] for a definition of \( S_{\min} \) and \( S_{\max} \).

Next we give definitions of continuous spectrum and pure point spectrum.

**Definition 4.** (See [18].) Let \( T \) be a self-adjoint operator on \( H \). Let \( H_p \) denote the closed linear hull of all eigenelements of \( T \), we call \( H_p = H_p(T) \) the discontinuous subspace of \( H \) with respect to \( T \). The orthogonal complement of \( H_p \) is called the continuous subspace of \( H \) with respect to \( T \). This is denoted by \( H_c = H_c(T) \).

We denote by \( T_p, T_c \) the restrictions of \( T \) to \( H_p, H_c \), respectively. These operators are called the (spectral) discontinuous, and continuous parts of \( T \), respectively.

**Definition 5.** (See [18].) The continuous spectrum \( \sigma_c(T) \) of \( T \) is defined as the spectrum of \( T_c \). The point spectrum \( \sigma_p(T) \) is defined as the set of eigenvalues of \( T \).

**Remark.** The point spectrum \( \sigma_p(T) \) is also the eigenvalues of \( T_p \); however, in general, we only have \( \sigma(T_p) = \overline{\sigma_p(T)} \). The set \( \sigma_c(T) \) is closed, and \( \sigma(T) = \overline{\sigma_p(T)} \cup \sigma_c(T) \). We say that \( T \) has a pure point spectrum if \( H_p = H \), i.e. \( \sigma(T) = \overline{\sigma_p(T)} \) (see [18, p. 209]).

Another basic partition of the spectrum is in terms of the discrete spectrum and the essential spectrum.

**Definition 6.** (See [7].) The discrete spectrum of \( T \), \( \sigma_d(T) \), is the set of all isolated eigenvalues of \( T \) with finite multiplicity, the essential spectrum of \( T \) is the complement in \( \sigma(T) \) of \( \sigma_d(T) \).

The next theorem is our main result; the special case of it when \( d = k \) and \( w = 1 \) is contained in Theorem 11.7 of Weidmann’s monograph [19].
Theorem 1. Let $M = MQ$, $Q \in \mathbb{Z}_n(J, \mathbb{R})$, $n = 2k$, $k > 1$, be a symmetric differential expression, $w \in L_{\text{loc}}(\mathbb{R})$, $w > 0$ on $J$, and let the endpoint $a$ of $J$ be regular. Let $d$ be the deficiency index of $(M, w)$ and suppose that $k < d < 2k$. Assume there exists an open interval $I = (\mu_1, \mu_2)$, $-\infty \leq \mu_1 < \mu_2 \leq \infty$, of the real line such that Eq. (2.1) has $d$ linearly independent solutions which lie in $H$ for every $\lambda \in I$. Then we have:

1. There is a self-adjoint realization $S$ of (2.1), with strictly separated boundary conditions, such that the intersection $\sigma_c(S) \cap I$ is empty.
2. For every self-adjoint realization $S$ of (2.1), the point spectrum $\sigma_p(S)$ is nowhere dense in $I$.

Proof. This will be given below, it is long and technical.

Definition 7. For real $\lambda$, let $r(\lambda)$ denote the number of linearly independent solutions of (2.1) which lie in $H = L^2(J, w)$.

Theorem 1 adds to our understanding of the relationship between the spectrum of self-adjoint realizations of Eq. (2.1) and the number of linear independent solutions of this equation for real-values of the spectral parameter $\lambda$. We collect some of these results in the next theorem, make some remarks and state a conjecture.

Theorem 2. Let $M = MQ$, $Q \in \mathbb{Z}_n(J, \mathbb{R})$, $n = 2k$, $k > 1$, be a symmetric differential expression, $w \in L_{\text{loc}}(\mathbb{R})$, $w > 0$ on $J$, and let the endpoint $a$ of $J$ be regular. Let $d$ be the deficiency index of $(M, w)$ and let $r(\lambda)$ be defined by Definition 7 for $\lambda \in \mathbb{R}$. The following results hold:

1. For every $\lambda \in \mathbb{R}$ we have $r(\lambda) \leq d$.
2. If $r(\lambda) < d$, then $\lambda$ is in the essential spectrum of every self-adjoint realization of Eq. (2.1).
3. If $r(\lambda) = d$ for all $\lambda$ in an open interval $I$, then there is a self-adjoint realization $S$, with strictly separated boundary conditions, such that $\sigma_c(S) \cap I$ is empty.
4. For any $\lambda \in \mathbb{R}$, if Eq. (2.1) has $m$ linearly independent solutions in $H$, then it has $m$ linearly independent real-valued solutions in $H$.

Proof. Part (1) is established in [1], see p. 1398, for smooth coefficients but the proof given there can be extended to the much more general symmetric expressions $M$ studied here. Part (2) is well known, see [19], also the proof given in [10] for a special case extends readily to our hypotheses. Part (3) is given by part (1) of our Theorem 1 when $k < d < 2k$. The case $d = k$ is established in Weidmann [19]; the case $d = 2k$ is the maximal deficiency or limit-circle (LC) case for which it is well known that the spectrum of every self-adjoint realization of (2.1) is discrete. Part (4) is proven in [17].

Remark. Part (1) of Theorem 1 shows that there is a self-adjoint realization $S$ which has no continuous spectrum in $I$. We conjecture that this result is true for every self-adjoint realization $S$. In our proof, given below, we construct such an $S$ with strictly separated boundary conditions. When combined with some results from [16] this construction can be used to show that there is no continuous spectrum in $I$ for any a self-adjoint realization $S$ of (2.1) determined by strictly separated boundary conditions. For coupled and mixed boundary conditions our construction needs some further refinements. We plan to pursue this in a subsequent paper.
In his book [19] Weidmann suggests that, for the case when \( d = k \), it should be possible to strengthen the conclusion that there is no continuous spectrum in \( I \) to ‘there is no essential spectrum in \( I \)’ and refers to some work of Hartman and Wintner [5,6] for the second order, i.e. Sturm–Liouville, case. We conjecture that this result holds for the general deficiency case.

**Conjecture 1.** Let \( M = MQ, Q \in Z_n(J, \mathbb{R}), n = 2k, k > 1 \), be a symmetric differential expression, \( w \in L_{\text{loc}}(\mathbb{R}), w > 0 \) on \( J \), and let the endpoint \( a \) of \( J \) be regular. Let \( d \) be the deficiency index of \( (M, w) \) and assume that Eq. (2.1) has \( d \) linearly independent solutions which lie in \( H \) for every \( \lambda \) in an open interval \( I \) of the real line. Then there is no essential spectrum in \( I \) for any self-adjoint realization \( S \) of (2.1).

The proof of the main result, Theorem 1, depends on our construction of ‘LC’ and ‘LP’ solutions given in [17]. These can be viewed as higher order analogues of the well-known ‘limit-point’ (LP) and ‘limit-circle’ (LC) solutions in the second order case. Next we comment on this and also on the general symmetric differential expressions \( M \) used here and in [16,17].

**Remark.** In the second order case \( n = 2 \) with one regular and one singular endpoint \( d = 1 \) or \( d = 2 \). This is the celebrated Weyl alternative: \( d = 1 \) is the Weyl limit-point (LP) case and \( d = 2 \) is the Weyl limit-circle (LC) case. In an attempt to extend Weyl’s alternative to higher order problems, Windau [20] for \( n = 4 \) and Shin [12–15] for general even order \( n = 2k \), erroneously reached the conclusion that the only values of \( d \) are \( d = k \) or \( d = n = 2k \). Glazman, see [10], showed that all possible values of \( d \) in (2.9) occur. Simpler examples were later given by Orlov [11] and by Kauffman, Read and Zettl [8]. Although Shin reached the wrong conclusion about the deficiency index, he also discovered the general symmetric expressions discussed in Section 2 above. These were rediscovered by Zettl [21].

3. **Local absence of continuous spectrum**

In this section we prove part (1) of Theorem 1. Specifically we show that if, for all \( \lambda \) in an open interval \( I \), the number of linearly independent solutions of Eq. (2.1) is equal to the deficiency index \( d \), then there is a self-adjoint realization \( S \) with strictly separated boundary conditions, and \( S \) has no continuous spectrum in the interval \( I \). The construction of separated boundary conditions in terms of real solutions given in [16] plays an important role in our proof. We continue to use the notations and definitions from Sections 1 and 2 above.

Our proof has three basic components: (i) The construction of a self-adjoint operator realization \( S \) on \( H \) determined by strictly separated boundary conditions; as shown in [16] there must be exactly \( k \) separated conditions at the regular endpoint \( a \) and exactly \( s = \frac{m}{2} = d - k \) separated conditions at the singular endpoint \( b \). (ii) For \( t \in (a, b) \) the construction of an operator realization \( S_t \) of \((M, w)\) in the (truncated) Hilbert space \( H_t = L^2((a, t), w) \) also with strictly separated boundary conditions; in this case, because both endpoints are regular, there must be exactly \( k \) conditions at \( a \) and exactly \( k \) at \( t \), as shown in [16]. (iii) The separated conditions on \((a, t)\) must be selected such that \( S_t \) converges to \( S \) as \( t \to b \) in the sense of strong resolvent convergence. Since there are exactly \( k \) separated conditions at the regular endpoint \( t \) and only \( s = d - k \) separated conditions at the singular endpoint \( b \), only \( d - k \) separated conditions at \( t \) can be ‘inherited’ from those at \( b \) and the additional ones are chosen so that they ‘disappear’ in the limit as \( t \to b \). This is achieved by a careful use of the limit-circle (LC) and limit-point (LP) solutions constructed in [17].
Recall that $d$ denotes the deficiency index and let $m = 2d - 2k$. Assume for some $\lambda \in \mathbb{R}$ Eq. (2.1) has $d$ linearly independent solutions lying in $H$. By Theorem 2 we may assume that these solutions are real-valued. In [17] these $d$ solutions are classified into two disjoint classes: LC and LP. Denote the $m$ LC solutions by $u_1, \ldots, u_m$ and the $d - m$ LP solutions by $u_{m+1}, \ldots, u_d$. Let $y_1, \ldots, y_n$ denote a solution bases which contains the LC and LP solutions. (The $u_j$ are now renamed $y_j$.)

In [17] we characterized the self-adjoint domains as follows.

**Theorem 3.** Let $d$ be the deficiency index. Assume there exists $\lambda \in \mathbb{R}$ such that (2.1) has $d$ linearly independent solutions lying in $H$. By Lemma 2.1 there exist $d$ linearly independent real-valued solutions $u_1, \ldots, u_d$ in $H$. Let $u_1, \ldots, u_m$ be the LC solutions constructed in [17]. Then a linear submanifold $D(S)$ of $D_{\text{max}}$ is the domain of a self-adjoint extension $S$ of $S_{\text{min}}$ if and only if there exist a complex $d \times n$ matrix $A$ and a complex $d \times m$ matrix $B$ such that the following three conditions hold:

1. $\text{rank}(A : B) = d$;
2. $AE_n A^* = B E_m B^*$;
3. $D(S) = \left\{ y \in D_{\text{max}} : A \begin{pmatrix} y(a) \\ \vdots \\ y^{[n-1]}(a) \end{pmatrix} + B \begin{pmatrix} [y, u_1](b) \\ \vdots \\ [y, u_m](b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$. (3.1)

Here $E_j$ is the symplectic matrix (2.7) of order $j$ and $[y, u]$ denotes the Lagrange bracket. (See Section 4 below for a detailed definition of $[y, u_j]$.)

The next three lemmas are established in [16]; we state them here for the convenience of the reader since they are used in the construction of a self-adjoint operator $S$ acting on $H$ by means of strictly separated boundary conditions.

**Lemma 1.** Let $s = \frac{m}{2} = d - k$ and suppose the matrices

$$A_1 = \begin{pmatrix} A_{k \times n} \\ 0_{s \times n} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0_{k \times m} \\ D_{s \times m} \end{pmatrix},$$

have $\text{rank}(A_1) = k$ and $\text{rank}(B_1) = s$. Then

$$A_1 \begin{pmatrix} y(a) \\ \vdots \\ y^{[n-1]}(a) \end{pmatrix} = 0 \quad \text{and} \quad B_1 \begin{pmatrix} [y, u_1](b) \\ \vdots \\ [y, u_m](b) \end{pmatrix} = 0$$

are strictly separated self-adjoint boundary conditions if and only if $A_1 E_n A_1^* = 0_{m \times m} = B_1 E_m B_1^*$. Here $E_n$ and $E_m$ are the symplectic matrices (2.7) of orders $n$ and $m$, respectively.

**Lemma 2.** Let $h$ be any even number and let $C$ be an $r \times h$ matrix with $\text{rank} C = r$. Assume $CE_h C^* = 0$, where $E_h$ is the symplectic matrix (2.7) of order $h$. Then $r \leq \frac{h}{2}$.
Lemma 3. There exist vectors $\gamma_1, \ldots, \gamma_{h/2}$ in $\mathbb{R}^h$ such that $CE_hC^* = 0$, where

$$C = \begin{pmatrix}
\gamma_1 \\
\vdots \\
\gamma_{h/2}
\end{pmatrix}.$$ 

For convenience, for $n$-tuples $\alpha = (\alpha_0, \ldots, \alpha_{n-1})$, $\beta = (\beta_0, \ldots, \beta_{n-1})$, $\alpha_j, \beta_j \in \mathbb{C}$, and $u \in D(S_{\text{max}})$, we define $[\alpha, \beta](a)$ and $[\alpha, u](a)$ such that

$$[\alpha, \beta](a) = [v, w](a), \quad [\alpha, u](a) = [v, u](a),$$

where $v, w \in D(S_{\text{max}})$ satisfy

$$v^{[j]}(a) = \alpha_j, \quad w^{[j]}(a) = \beta_j, \quad j = 0, 1, \ldots, n - 1.$$ 

We now give the proof of part (1) of Theorem 1, i.e. assume that for every $\lambda \in I$, there exist $d$ linearly independent solutions of $Mu = \lambda wu$ in $H$. Then there is a self-adjoint extension $S$ of $S_{\text{min}}$ with strictly separated boundary conditions, such that the intersection $\sigma_c(S) \cap I$ is the empty set.

Proof.

(i) Construction of a self-adjoint extension $S$ in $H$ with strictly separated regular and singular boundary conditions. Let $s = \frac{n}{2} = d - k$. From [16] for every $\lambda \in I$, we have $d$ linearly independent real solutions $u_1, \ldots, u_d$ of (2.1) in $H$, where $u_1, \ldots, u_s, u_{s+1}, \ldots, u_m$ are LC solutions and $u_{m+1}, \ldots, u_d$ are LP solutions.

By Lemmas 2 and 3 we can find a real matrix $C_{k \times n}$ such that $CE_n C^* = 0$ and rank $C = k$. Let $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in})$, $i = 1, \ldots, k$, be the row vectors of $C$. Then $CE_n C^* = 0$ is equivalent to

$$(\alpha_i, \alpha_j E_n) = 0, \quad i, j = 1, 2, \ldots, k,$$

where $(\cdot, \cdot)$ denote the usual inner product in $\mathbb{R}^n$.

Since $d > k$, there exists at least one $u_i$, $i = 1, 2, \ldots, d$, such that

$$\beta_i = (u_i(a), u_i^{[1]}(a), \ldots, u_i^{[n-1]}(a)) \notin \text{span}\{\alpha_1, \ldots, \alpha_k\} = \{\alpha_1 E_n, \ldots, \alpha_k E_n\}^\perp.$$ (3.2)

The function $u_i$ may be an LP solution, or an LC solution. If $u_i$ is an LC solution, then it must be in one of the sets: $\{u_1, \ldots, u_s\}$ or $\{u_{s+1}, \ldots, u_m\}$.

Using Lemma 1 we construct strictly separated boundary conditions as follows. Choose

$$A = \begin{pmatrix}
A_{1k \times n} \\
0_{s \times n}
\end{pmatrix},$$ (3.3)

where $A_1 = (-1)^k CE_n$. 

When $u_i$ is an LC solution which is in the set $\{u_1, \ldots, u_s\}$, we choose

$$B = \begin{pmatrix} 0_{k \times s} & 0_{k \times s} \\ I_{s \times s} & 0_{s \times s} \end{pmatrix},$$  

where $I$ denotes the identity matrix. When $u_i$ is an LC solution which is in the set $\{u_{s+1}, \ldots, u_m\}$, we choose

$$B = \begin{pmatrix} 0_{k \times s} & 0_{k \times s} \\ 0_{s \times s} & I_{s \times s} \end{pmatrix}.$$  

(3.4)

(3.5)

When $u_i$ is an LP solution, we choose either (3.4) or (3.5).

Next, without loss of generality, we assume that $u_i$ is an LC solution in the set $\{u_{s+1}, \ldots, u_m\}$. Then we choose the matrix $B$ defined by (3.5). Clearly $BE_nB^* = 0$. By $CE_nC^* = 0$, we have $AE_nA^* = 0$. So we obtain the separated self-adjoint boundary conditions:

$$A \begin{pmatrix} y(a) \\ \vdots \\ y^{[n-1]}(a) \end{pmatrix} + B \begin{pmatrix} [y, u_1](b) \\ \vdots \\ [y, u_m](b) \end{pmatrix} = 0,$$

i.e.

$$D(S) = \{ y \in D(S_{\text{max}}) \mid [y, \alpha_i](a) = 0, \ i = 1, \ldots, k, \ [y, u_j](b) = 0, \ j = s + 1, \ldots, m \}.$$  

(3.6)

The other two cases when $u_i$ is in the set $\{u_1, \ldots, u_s\}$ or is an LP solution are established similarly.

**Remark.** To construct a self-adjoint extension $S$ we must impose exactly $k$ separated conditions at the regular endpoint $a$ and exactly $d - k$ at the singular endpoint $b$. These singular conditions are constructed using LC solutions and (3.4) or (3.5).

**(ii) Construction of the operators $S_t$.** For $t \in (a, b)$ we now define a regular self-adjoint operator $S_t$ acting in the Hilbert space $H_t = L^2((a, t), w)$ with separate boundary conditions, these conditions are ‘inherited’ from those of $S$. In particular, the boundary conditions of $S_t$ at $a$ are the same as those of $S$ at $a$. Since $t$ is regular, there must be exactly $k$ linearly independent conditions imposed at $t$ to get strictly separated self-adjoint boundary conditions on the interval $(a, t)$. Comparing these with the boundary conditions for $S$ in (3.6) note that we have to add $2k - d = n - d$ conditions at the endpoint $t$. These additional conditions are constructed by using the LP solutions.

We use the same solutions $u_1, \ldots, u_d$ as in part (i). In addition we add solution $v_1, \ldots, v_{n-d}$ to form a bases of solutions of Eq. (2.1). Note that $v_1, \ldots, v_{n-d}$ are not in $H$ but they are in $L^2((a, t), w)$ for any $t \in (a, b)$. The self-adjoint operator $S_t$ is constructed as follows.

Let

$$A_1 = \begin{pmatrix} \tilde{A}_{1k \times n} \\ 0_{k \times n} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0_{k \times k} & 0_{k \times k} \\ 0_{k \times k} & I_{k \times k} \end{pmatrix},$$  

(3.7)

where $\tilde{A}_1 = (-1)^kCE_n = \tilde{A}$. From Theorem 3 it follows that
are separated self-adjoint boundary conditions on the interval \((a, t)\), i.e.

\[
D(S_t) = \{ y \in D[(S_t)_{\text{max}}] \mid [y, \alpha_i](a) = 0, \ i = 1, \ldots, k, \ [y, u_j](t) = 0, \ j = s + 1, \ldots, m, \ [y, v_j](t) = 0, \ j = m + 1, \ldots, d \}.
\]

(3.8)
is a self-adjoint domain in \(H_t\).

**Remark.** In definition (3.8) note that the boundary conditions at \(t\) involve only the LP and LC solutions \(u_j\); the solutions \(v_j\), which are not in \(H\), are not present in (3.8). The conditions involving the LC solutions can be considered as being ‘inherited’ from the corresponding conditions defining \(S\), while those involving LP solutions are the additional conditions needed at the regular point \(t\). From [17] we know that if \(u\) is an LP solution then \([y, u_j](t) \to 0\) as \(t \to b\) for any \(y \in D_{\text{max}}\). This property of LP solutions \(u\) is used in showing that \(S_t \to S\) in the sense of strong resolvent convergence.

In the following, for convenience, we let the functions \(y_1, \ldots, y_k\) denote the LC solutions \(u_{s+1}, \ldots, u_m\) and the LP solutions \(u_{m+1}, \ldots, u_d\), respectively. Then (3.8) can be rewritten as

\[
D(S_t) = \{ y \in D[(S_t)_{\text{max}}] \mid [y, \alpha_i](a) = 0, \ i = 1, \ldots, k, \ [y, u_j](t) = 0, \ j = m + 1, \ldots, d \}.
\]

(3.9)

**(iii) Local absence of continuous spectrum.** We now prove that the operators \(S_t\) converge to \(S\) in the sense of strong resolvent convergence; actually that the operators \(B_t := S_t \ominus 0_{L^2(t, b)}\) converge to \(S\) in the sense of strong resolvent convergence since \(S_t\) and \(S\) are operators acting in different spaces \(H_t\) and \(H\), respectively.

To prove that \(B_t \to S\) in the sense of strong resolvent convergence it suffices [19] to find a core \(\tilde{D}\) of \(D(S)\) such that for each \(f \in \tilde{D}\) there exists a \(t_0\) with \(f \in D(B_t)\) for \(t > t_0\) and \(B_tf \to Sf\) as \(t \to b\). Here we choose

\[
\tilde{D} = \{ f \in D(S) : f(x) = 0 \text{ for } x \text{ close to } b \}.
\]

**Remark.** Since \(u_{m+1}, \ldots, u_d\) are LP solutions we know from [17] that \([y, u_j](t) \to 0\) as \(t \to b\) for any \(y \in D_{\text{max}}\) and \(u_j, \ j = m + 1, \ldots, d\). This means that the last \(d - m = n - d\) boundary conditions of (3.8) are not present in the definition (3.6) of \(S\). This is a key point in the proof of \(B_t \to S\).

Since \(S_t\) is a regular self-adjoint operator, its spectrum is discrete. Let \(\rho_i, i = 1, 2, \ldots, \) denote its eigenvalues and \(\eta_i\) the corresponding orthonormalized eigenfunctions. These depend on \(t\) but we will not indicate their \(t\) dependence in the notation in the interest of simplicity. Since the
\( \eta_i \) are in \( D(S_t) \), they satisfy the boundary conditions at \( a \) by the above construction of \( S_t \). By Lemmas 2 and 3, they satisfying the following normalized initial conditions

\[
(\eta_i(a) \cdots \eta_i^{[n-1]}(a)) = \sum_{j=1}^{k} \gamma_{ij} \alpha_j, \quad \sum_{j=1}^{k} |\gamma_{ij}|^2 = 1.
\]

Having constructed \( S \) in terms of separated boundary conditions and then \( S_t \) in terms of separated boundary conditions ‘inherited’ from \( S \), we are now in position to adapt Weidmann’s proof, used in [19] for the case \( d=k \) when there are no singular conditions at the endpoint \( b \), to our situation when \( d>k \) and there are singular conditions at \( b \).

The symbols \((\cdot,\cdot)\) and \( \| \cdot \|_r \) denote the usual inner product and the norm in \( L^2((a,t),w) \), respectively. Obviously, for \( y_j, j = 1, \ldots, k \), from (3.9) we have

\[
\sum_{j=1}^{k} \|y_j\|^2 \geq \sum_{j=1}^{k} \|y_j\|_r^2.
\]

By the Bessel inequality, we have

\[
\sum_{j=1}^{k} \|y_j\|_r^2 \geq \sum_{j=1}^{k} \sum_{i} \left| \left\langle y_j, \eta_i \right\rangle \right|_r^2 \geq \sum_{j=1}^{k} \sum_{i \in \{i: |\lambda - \rho_i| \leq \varepsilon\}} \left| \left\langle y_j, \eta_i \right\rangle \right|_r^2 \|\eta_i\|_r^{-2}.
\]

Since \( \{y_j\}_{j=1}^{k} \) are solutions of (2.1), in terms of the Lagrange brackets \([\cdot,\cdot]_{a}^{t} \), we have

\[
\sum_{j=1}^{k} \sum_{i \in \{i: |\lambda - \rho_i| \leq \varepsilon\}} \left| \left\langle y_j, \eta_i \right\rangle \right|_a^t \|\eta_i\|_r^{-2} = \sum_{i \in \{i: |\lambda - \rho_i| \leq \varepsilon\}} \|\eta_i\|_r^{-2} (\lambda - \rho_i)^{-2} \sum_{j=1}^{k} \left| \left\langle y_j, \eta_i \right\rangle \right|_a^t.
\]

Because \( \eta_i \) are the eigenfunctions of \( S_t \), \( \eta_i \) satisfy the boundary conditions of \( S_t \) at \( t \). Hence

\[
[\eta_i, y_j](t) = 0, \quad j = 1, \ldots, k.
\]

So

\[
\sum_{i \in \{i: |\lambda - \rho_i| \leq \varepsilon\}} \|\eta_i\|_r^{-2} (\lambda - \rho_i)^{-2} \sum_{j=1}^{k} \left| \left\langle y_j, \eta_i \right\rangle \right|_a^t = \sum_{i \in \{i: |\lambda - \rho_i| \leq \varepsilon\}} \|\eta_i\|_r^{-2} (\lambda - \rho_i)^{-2} \sum_{j=1}^{k} \left| \left\langle y_j, \eta_i \right\rangle \right|_a^t.
\]

Let

\[
C(\lambda) = \inf \left\{ \sum_{j=1}^{k} \left[ \sum_{i=1}^{k} \delta_i \alpha_i \right](a) \left| \sum_{i=1}^{k} |\delta_i|^2 = 1 \right\}.
\]

By (3.2) and (3.6) in part (i) of the proof, we note that there is at least one solution, say \( y_r \), in the set \( y_1, \ldots, y_k \), such that
Then by Lemma 2, we have $C(\lambda) > 0$.

Thus

$$\sum_{\{i: |\lambda - \rho_i| \leq \varepsilon\}} \|\eta_i\|_t^{-2} (\lambda - \rho_i)^{-2} \geq C(\lambda) \sum_{\{i: |\lambda - \rho_i| \leq \varepsilon\}} \|\eta_i\|_t^{-2} (\lambda - \rho_i)^{-2}.$$ 

And from the preceding analysis we have

$$\sum_{j=1}^{k} \|y_j\|^2 \geq C(\lambda) \sum_{\{i: |\lambda - \rho_i| \leq \varepsilon\}} \|\eta_i\|_t^{-2} (\lambda - \rho_i)^{-2}.$$ 

Therefore

$$\sum_{\{i: |\lambda - \rho_i| \leq \varepsilon\}} \|\eta_i\|_t^{-2} \leq C_1(\lambda) \varepsilon^2. \quad (3.12)$$

Let $u \in H$ have compact support in $[a, b)$, and let $E_t$ denote the spectral family of $S_t$. Since $S_t$ is a regular self-adjoint operator, for $t$ sufficiently large (such that the support of $u$ is contained in $(a, t)$), we have

$$\langle (E_t(\lambda + \varepsilon) - E_t(\lambda - \varepsilon))u, u \rangle = \sum_{\{i: |\lambda - \rho_i| \leq \varepsilon\}} |\langle u, \eta_i \rangle|^2 \|\eta_i\|_t^{-2}. \quad (3.13)$$

Because the eigenfunctions $\{\eta_i\}$ satisfy the initial conditions (3.10), the functions $\{\eta_i\}$ are uniformly bounded on the support of $u$ for all $\rho_i \in I$. From this and (3.12) we get

$$\sum_{\{i: |\lambda - \rho_i| \leq \varepsilon\}} |\langle u, \eta_i \rangle|^2 \|\eta_i\|_t^{-2} \leq C(u) \sum_{\{i: |\lambda - \rho_i| \leq \varepsilon\}} \|\eta_i\|_t^{-2} \leq C(\lambda, u) \varepsilon^2,$$

i.e.,

$$\langle (E_t(\lambda + \varepsilon) - E_t(\lambda - \varepsilon))u, u \rangle \leq C(\lambda, u) \varepsilon^2. \quad (3.14)$$

From $B_t \to S$ in the sense of strong resolvent convergence and [18, Theorem 9.19] we obtain that $E_t(s) \to E(s)$ in the sense of strong convergence if $s$ is not an eigenvalue of $S$. And hence

$$\langle (E(\lambda + \varepsilon) - E(\lambda - \varepsilon))u, u \rangle \leq C(\lambda, u) \varepsilon^2 \quad (3.15)$$

whenever $\lambda \pm \varepsilon$ are not eigenvalues of $S$.

With $P_c$ the orthogonal projection onto the continuous subspace $H_c$ (see Definition 4) then (3.15) implies that for $u$ with compact support the function $\langle E(\cdot)P_cu, u \rangle$ is continuous and differentiable at every $\lambda \in I$. It follows that $\langle E(\cdot)P_cu, u \rangle$ is absolutely continuous for $u$ with compact support. By (3.15) we know the derivative is zero almost everywhere and this shows that the function $\langle E(\cdot)P_cu, u \rangle$ is constant in $I$. By [18, Theorems 7.29 and 7.22] this means there is no continuous spectrum in the interval $I$. This completes the proof of part (1) of Theorem 1.
Remark. The functions $y_1, \ldots, y_k$ used in the construction of the separated self-adjoint boundary conditions of $S_t$ on the truncated interval $(a, t)$ are composed of the LC solutions $u_{s+1}, \ldots, u_m$ and the LP solutions $u_{m+1}, \ldots, u_d$ of (2.1). When the deficiency index $d$ is maximal, then all the $y_1, \ldots, y_k$ are LC solutions. If $k < d < 2k$, then $\{y_i\}_{i=1}^k$ are composed of $d - k$ LC solutions and $n - d$ LP solutions.

Here, in the case of middle deficiency indices, $k < d < 2k = n$, we proved that if for every $\lambda$ in an open interval $I$, the number of linearly independent solutions of (2.1) is $d$, then there is a self-adjoint extension $S$ of $S_{\text{min}}$, with strictly separated boundary conditions, such that $S$ has no continuous spectrum in $I$. Combining this with the case $d = k$ established in [19] and the well-known result that the spectrum is discrete when $d = 2k$ we obtain:

**Corollary 1.** If for every $\lambda$ in an open interval $I$ the number of linearly independent solutions of Eq. (2.1) is equal to the deficiency index $d$, then there is a self-adjoint extension $S$ of $S_{\text{min}}$, determined by strictly separated boundary conditions, such that $S$ has no continuous spectrum in $I$.

4. Density of eigenvalues

In this section we prove part (2) of Theorem 1. Fundamental to the study of boundary value problems is the Lagrange identity which, in our setting, reads as follows.

**Lemma 4 (Lagrange identity).** Suppose $Q \in Z_n(J, \mathbb{R})$ and $M = MQ$ is a symmetric differential expression as defined in Definition 1. Then for any $y, z \in D(Q)$ we have

$$\bar{z}My - yMZ = [y, z]'$$

where

$$[y, z] = (-1)^k \sum_{r=0}^{n-1} (-1)^{n+1-r} z^{[n-r-1]} y^{[r]} = (-1)^k (Z^* EY),$$

$$Y = \begin{pmatrix} y \\ y^{[1]} \\ \vdots \\ y^{[n-1]} \end{pmatrix}, \quad Z = \begin{pmatrix} z \\ z^{[1]} \\ \vdots \\ z^{[n-1]} \end{pmatrix}.$$ 

**Corollary 2.** If $My = \lambda wy$ and $Mz = \bar{\lambda} wz$, then $[y, z]$ is constant on $J$. In particular, if $\lambda$ is real, $My = \lambda wy$ and $Mz = \lambda wz$, then $[y, z]$ is constant on $J$.

**Proof.** This follows directly from (4.1). Since the coefficients are real, the last statement follows from the observation that if $My = \lambda wy$ then $M\bar{y} = \lambda w\bar{y}$. □

For the convenience of the reader we state the GKN theorem first and then establish several lemmas; some of these are of independent interest.
Theorem 4 (GKN). A linear submanifold $D(S)$ of $D_{\text{max}}$ is the domain of a self-adjoint extension $S$ of $S_{\text{min}}$ if and only if there exist functions $v_1, v_2, \ldots, v_d$ in $D_{\text{max}}$ satisfying the following conditions:

(i) $v_1, v_2, \ldots, v_d$ are linearly independent modulo $D_{\text{min}}$;
(ii) $[v_i, v_j](b) - [v_i, v_j](a) = 0$, $i, j = 1, \ldots, d$;
(iii) $D(S) = \{ y \in D_{\text{max}} : [y, v_j](b) - [y, v_j](a) = 0, \ j = 1, \ldots, d \}$, where $d$ is the deficiency index of $S_{\text{min}}$.

Proof. See [17] and its references. \qed

Lemma 5. Suppose $v_1, v_2, \ldots, v_d$ in $D_{\text{max}}$ satisfy conditions (i) and (ii) of the GKN theorem. Then

$$D(S) = \{ y \in D_{\text{max}} : [y, v_j](b) - [y, v_j](a) = 0, \ j = 1, \ldots, d \} = D_{\text{min}} + \text{span}\{v_1, \ldots, v_d\}. \quad (4.2)$$

Proof. Let $D' = D_{\text{min}} + \text{span}\{v_1, \ldots, v_d\}$ and $D(S) = \{ y \in D_{\text{max}} : [y, v_j](b) - [y, v_j](a) = 0, \ j = 1, \ldots, d \}$. Assume $y \in D'$. Then there exist constants $c_1, \ldots, c_d$ and function $y_0 \in D_{\text{min}}$ such that

$$y = c_1 v_1 + \cdots + c_d v_d + y_0.$$

From $[v_i, v_j]^b_a = 0$ ($i, j = 1, \ldots, d$), it follows that

$$[v_j, y]^b_a = 0, \quad j = 1, \ldots, d.$$

This implies that $D' \subset D(S)$. Since the dimension of $D'$ modulo $S_{\text{min}}$ is $d$, and the dimension of $D(S)$ modulo $D_{\text{min}}$ is also $d$, it follows that $D' = D(S)$. \qed

Remark. By Lemma 5, given any self-adjoint extension $S$ of $S_{\text{min}}$, its domain $D(S)$ is given by

$$D(S) = D_{\text{min}} + \text{span}\{v_1, \ldots, v_d\}. \quad (4.3)$$

Notation 1. Let the hypotheses and notation of Lemma 5 hold and let $v = (v_1, \ldots, v_d)$. We use the notation $S_v$ to denote the self-adjoint operator whose domain is given by

$$D(S_v) = D_{\text{min}} + \text{span}\{v_1, \ldots, v_d\}.$$
Lemma 7. Let the hypotheses and notation of Theorem 1 hold. Assume that for some real \( \lambda \) Eq. (2.1) has \( d \) linearly independent real-valued solutions \( u_1, \ldots, u_d \) in \( H \). Let \( u = (u_1, \ldots, u_d) \). Then \( \lambda \) is an eigenvalue of multiplicity \( d \) of the self-adjoint operator \( S_u \).

Proof. This follows from the GKN theorem and the observation that each \( v_j, \ j = 1, \ldots, d \), is an eigenfunction with eigenvalue \( \lambda \). \( \square \)

Lemma 8. Assume \( S \) is a self-adjoint realization of (2.1) with domain 

\[
D(S) = D_{\min} \overset{\perp}\cup \text{span}\{v_1, \ldots, v_d\}.
\]

Let the hypotheses and notation of Theorem 1 hold. Assume that \( u_1, \ldots, u_d \) are linearly independent real-valued solutions in \( H \) for \( \lambda = \mu \in \mathbb{R} \). Let \( S_\mu \) denote the self-adjoint operator with domain 

\[
D(S_\mu) = D_{\min} \overset{\perp}\cup \text{span}\{u_1, \ldots, u_d\}.
\]

If \( \mu \) is not an eigenvalue of \( S \), then

1. \( v_1, \ldots, v_d, u_1, \ldots, u_d \) are linearly independent modulo \( D_{\min} \);
2. \( D(S_\mu) \cap D(S) = D_{\min} \).

Proof. We prove part (1) by contradiction. If not, then there exist constants \( c_1, \ldots, c_d, e_1, \ldots, e_d \) which are not all zero, such that

\[
c_1 u_1 + \cdots + c_d u_d + e_1 v_1 + \cdots + e_d v_d = y \in D_{\min}.
\]

From the characterization of the domain of the minimal operator [17], for any \( z \in D_{\max} \), we have

\[
[y, z](a) = [y, z](b) = 0.
\]

Therefore

\[
c_1 [u_1, u_i]^b_a + \cdots + c_d [u_d, u_i]^b_a + e_1 [v_1, u_i]^b_a + \cdots + e_d [v_d, u_i]^b_a = 0, \quad i = 1, \ldots, d,
\]

and

\[
c_1 [u_1, v_i]^b_a + \cdots + c_d [u_d, v_i]^b_a + e_1 [v_1, v_i]^b_a + \cdots + e_d [v_d, v_i]^b_a = 0, \quad i = 1, \ldots, d,
\]

i.e.

\[
\begin{pmatrix}
[u_1, u_1]^b_a & \cdots & [u_d, u_1]^b_a & [v_1, u_1]^b_a & \cdots & [v_d, u_1]^b_a \\
\vdots & \cdots & \vdots & \cdots & \cdots & \cdots \\
[u_1, u_d]^b_a & \cdots & [u_d, u_d]^b_a & [v_1, u_d]^b_a & \cdots & [v_d, u_d]^b_a \\
[u_1, v_1]^b_a & \cdots & [u_d, v_1]^b_a & [v_1, v_1]^b_a & \cdots & [v_d, v_1]^b_a \\
\vdots & \cdots & \vdots & \cdots & \cdots & \cdots \\
[u_1, v_d]^b_a & \cdots & [u_d, v_d]^b_a & [v_1, v_d]^b_a & \cdots & [v_d, v_d]^b_a
\end{pmatrix}
\begin{pmatrix}
c_1 \\
\vdots \\
c_d \\
e_1 \\
\vdots \\
e_d
\end{pmatrix} = 0. \quad (4.4)
\]
Set

\[ U = \begin{pmatrix}
[u_1, v_1]^b_a & \cdots & [u_d, v_1]^b_a \\
\vdots & \ddots & \vdots \\
[u_1, v_d]^b_a & \cdots & [u_d, v_d]^b_a
\end{pmatrix} \]

and note that the coefficient matrix of equations (4.4) is equal to \( \tilde{U} = \begin{pmatrix} 0 & -U^* \\ U & 0 \end{pmatrix} \), where \( U^* \) denotes the complex conjugate transpose of \( U \).

Next we prove that the rank of \( U \) is \( d \). Suppose rank \( U < d \), and let

\[ \beta_i = ([u_1, v_i]^b_a, \ldots, [u_d, v_i]^b_a), \quad i = 1, \ldots, d, \]

denote the \( d \) row vectors of the matrix \( U \). Then \( \beta_1, \ldots, \beta_d \) are linearly dependent, i.e., there exists at least one vector in the set \( \{\beta_i\}_{i=1}^d \), which is a linear combination of the others. Without loss of generality, we may assume that

\[ \beta_1 = k_2 \beta_2 + \cdots + k_d \beta_d, \quad k_i \in \mathbb{C}. \]

Therefore

\[ [u_1, v_i]^b_a = k_2[u_2, v_i]^b_a + \cdots + k_d[u_d, v_i]^b_a, \quad i = 1, \ldots, d. \]

Thus

\[ [u_1 - k_2 u_2 - \cdots - k_d u_d, v_i]^b_a = 0, \quad i = 1, \ldots, d. \]

Hence

\[ u_1 - k_2 u_2 - \cdots - k_d u_d \in D(S). \]

Since \( u_1, \ldots, u_d \) are linearly independent, we know that \( u_1 - k_2 u_2 - \cdots - k_d u_d \) is a non-trivial solution of \( My = \mu wy \). This contradicts the fact that \( \mu \) is not an eigenvalue of \( S \). Therefore rank \( U = d \).

Therefore the determinant of \( \tilde{U} \) is not zero. Thus \( c_i = e_i = 0 \) (\( i = 1, \ldots, d \)) which contradicts the fact that the complex constants \( c_1, \ldots, c_d, e_1, \ldots, e_d \) are not all zero. Hence we see that the functions \( u_1, \ldots, u_d, v_1, \ldots, v_d \) are linearly independent modulo \( D_{\min} \).

To prove part (2), let \( y \in D(S_\mu) \cap D(S) \). Then there exist constants \( \tilde{c}_i, \tilde{e}_i \) (\( i = 1, \ldots, d \)) and functions \( u_0, v_0 \in D_{\min} \) such that

\[ y = \tilde{c}_1 u_1 + \cdots + \tilde{c}_d u_d + u_0 \quad \text{and} \quad y = \tilde{e}_1 v_1 + \cdots + \tilde{e}_d v_d + v_0. \]

Hence

\[ \tilde{c}_1 u_1 + \cdots + \tilde{c}_d u_d - \tilde{c}_1 v_1 - \cdots - \tilde{e}_d v_d + u_0 - v_0 = 0. \]

By part (1), the linear manifolds span \( \{u_1, \ldots, u_d, v_1, \ldots, v_d\} \) and \( D_{\min} \) are linearly independent. Therefore \( \tilde{c}_i = \tilde{e}_i = 0 \) (\( i = 1, \ldots, d \)) and \( u_0 = v_0 \). Thus \( y = u_0 \in D_{\min} \). This shows that \( D(S_\mu) \cap D(S) \subset D_{\min} \). Since clearly \( D_{\min} \subset D(S_\mu) \cap D(S) \) we conclude that \( D(S_\mu) \cap D(S) = D_{\min} \). \( \Box \)
Remark. If two self-adjoint realizations of Eq. (2.1) have different eigenvalues $\mu$ and $\lambda$ and $\mu$ has geometric multiplicity $d$, then the intersection of their domains is $D_{\text{min}}$. This follows directly from Lemma 8. Also note that, in Lemma 8, we did not assume that the $v_j$ are solutions but only that they are maximal domain functions satisfying conditions (i) and (ii) of the GKN theorem.

We now proceed to the proof of part (2) of Theorem 1.

Proof. Let $I$ be an open interval of the real line. Suppose $S$ is a self-adjoint realization of (2.1) with domain

$$D(S) = D_{\text{min}} + \text{span}\{v_1, \ldots, v_d\}.$$ 

Assume that $\lambda \in I$ is not an eigenvalue of $S$ and that $My = \lambda wy$ has $d$ linearly independent solutions $u_1, \ldots, u_d$ in $H$. By Lemma 6 we may assume that $u_1, \ldots, u_d$ are real-valued. Let $u = (u_1, \ldots, u_d)$ and consider the self-adjoint operator $S_1 = S_u$ with domain

$$D(S_1) = \{y \in D_{\text{max}}: [u_j, y]^b_a = 0, \ j = 1, \ldots, d\}.$$ 

By Lemma 8 we know that $u_1, \ldots, u_d, v_1, \ldots, v_d$ are linearly independent modulo $D_{\text{min}}$ and $D(S) \cap D(S_1) = D_{\text{min}}$.

Let $\lambda_i$ denote the eigenvalues of $S$ lying in $I$, and let $y_i$ be the corresponding orthonormalized eigenfunctions in $H$. Then we have

$$\langle u_j, y_i \rangle_w = \int_a^b u_j \bar{y}_i w \, dx = (\lambda - \lambda_i)^{-1} \int_a^b [Mu_j \bar{y}_i - u_j \bar{M}y_i] \, dx = (\lambda - \lambda_i)^{-1} [u_j, y_i]^b_a. \quad (4.5)$$

For any given $i$, at least one of $[u_1, y_i]^b_a, \ldots, [u_d, y_i]^b_a$ is not zero. If not, for $j = 1, \ldots, d, [u_j, y_i]^b_a = 0$, then $y_i \in D(S_1)$. Since $y_i$ is an eigenfunction of $S$, we know $y_i \in D(S)$. Thus by Lemma 8 we have

$$y_i \in D(S_1) \cap D(S) = D_{\text{min}}.$$ 

Since $a$ is the regular endpoint we know from the characterization of the minimal domain [10] that

$$y_i(a) = y_i^{[1]}(a) = \cdots = y_i^{[n-1]}(a) = 0,$$

and consequently, by the theory of existence and uniqueness for ordinary differential equations, it follows that $y_i = 0$. This contradicts the fact that $y_i$ is an eigenfunction. Therefore for any given $i$ there is at least one $j \in \{1, \ldots, d\}$ such that $[u_j, y_i]^b_a$ does not vanish.

Set $c_i = \sum_{j=1}^d |[u_j, y_i]^b_a|^2 > 0$. Then

$$\sum_{j=1}^d |\langle u_j, y_i \rangle_w|^2 = (\lambda - \lambda_i)^{-2} \sum_{j=1}^d |[u_j, y_i]^b_a|^2 = (\lambda - \lambda_i)^{-2} c_i.$$
Thus

\[ \sum_{j=1}^{d} \|u_j\|^2 \geq \sum_{j=1}^{d} \sum_{i} |\langle u_j, y_i \rangle_w|^2 = \sum_{i} (\lambda - \lambda_i)^{-2} c_i. \]

Therefore

\[ \sum_{i} (\lambda - \lambda_i)^{-2} c_i < \infty, \quad \lambda \in I \setminus \{\lambda_i\}. \] (4.6)

Next we show that the formula (4.6) and \( c_i > 0 \) imply that the set of eigenvalues \( \{\lambda_i\} \) is nowhere dense in \( I \). The proof is similar to one given in [19]. Here we present it for the sake of completeness.

Assume that \( \{\lambda_i\} \) is dense in the subinterval \([h, k]\) of \( I \). Set

\[ K_N = \left\{ \lambda \left| \sum_{i} (\lambda - \lambda_i)^{-2} c_i \leq N \right. \right\} = \bigcap_{j=1}^{\infty} \left\{ \lambda \left| \sum_{i=1}^{j} (\lambda - \lambda_i)^{-2} c_i \leq N \right. \right\}. \]

Then \( K_N \) is closed and

\[ \bigcup_{N=1}^{\infty} K_N = \left\{ \lambda \left| \sum_{i} (\lambda - \lambda_i)^{-2} c_i < \infty \right. \right\} = [h, k] \setminus \{\lambda_i\}. \]

Every \( K_N \) is nowhere dense in \([h, k]\). Since every subinterval of \([h, k]\) contains some eigenvalue \( \lambda_i \) and therefore a sufficiently small neighborhood of \( \lambda_i \), which is not contained in \( K_N \). Hence

\[ [h, k] \setminus \bigcup_{N=1}^{\infty} K_N = \left\{ \lambda \left| \sum_{i} (\lambda - \lambda_i)^{-2} c_i = \infty \right. \right\} = \{\lambda_i\} \]

is of second category. This contradicts the fact that the set of eigenvalues \( \{\lambda_i\} \) is countable and completes the proof of part (2). \( \square \)

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References


Further reading