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Generic and Cogeneric Monomial Ideals

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Monomial ideals which are generic with respect to either their generators or irreducible components have minimal free resolutions encoded by simplicial complexes. There are numerous equivalent ways to say that a monomial ideal is generic or cogeneric. For a generic monomial ideal, the associated primes satisfy a saturated chain condition, and the Cohen–Macaulay property implies shellability for both the Scarf complex and the Stanley–Reisner complex. Reverse lexicographic initial ideals of generic lattice ideals are generic. Cohen–Macaulayness for cogeneric ideals is characterized combinatorially; in the cogeneric case, the Cohen–Macaulay type is greater than or equal to the number of irreducible components. Methods of proof include Alexander duality and Stanley's theory of local h-vectors.

1. Genericity of Monomial Ideals

This paper is a study of genericity properties of monomial ideals, initiated by Bayer *et al.* (1998). We will often use results from prior papers on this subject, although we have tried to make the exposition as self-contained as possible. The interested reader is encouraged to consult Bayer *et al.* (1998), Bayer and Sturmfels (1998), Miller (1998) and Peeva and Sturmfels (1998b) for the background. While the present paper is theoretical rather than algorithmic, we expect that our results on genericity will play a role for future implementations in Gröbner basis systems.

Let M be a monomial ideal minimally generated by monomials $m_1, \dots, m_r$ in a polynomial ring $S = k[x_1, \dots, x_n]$ over a field k. For a subset $\sigma \subseteq \{1, \dots, r\}$, we set $m_\sigma := \mathrm{lcm}(m_i | i \in \sigma)$, and let $\mathbf{a}_\sigma := \deg m_\sigma \in \mathbb{N}^n$ denote the exponent vector of m_σ. Here $m_\emptyset = 1$. For a monomial $\mathbf{x}^\mathbf{a} = x_1^{a_1} \cdots x_n^{a_n}$, we set $\deg_{x_i}(\mathbf{x}^\mathbf{a}) := a_i$, and we call $\mathrm{supp}(\mathbf{x}^\mathbf{a}) := \{i | a_i \neq 0\} \subseteq \{1, \dots, n\}$ the *support* of $\mathbf{x}^\mathbf{a}$. We say a monomial $m \in S$ *strictly divides* $m' \in S$, if m divides m' and $\mathrm{supp}(m'/m) = \mathrm{supp}(m')$.

Definition 1.1. A monomial ideal $M = \langle m_1, \dots, m_r \rangle$ is called *generic* if the following condition holds: if two distinct minimal generators m_i and m_j have the same positive degree in some variable x_s, there is a third generator m_l which strictly divides $m_{\{i,j\}} = \mathrm{lcm}(m_i, m_j)$.

The above definition of genericity is more inclusive than the one given by Bayer *et al.* (1998), which we propose to call "strongly generic". That is, we call a monomial ideal M *strongly generic* if no two distinct minimal generators m_i and m_j have the same positive degree in any variable x_s. Our new definition of "generic" is justified by the completeness of the results in Theorems 1.5 and 3.1.

There are many interesting monomial ideals which are generic but not strongly generic. One such ideal is discussed in Example 3.2. Here is another one:

EXAMPLE 1.2. The *tree ideal* $M = \langle \left(\prod_{s\in I} x_s\right)^{n-|I|+1} \mid \emptyset \neq I \subseteq \{1,\ldots,n\}\rangle$ is generic but not strongly generic. This ideal is artinian of colength $(n+1)^{n-1}$, the number of trees on $n+1$ labeled vertices.

Recall that every monomial ideal $M \subset S$ can be uniquely written as a finite irredundant intersection $M = \bigcap_{i=1}^{r} M_i$ of irreducible monomial ideals (i.e. ideals generated by powers of variables). We say that M_i is an *irreducible component* of S/M.

DEFINITION 1.3. A monomial ideal with irreducible decomposition $M = \bigcap_{i=1}^{r} M_i$ is called *cogeneric* if the following condition holds: if distinct irreducible components M_i and M_j have a minimal generator in common, there is an irreducible component $M_l \subset M_i + M_j$ such that M_l and $M_i + M_j$ do not have a minimal generator in common.

A monomial ideal M is cogeneric if and only if its *Alexander dual* $M^{\mathbf{a}}$ is generic. See Miller (1998) or Section 4 for the relevant definitions. Cogeneric monomial ideals will be studied in detail in Section 4. The remainder of this section is devoted to the statement and proof of the equivalent characterizations of genericity in Theorem 1.5.

Let $M \subset S$ be the ideal minimally generated by monomials $m_1,\ldots,m_r$ again. The following simplicial complex on r vertices, called the *Scarf complex* of M, was introduced by Bayer *et al.* (1998):

$$\Delta_M := \{\sigma \subseteq \{1,\ldots,r\} \mid m_\sigma \neq m_\tau \text{ for all } \tau \neq \sigma\}.$$

Let $S(-\mathbf{a}_\sigma)$ denote the free S-module with one generator e_σ in multidegree $\mathbf{a}_\sigma$. The *algebraic Scarf complex* F_{Δ_M} is the free S-module $\bigoplus_{\sigma\in\Delta_M} S(-\mathbf{a}_\sigma)$ with the differential

$$d(e_\sigma) = \sum_{i\in\sigma} \operatorname{sign}(i,\sigma) \cdot \frac{m_\sigma}{m_{\sigma\setminus\{i\}}} \cdot e_{\sigma\setminus\{i\}},$$

where $\operatorname{sign}(i,\sigma)$ is $(-1)^{j+1}$ if i is the jth element in the ordering of σ. It is known that F_{Δ_M} is always a subcomplex of the minimal free resolution of S/M (Bayer *et al.*, 1998, Section 3), although F_{Δ_M} is not acyclic in general. However, it will follow from Theorem 1.5 that it is acyclic if M is generic, as was the case for strongly generic ideals.

Since the definition of the Scarf complex depends only on the coordinatewise order of the exponents of the generators, it also makes sense for (formal) monomials with real exponents in $\mathbb{R}^n$. This makes way for the following definition.

DEFINITION 1.4. A *deformation* ϵ of a monomial ideal $M = \langle m_1,\ldots,m_r\rangle \subset S$ is a choice, for each $i \in \{1,\ldots,r\}$, of vectors $\epsilon_i = (\epsilon_1^i,\ldots,\epsilon_n^i) \in \mathbb{R}^n$ satisfying

$$a_s^i < a_s^j \Rightarrow a_s^i + \epsilon_s^i < a_s^j + \epsilon_s^j \qquad \text{and} \qquad a_s^i = 0 \Rightarrow \epsilon_s^i = 0,$$

where $\mathbf{a}_i = (a_1^i,\ldots,a_n^i)$ is the exponent vector of m_i. We formally introduce the monomial ideal (in a polynomial ring with real exponents):

$$M_\epsilon := \langle m_1 \cdot \mathbf{x}^{\epsilon_1}, m_2 \cdot \mathbf{x}^{\epsilon_2},\ldots,m_r \cdot \mathbf{x}^{\epsilon_r}\rangle = \langle \mathbf{x}^{\mathbf{a}_1+\epsilon_1}, \mathbf{x}^{\mathbf{a}_2+\epsilon_2},\ldots,\mathbf{x}^{\mathbf{a}_r+\epsilon_r}\rangle.$$

The Scarf complex Δ_{M_ϵ} of the deformation M_ϵ has the same vertex set $\{1,\dots,r\}$ as Δ_M. For a suitable ϵ, Δ_{M_ϵ} gives a simple (but typically non-minimal) free resolution of M; see Bayer *et al.* (1998, Theorem 4.3). Definition 1.4 is slightly different from the one given by Bayer *et al.* (1998, Construction 4.1). We require that the zeros remain unchanged, but we do not assume that M_ϵ is "(strongly) generic".

It is frequently convenient to add in high powers $x_1^D,\dots,x_n^D$ of the variables to M, where D is larger than any exponent of any minimal generator of M. One obtains an artinian ideal

$$M^* := M + \langle x_1^D,\dots,x_n^D\rangle, \tag{1.1}$$

the Scarf complex of which is called the *extended Scarf complex* Δ_{M^*} of M. Note that if $x_i^d \in M$ for some d, then x_i^D is not a minimal generator of M. As a simplicial complex, Δ_{M^*} does not depend on D. The monomial ideal M is generic if and only if M^* is generic.

The following theorem provides a justification for our new definition of "generic". It provides appropriate converses to results of Bayer *et al.* (1998, Theorems 3.2 and 3.7) and Bayer and Sturmfels (1998, Theorem 2.9). All statements are independent of the particular choice of D used to define M^*. In parts (d) and (e), $\mathfrak{m}^{\mathbf{b}} := \langle x_s^{b_s} \mid b_s \geq 1\rangle$ is an irreducible ideal for $\mathbf{b}\in\mathbb{N}^n$. For the definition of the *hull complex* $\mathrm{hull}(M)$ of a monomial ideal M, see Bayer and Sturmfels (1998, p. 131).

Theorem 1.5. *The following are equivalent for a monomial ideal M:*

(a) M is generic.
(b) $F_{\Delta_{M^*}}$ is a minimal free resolution of S/M^*.
(c) $\Delta_{M^*} = \mathrm{hull}(M^*)$.
(d) $M^* = \bigcap\{\mathfrak{m}^{\mathbf{a}_\sigma} \mid \sigma\in\Delta_{M^*},\ \#\sigma = n\}$ is an irredundant irreducible decomposition.
(e) For each irreducible component $\mathfrak{m}^{\mathbf{b}}$ of S/M^*, there is a face $\sigma\in\Delta_{M^*}$ with $\mathbf{a}_\sigma = \mathbf{b}$.
(f) F_{Δ_M} is a free resolution of S/M, and no variable x_s appears with the same non-zero exponent in m_i and m_j for any edge $\{i,j\}$ of the Scarf complex Δ_M.
(g) If $\sigma\notin\Delta_{M^*}$, then there is some monomial $m\in M$ which strictly divides m_σ.
(h) The extended Scarf complex Δ_{M^*} does not change under arbitrary deformations of M^*.

Proof. The scheme of the proof is

$$(b)\Rightarrow(c)\Rightarrow(d)\Rightarrow(e)\Rightarrow(b) \qquad\text{and}\qquad (c)\Rightarrow(f)\Rightarrow(a)\Rightarrow(g)\Rightarrow(h)\Rightarrow(b).$$

(b) $\Rightarrow$ (c). By induction on n; if $n=2$, this is obvious, so suppose (b) $\Rightarrow$ (c) for $\leq n-1$ variables. The fact that S/M^* is artinian in n variables implies that Δ_{M^*} is pure of dimension $n-1$ by Miller (1998, Lemma 5.11) or Yanagawa (1999, Proposition 2.9). The restriction of Δ_{M^*} to those vertices whose monomial labels are not divisible by x_s is the Scarf complex of the ideal $M_s^* = (M^* + \langle x_s\rangle)/\langle x_s\rangle$ in $k[x_1,\dots,x_n]/\langle x_s\rangle$. By induction, $\Delta_{M_s^*} = \mathrm{hull}(M_s^*)$ because $F_{\Delta_{M_s^*}}$ is acyclic by Bayer *et al.* (1998, Lemma 2.2). The topological boundary of Δ_{M^*} is the union of the complexes $\Delta_{M_s^*}$, and the topological boundary of $\mathrm{hull}(M^*)$ is the union of the complexes $\mathrm{hull}(M_s^*)$, where s runs over $\{1,2,\dots,n\}$. On the other hand, by Bayer and Sturmfels (1998, Proposition 2.6), we know that the acyclic simplicial complex Δ_{M^*} is a subcomplex of the polyhedral complex $\mathrm{hull}(M^*)$. The latter being a subdivision of the $(n-1)$-ball, and both complexes containing the boundary of $\mathrm{hull}(M^*)$, we can conclude that $\Delta_{M^*} = \mathrm{hull}(M^*)$.

(c) $\Rightarrow$ (d). Holds for any minimal simplicial resolution by Miller (1998, Theorem 5.12).
(d) $\Rightarrow$ (e). Trivial.

Set $\beta_{i,\mathbf{b}}(N) = \dim_k[\mathrm{Tor}_i^S(N,k)]_{\mathbf{b}}$, the ith Betti number of a module N in degree $\mathbf{b}$.

LEMMA 1.6. *If* $\mathbf{b} \in \mathbb{Z}^n$ *and* $\beta_{i,\mathbf{b}}(S/M^*) \neq 0$ *for some* i, *then there is an irreducible component* $\mathfrak{m}^{\mathbf{a}}$ *of* S/M^* *such that* $\mathbf{b} \preceq \mathbf{a}$.

PROOF. An irreducible ideal $\mathfrak{m}^{\mathbf{a}}$ is an irreducible component of S/M^* if and only if $\beta_{n,\mathbf{a}}(S/M^*) \neq 0$ by Miller (1998, Proposition 4.12 and Theorem 3.12), using the fact that S/M^* is artinian so that $\mathbf{a}$ has full support. Let $F.$ be a minimal free resolution of S/M^* and $F.^* := \mathrm{Hom}_S(F., S)$ its dual. Then $F.^*$ is a minimal free resolution of some $\mathbb{Z}^n$-graded module N with $\beta_{i,\mathbf{b}}(S/M^*) = \beta_{n-i,-\mathbf{b}}(N)$ by local duality. It follows that $-\mathbf{b} \succeq -\mathbf{a}$ for some $\mathbf{a}$ with $0 \neq \beta_{0,-\mathbf{a}}(N) = \beta_{n,\mathbf{a}}(S/M^*)$.□

(e) $\Rightarrow$ (b). Suppose $\beta_{i,\mathbf{a}}(S/M^*) \neq 0$. Since the Taylor complex of M^* is acyclic, it follows that $\mathbf{a} = \mathbf{a}_\sigma$ for some $\sigma \subset \{1, \ldots, r+n\}$. It suffices to prove $\sigma \in \Delta_{M^*}$ by Bayer *et al.* (1998, Lemma 3.1). From Lemma 1.6 and (e), there is some $\tau \in \Delta_{M^*}$ such that $\mathbf{a}_\sigma (= \mathbf{a}) \preceq \mathbf{a}_\tau$, that is, m_σ divides m_τ. Since $\tau \in \Delta_{M^*}$, we have $\sigma \subset \tau$. Thus $\sigma \in \Delta_{M^*}$.

(c) $\Rightarrow$ (f). Acyclicity follows from the criterion of Bayer *et al.* (1998, Lemma 2.2), because Δ_M is the subcomplex of Δ_{M^*} consisting of the faces whose labels divide $x_1^{D-1} \cdots x_n^{D-1}$. It therefore suffices to show the condition on edges when $M = M^*$ (note that Δ_M is a subcomplex of Δ_{M^*}).

If σ is any facet of Δ_{M^*}, then $\#\sigma = n$, $\mathrm{supp}(m_\sigma) = \{1, \ldots, n\}$, and

$$\text{each exponent vector } \mathbf{a}_i, i \in \sigma, \text{ shares a different coordinate with } \mathbf{a}_\sigma. \qquad (*)$$

Suppose now that $0 \neq \deg_{x_s} m_i = \deg_{x_s} m_j$ and $\{i,j\} \in \Delta_{M^*}$ is an edge. The end of Miller (1998, Remark 5.21) says that $m_{\{i,j\}} = \gcd(m_\sigma \mid \sigma \in \Delta_{M^*}$ is a facet containing $\{i,j\})$. In particular, there is some facet $\sigma \supseteq \{i,j\}$ with $\deg_{x_s} m_\sigma = \deg_{x_s} m_{\{i,j\}} = \deg_{x_s} m_i = \deg_{x_s} m_j$, contradicting $(*)$.

(f) $\Rightarrow$ (a). For any generator m_i let

$$A_i := \{m_j \mid m_j \neq m_i \text{ and } \deg_{x_s} m_j = \deg_{x_s} m_i > 0 \text{ for some } s\}.$$

The set A_i can be partially ordered by letting $m_j \preceq m_{j'}$ if $m_{\{i,j\}}$ divides $m_{\{i,j'\}}$. It is enough to produce a monomial m_l as in Definition 1.1 whenever $m_j \in A_i$ is a minimal element for this partial order. Supposing that m_j is minimal, use acyclicity to write

$$\frac{m_{\{i,j\}}}{m_i} \cdot e_i - \frac{m_{\{i,j\}}}{m_j} \cdot e_j = \sum_{\{u,v\} \in \Delta_M} b_{u,v} \cdot d(e_{\{u,v\}}), \qquad (1.2)$$

where we may assume (by picking such an expression with a minimal number of non-zero terms) that the monomials $b_{u,v}$ are 0 unless $m_{\{u,v\}}$ divides $m_{\{i,j\}}$. There is at least one monomial m_l such that $b_{l,j} \neq 0$, and we claim $m_l \notin A_i$. Indeed, m_l divides $m_{\{i,j\}}$ because $m_{\{l,j\}}$ does, so if $\deg_{x_t} m_i < \deg_{x_t} m_j$ (which must occur for some t because m_j does not divide m_i), then $\deg_{x_t} m_l \leq \deg_{x_t} m_j$. Applying the second half of (f) to $m_{\{l,j\}}$ we obtain $\deg_{x_t} m_l < \deg_{x_t} m_j$, and, furthermore, $\deg_{x_t} m_{\{i,l\}} < \deg_{x_t} m_{\{i,j\}}$, whence $m_l \notin A_i$ by minimality of m_j. So if $\deg_{x_s} m_{\{i,j\}} > 0$ for some s, then either $\deg_{x_s} m_l < \deg_{x_s} m_j$ by the second half of (f), or $\deg_{x_s} m_l < \deg_{x_s} m_i$ because $m_l \notin A_i$.

(a) $\Rightarrow$ (g). Choose $\sigma \notin \Delta_{M^*}$ maximal among subsets with label m_σ. Then $m_\sigma = m_{\sigma \setminus \{i\}}$

for some $i \in \sigma$. If $\mathrm{supp}(m_\sigma/m_i) = \mathrm{supp}(m_\sigma)$, the proof is done. Otherwise, there is some $j \in \sigma \setminus \{i\}$ with $\deg_{x_s} m_i = \deg_{x_s} m_j > 0$ for some x_s. Then neither m_i nor m_j is a power of a variable, so $m_i, m_j \in M$. Since M is generic, there is a monomial $m \in M$ which strictly divides $m_{\{i,j\}}$, which in turn strictly divides m_σ.

(g) $\Rightarrow$ (h). The strict inequalities which define the conditions "m_i does not divide m_σ" and "m_i strictly divides m_σ" persist after deformation. Persistence of the former implies that $\sigma \in \Delta_{M^*}$ remains a face in the deformation, while persistence of the latter implies that $\sigma \notin \Delta_{M^*}$ remains a non-face.

(h) $\Rightarrow$ (b). By Bayer *et al.* (1998, Theorem 4.3), there is a deformation ϵ of M^* such that $\Delta_{M^*_\epsilon}$ gives a free resolution of S/M^*. Since $\Delta_{M^*} = \Delta_{M^*_\epsilon}$, $F_{\Delta_{M^*}}$ is a free resolution. This resolution is automatically minimal.$\square$

Remark 1.7. (i) Conditions (b), (d), and (h) in Theorem 1.5 can be more naturally phrased (without referring to algebraic properties of M^*) in terms of $\mathbb{Z}^n$-graded *injective* resolutions of S/M, which are equivalent to free resolutions of S/M^* by Miller (2000). For instance, (d) says that the zeroth Bass numbers of S/M are determined by Δ_{M^*}, and (b) says that the entire $\mathbb{Z}^n$-graded injective resolution is determined by Δ_{M^*}.

(ii) The equivalence (g) $\Leftrightarrow$ (h) remains true, even if every occurrence of M^* is replaced by M. However, if M^* is replaced by M, then the conditions are not equivalent to genericity. A counterexample is $M = \langle xy, xz, xw \rangle$, whose Scarf complex does not change under deformations, and gives a minimal free resolution of S/M.

For generic M, $\mathrm{hull}(M)$ (resp. Δ_M) is the restriction of $\mathrm{hull}(M^*)$ (resp. Δ_{M^*}) to $\{1, \ldots, r\}$. Therefore, the next result follows from $(a) \Rightarrow (c)$ of Theorem 1.5.

Corollary 1.8. *If M is a generic monomial ideal, then* $\mathrm{hull}(M)$ *coincides with* Δ_M*, and the hull resolution* $F_{\mathrm{hull}(M)} = F_{\Delta_M}$ *is minimal.*

Example 1.2. (continued) The Scarf complex Δ_M of M is the first barycentric subdivision of the $(n-1)$-simplex. By Theorem 1.5, F_{Δ_M} gives a minimal free resolution of S/M. Miller (1998) also constructed a minimal free resolution of S/M as a *cohull resolution*, derived essentially from the coboundary complex of a permutahedron.

2. Associated Primes and Irreducible Components

In this section we study the primary decomposition of a generic monomial ideal M. For a monomial prime ideal P in S, we identify the homogeneous localization $(S/M)_{(P)}$ with the algebra $k[x_i \mid x_i \in P]/M_{(P)}$, where $M_{(P)}$ is the monomial ideal of $k[x_i \mid x_i \in P]$ gotten from M by setting all the variables not in P equal to 1.

Remark 2.1. If M is a generic monomial ideal, then so is $M_{(P)}$.

Let $M = \bigcap_{i=1}^r M_i$ be the irreducible decomposition of a monomial ideal M. Then $\{\mathrm{rad}(M_i) \mid 1 \le i \le r\} = \mathrm{Ass}(S/M)$ is the set of associated primes. Note that distinct irreducible components may have the same radical.

Recall that $\mathrm{codim}(I) \le \mathrm{codim}(P) \le \text{proj-dim}_S(S/I) \le n$ for any graded ideal $I \subset S$ and any associated prime $P \in \mathrm{Ass}(S/I)$, and $\mathrm{codim}(I) = \text{proj-dim}_S(S/I)$ if and only if S/I is Cohen–Macaulay. There always exists a minimal prime $P \in \mathrm{Ass}(S/I)$

with $\operatorname{codim}(P) = \operatorname{codim}(I)$, but in general there is no $P \in \operatorname{Ass}(S/I)$ with $\operatorname{codim}(P) = \operatorname{proj-dim}_S(S/I)$. For example, if $I = \langle x_1, x_2 \rangle \cap \langle x_3, x_4 \rangle$, then $\operatorname{proj-dim}_S(S/I) = 3$.

THEOREM 2.2. *Let $M \subset S$ be a generic monomial ideal.*

(a) For each integer i with $\operatorname{codim}(M) < i \leq \operatorname{proj-dim}_S(S/M)$, there is an embedded associated prime $P \in \operatorname{Ass}(S/M)$ with $\operatorname{codim}(P) = i$.

(b) For all $P \in \operatorname{Ass}(S/M)$ there is a chain of associated primes $P = P_0 \supset P_1 \supset \cdots \supset P_t$ with P_t a minimal prime of M and $\operatorname{codim}(P_i) = \operatorname{codim}(P_{i-1}) - 1$ for all i.

PROOF. (a) This was proved by Yanagawa (1999) for strongly generic ideals. The argument used there also works here.

(b) It suffices to show that for any embedded prime P of M there is an associated prime $P' \in \operatorname{Ass}(S/M)$ with $\operatorname{codim}(P') = \operatorname{codim}(P) - 1$ and $P' \subset P$. The localization $P_{(P)}$ of P is a maximal ideal of $S_{(P)} := k[x_i \mid x_i \in P]$, and an embedded prime of $M_{(P)}$, so there is a prime $P'_{(P)} \subset S_{(P)}$ such that $P'_{(P)} \in \operatorname{Ass}(S/M)_{(P)}$, $\operatorname{codim}(P'_{(P)}) = \operatorname{codim}(P_{(P)}) - 1$ and $P'_{(P)} \subset P_{(P)}$ by (a) applied to the generic ideal $M_{(P)}$. The preimage $P' \subset S$ of $P'_{(P)} \subset S_{(P)}$ has the expected properties.□

REMARK 2.3. Let $M \subset S$ be a generic monomial ideal, and $P, P' \in \operatorname{Ass}(S/M)$ such that $P \supset P'$ and $\operatorname{codim} P \geq \operatorname{codim} P' + 2$. Theorem 2.2 does *not* state that there is an associated prime between P and P'. For example, set $M = \langle ac, bd, a^3b^2, a^2b^3 \rangle$. Then $\langle a, b \rangle, \langle a, b, c, d \rangle \in \operatorname{Ass}(S/M)$, but there is no associated prime between them.

Recall the definition of the *extended Scarf complex* Δ_{M^*} of M after equation (1.1). From here on, we index the new monomials x_s^D just by their variables x_s; so the vertex set of Δ_{M^*} is a subset of $\{1, \ldots, r\} \cup \{x_1, \ldots, x_n\}$. This subset is proper if M contains a power of a variable. Recall that Δ_{M^*} is a regular triangulation of an $(n-1)$-simplex Δ. This was proved in Bayer *et al.* (1998, Corollary 5.5) for strongly generic ideals or Miller (1998, Proposition 5.16) for generic ideals. The vertex set of Δ equals $\{x_1, \ldots, x_n\}$ unless M contains a power of a variable. The restriction of Δ_{M^*} to $\{1, \ldots, r\}$ equals the Scarf complex Δ_M of M. We next determine the restriction of Δ_{M^*} to $\{x_1, \ldots, x_n\}$.

The radical $\operatorname{rad}(M)$ of M is a squarefree monomial ideal. Let $V(M)$ denote the corresponding *Stanley–Reisner complex*, which consists of all subsets of $\{x_1, \ldots, x_n\}$ which are not support sets of monomials in M. Then we have the following.

LEMMA 2.4. *For a generic monomial ideal M, the restriction of the extended Scarf complex Δ_{M^*} to $\{x_1, \ldots, x_n\}$ coincides with the Stanley–Reisner complex $V(M)$.*

PROOF. Each facet σ of Δ_{M^*} gives an irreducible component of S/M (Theorem 1.5), the radical of which represents the face $\sigma \cap \{x_1, \ldots, x_n\}$ of $V(M)$. The facets of $V(M)$ arise in this way from irreducible components whose associated primes are minimal.□

The following theorem generalizes a result of Yanagawa (1999, Corollary 2.4). For the definition of shellability, see Stanley (1996, Section III.2) or Ziegler (1995, Lecture 8).

THEOREM 2.5. *Let M be a generic monomial ideal. If M has no embedded associated primes, then M is Cohen–Macaulay. For this case, both Δ_M and $V(M)$ are shellable.*

PROOF. The first statement immediately follows from Theorem 2.2. For the second statement we note that all facets σ of Δ_{M^*} have the following property:

$$|\sigma \cap \{1, \ldots, r\}| = \mathrm{codim} M \qquad \text{and} \qquad |\sigma \cap \{x_1, \ldots, x_n\}| = \dim S/M. \tag{2.1}$$

In particular, both cardinalities in (2.1) are independent of the facet σ. On the other hand, Δ_{M^*} is shellable since it is a regular triangulation of a simplex. A theorem of Björner (1995, Theorem 11.13) implies that the restrictions of Δ_{M^*} to $\{1, 2, \ldots, r\}$ and to $\{x_1, \ldots, x_n\}$ are both shellable. We are done in view of Lemma 2.4.□

REMARK 2.6. (a) The shellability of Δ_{M^*} also implies the following result. If M is generic and $P, P' \in \mathrm{Ass}(S/M)$, then there is a sequence of associated primes $P = P_0, P_1, \ldots, P_t = P'$ with $\mathrm{codim}(P_i + P_{i-1}) = \min\{\mathrm{codim}(P_i), \mathrm{codim}(P_{i-1})\} + 1$ for all $1 \leq i \leq t$. If M is pure dimensional, this simply says that S/M is connected in codimension 1.

(b) A shelling of the boundary complex of a polytope can start from a shelling of the subcomplex consisting of all facets containing a given face; see Ziegler (1995, Theorem 8.12). The complex $V(M)$ of a generic Cohen–Macaulay monomial ideal M inherits this property, so $V(M)$ has stronger properties than general shellable complexes.

Theorem 2.5 and Remark 2.6 suggest the following combinatorial problems.

PROBLEM 2.7. (i) Characterize all collections $\mathcal{A}$ of monomial primes for which there exists a generic monomial ideal M with $\mathcal{A} = \mathrm{Ass}(S/M)$.

(ii) Characterize the Stanley–Reisner complexes $V(M)$ of Cohen–Macaulay generic monomial ideals M.

A necessary condition for (i) is that $\mathcal{A}$ satisfy the connectivity in Remark 2.6(a). However, this is not sufficient: for instance, take $\mathcal{A}$ to be the minimal primes of a Stanley–Reisner ring which is Cohen–Macaulay but whose simplicial complex is not shellable.

For problem (ii), the Cohen–Macaulayness assumption is essential. Indeed, for any simplicial complex Σ on $\{1, 2, \ldots, n\}$, a deformation of the Stanley–Reisner ideal I_Σ gives a (not necessarily Cohen–Macaulay) generic monomial ideal M with $V(M) = \Sigma$. By Theorem 2.5, shellability is a necessary condition for problem (ii), but it is not sufficient as Remark 2.6(b) shows.

If we put further restrictions on the generators of a generic monomial ideal M, then, since the extended Scarf complex Δ_{M^*} is a triangulation of a simplex, we can apply Stanley's theory of local h-vectors (Stanley, 1992). The next two results will be reinterpreted in Section 4 in terms of cogeneric ideals using Alexander duality (Miller, 1998).

Again, let M^* be as in (1.1), and define the *excess* $e(\sigma)$ of a face $\sigma \in \Delta_{M^*}$ to be $e(\sigma) := \#\mathrm{supp}(m_\sigma) - \#\sigma$. This agrees, in our case, with the definition in Stanley (1992).

THEOREM 2.8. *If M is generic and all r generators $m_1, \ldots, m_r$ have support of size c, i.e. $\#\mathrm{supp}(m_i) = c$ for all i, then M has at least $(c-1) \cdot r + 1$ irreducible components.*

EXAMPLE 2.9. This is false without the assumption that M is generic. For instance, the non-generic monomial ideal $M = \langle x_1, y_1 \rangle \cap \cdots \cap \langle x_n, y_n \rangle$ has $r = 2^n$ generators, and each generator has support of size $c = n$, but M has only n irreducible components.

PROOF. If $c = 1$, there is nothing to prove, so we may assume that $c \geq 2$. Set $\Gamma = \Delta_{M^*}$.

The hypothesis on the generators of M means that Γ has n vertices of excess 0 and r vertices of excess $c-1$. To prove the assertion, we use the decomposition

$$h(\Gamma, x) = \sum_{W \in \Delta} \ell_W(\Gamma_W, x) \tag{2.2}$$

of the h-polynomial of Γ into local h-polynomials (Stanley, 1992, equation (3)). Here, Δ denotes the simplex on $\{x_1, \dots, x_n\}$ and Γ_W the restriction of Γ to a face W of Δ. We have

$$\ell_W(\Gamma_W, x) = 1 \qquad \text{if } W = \emptyset. \tag{2.3}$$

Next, we consider the case $\#W = c$. In Γ_W, the vertices corresponding to generators of M have excess $c-1$, and all other faces have excess less than $c-1$. So we have

$$\ell_W(\Gamma_W, x) = \ell_1(\Gamma_W)x + \ell_2(\Gamma_W)x^2 + \cdots + \ell_{c-1}(\Gamma_W)x^{c-1} \qquad \text{if } \#W = c, \tag{2.4}$$

where $\ell_1(\Gamma_W)$ is the number of generators of M whose supports correspond to the face W of Δ by Stanley (1992, Example 2.3(f)). Moreover $\ell_i(\Gamma_W) \geq \ell_1(\Gamma_W)$ for all $1 \leq i \leq c-1$ by Stanley (1992, Theorems 5.2 and 3.3).

The coefficients of $\ell_W(\Gamma_W, x)$ are non-negative for all $W \in \Delta$ by Stanley (1992, Corollary 4.7). We now substitute the expressions in (2.3) and (2.4) into the sum on the right-hand side of (2.2) and then evaluate at $x = 1$. The number of irreducible components of M equals the number $f_{n-1}(\Gamma) = h(\Gamma, 1)$ of facets of Γ by Theorem 1.5, hence

$$h(\Gamma, 1) \geq 1 + \sum_{\#W=c} \left(\sum_{i=1}^{c-1} \ell_i(\Gamma_W) \right) \geq 1 + \sum_{\#W=c} (c-1) \cdot \ell_1(\Gamma_W) = (c-1) \cdot r + 1.$$

This yields the desired inequality.□

The inequality in Theorem 2.8 is sharp for all c and r; see Example 4.18 below.

Proposition 2.10. *Let M be a generic monomial ideal with r generators each of which is a bivariate monomial. Then M has exactly $r+1$ irreducible components if and only if $\#\mathrm{supp}(m_\sigma) \leq 3$ for all edges $\sigma \in \Delta_M$.*

Proof. By the assumption, Δ_{M^*} has n vertices of excess 0 and r vertices of excess 1. Adding a vertex to any face of Δ_{M^*} increases the excess by at most 1, so we conclude that the equality $\{\sigma \in \Delta_{M^*} \mid \#\sigma = e(\sigma)\} = \{\emptyset, \{1\}, \{2\}, \dots, \{r\}\}$ holds if and only if each edge of Δ_M has excess at most 1, equivalently, support of size at most 3. The result is now an immediate consequence of a result of Stanley (1992, Proposition 3.4).□

3. Initial Ideals of Lattice Ideals

One motivation for our new definition of genericity for monomial ideals is consistency with the notion of genericity for lattice ideals introduced by Peeva and Sturmfels (1998b). It is the purpose of this section to establish this connection. We fix a sublattice $\mathcal{L}$ of $\mathbb{Z}^n$ which contains no non-negative vectors. The *lattice ideal* $I_\mathcal{L}$ associated to $\mathcal{L}$ is defined by

$$I_\mathcal{L} := \langle \mathbf{x^a} - \mathbf{x^b} \mid \mathbf{a}, \mathbf{b} \in \mathbb{N}^n \quad \text{and} \quad \mathbf{a} - \mathbf{b} \in \mathcal{L} \rangle \subset S,$$

where $\mathbf{x^a} = x_1^{a_1} \cdots x_n^{a_n}$ for $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$. We have $\mathrm{codim}(I_\mathcal{L}) = \mathrm{rank}(\mathcal{L})$. The ideal $I_\mathcal{L}$ is homogeneous with respect to some grading where $\deg(x_s)$ is a positive integer for each s. In Theorem 3.1 below, "reverse lexicographic order" means a *degree* reverse lexicographic order with respect to this grading. Note that the ring $S/I_{\mathcal{L}}$ also has a fine grading by $\mathbb{Z}^n/\mathcal{L}$ (see Peeva and Sturmfels, 1998a).

The following three conditions are equivalent: (a) the abelian group $\mathbb{Z}^n/\mathcal{L}$ is torsion free, (b) $I_{\mathcal{L}}$ is a prime ideal, and (c) $I_{\mathcal{L}}$ is a toric ideal (i.e. $S/I_{\mathcal{L}}$ is an affine semigroup ring). Even if $I_{\mathcal{L}}$ is not prime, all monomials are non-zero divisors of $S/I_{\mathcal{L}}$, and all associated primes of $I_{\mathcal{L}}$ have the same codimension. If I_A is the toric ideal of an integer matrix A, as defined in Sturmfels (1995), then I_A coincides with the lattice ideal $I_{\mathcal{L}}$ where $\mathcal{L} \subset \mathbb{Z}^n$ is the kernel of A.

Following Peeva and Sturmfels (1998b), we call a lattice ideal $I_{\mathcal{L}}$ *generic* if it is generated by binomials with full support, i.e.

$$I_{\mathcal{L}} = \langle \mathbf{x}^{\mathbf{a}_1} - \mathbf{x}^{\mathbf{b}_1}, \mathbf{x}^{\mathbf{a}_2} - \mathbf{x}^{\mathbf{b}_2}, \ldots, \mathbf{x}^{\mathbf{a}_r} - \mathbf{x}^{\mathbf{b}_r} \rangle,$$

where none of the r vectors $\mathbf{a}_i - \mathbf{b}_i \in \mathbb{Z}^n$ has a zero coordinate.

Theorem 3.1. *Let $I_{\mathcal{L}}$ be a generic lattice ideal, and M the initial ideal of $I_{\mathcal{L}}$ with respect to a reverse lexicographic term order. Then M is a generic monomial ideal.*

Proof. Set $M = \mathrm{in}_{\mathrm{revlex}}(I_{\mathcal{L}}) = \langle m_1, \ldots, m_r \rangle$. Gasharov *et al.* (1999) proved that the algebraic Scarf complex F_{Δ_M} is a minimal free resolution of S/M. Using Theorem 1.5(f), it suffices to prove that no variable x_s appears with the same non-zero exponent in m_i and m_j for any $i \neq j$ with $\{i, j\} \in \Delta_M$. Assume the contrary, that is, $\deg_{x_s} m_i = \deg_{x_s} m_j > 0$ for some $\{i, j\} \in \Delta_M$. By Peeva and Sturmfels (1998b, Theorem 5.2), there are three monomials $m_i', m_j', m_l' \in S$ satisfying the following conditions.

(a) The set $\{m_i', m_j', m_l'\}$ is a *basic fiber* (Peeva and Sturmfels, 1998b, Section 2).
(b) $m_i = \frac{m_l'}{\gcd(m_i', m_l')}$ and $m_j = \frac{m_l'}{\gcd(m_j', m_l')}$.

From (b) and our assumption $\deg_{x_s} m_i' = \deg_{x_s} m_j > 0$, we have

$$\deg_{x_s}, (m_i') = \deg_{x_s}(\gcd(m_i', m_l')) = \deg_{x_s}(\gcd(m_j', m_l')) = \deg_{x_s},'_j\,.$$

Part of the requirement for (a) is that $\gcd(m_i', m_j', m_l') = 1$, so $\deg_{x_s} m_i' = \deg_{x_s} m_i' = 0$. Combining property (a) with Peeva and Sturmfels (1998b, Theorem 3.2), we see that the binomial

$$\frac{m_i'}{\gcd(m_i', m_j')} - \frac{m_j'}{\gcd(m_i', m_j')}$$

is a minimal generator of $I_{\mathcal{L}}$. Since $\deg_{x_s} m_i' = \deg_{x_s} m_j' = 0$, the variable x_s does not appear in the above binomial. This contradicts the genericity of $I_{\mathcal{L}}$, since a generic lattice ideal has a unique minimal set of homogeneous binomial generators; see Peeva and Sturmfels (1998b, Remark 4.4).□

Example 3.2. Theorem 3.1 is false if "generic" is replaced by "strongly generic". For example, consider the following generic lattice ideal in $k[a, b, c, d]$:

$$I_{\mathcal{L}} = \langle a^4 - bcd, a^3c^2 - b^2d^2, a^2b^3 - c^2d^2, ab^2c - d^3, b^4 - a^2cd, b^3c^2 - a^3d^2, c^3 - abd \rangle.$$

This ideal was featured in Peeva and Sturmfels (1998b, Example 4.5); it defines the toric curve $(t^{20}, t^{24}, t^{25}, t^{31})$. Consider a reverse lexicographic term order with $a > b > c > d$.

Then $M = \langle a^4, a^3c^2, a^2b^3, ab^2c, b^4, b^3c^2, c^3\rangle$. This ideal is not strongly generic since a^3c^2 and b^3c^2 are minimal generators. But M is generic since $ab^2c \in M$.

An important problem in combinatorial commutative algebra is to characterize those monomial ideals which are initial ideals of lattice ideals. The recent "Chain Theorem" of Hoşten and Thomas (1999) provides a remarkable necessary condition.

THEOREM 3.3. (HOŞTEN AND THOMAS, 1999) *Let M be the initial ideal of a lattice ideal $I_{\mathcal{L}}$ with respect to any term order. For each $P \in \mathrm{Ass}(S/M)$, there is a chain of associated primes $P = P_0 \supset P_1 \supset \cdots \supset P_t$ of M such that P_t is a minimal prime and* $\mathrm{codim}(P_i) = \mathrm{codim}(P_{i-1}) - 1$ *for all i.*

In other words, initial ideals of lattice ideals satisfy conclusion (b) of Theorem 2.2, even if they are not generic. We do not know whether part (a) holds as well.

CONJECTURE 3.4. *Let M be the initial ideal of $I_{\mathcal{L}}$ with respect to some term order. Then there is an associated prime $P \in \mathrm{Ass}(S/M)$ with* $\mathrm{codim}(P) = \text{proj-dim}_S(S/M)$.

COROLLARY 3.5. *Conjecture* 3.4 *holds for the reverse lexicographic term order if the lattice ideal $I_{\mathcal{L}}$ is generic.*

PROOF. Immediate from Theorems 2.2 and 3.1.□

The following result is implicit in Hoşten and Thomas (1999) and Peeva and Sturmfels (1998a).

LEMMA 3.6. *Let M be the initial ideal of a lattice ideal $I_{\mathcal{L}}$ with respect to any term order. Then we have* $\text{proj-dim}_S(S/M) \le 2^c - 1$ *where* $c := \mathrm{codim} I_{\mathcal{L}} = \mathrm{codim} M$.

PROOF. Following Peeva and Sturmfels (1998a, Algorithm 8.2), we construct a lattice ideal $I_{\mathcal{L}'}$ in $S[t] = k[x_1, \dots, x_n, t]$ whose images under the substitutions $t = 1$ and $t = 0$ are $I_{\mathcal{L}}$ and M, respectively. Moreover, t is a non-zero divisor of $S[t]/I_{\mathcal{L}'}$, and the codimension of $I_{\mathcal{L}'}$ in $S[t]$ is equal to $\mathrm{codim}(I_{\mathcal{L}})$. Since $S/M = S[t]/(I_{\mathcal{L}'} + \langle t\rangle)$, we have $\text{proj-dim}_S(S/M) = \text{proj-dim}_{S[t]}(S[t]/I_{\mathcal{L}'}) \le 2^c - 1$. The last inequality follows from Peeva and Sturmfels (1998a, Theorem 2.3).□

We note that Conjecture 3.4 is also true in codimension 2:

PROPOSITION 3.7. *Conjecture* 3.4 *holds for any term order if* $\mathrm{codim}(I_{\mathcal{L}}) = 2$.

PROOF. By Lemma 3.6, $\text{proj-dim}_S(S/M) \le 3$. We may assume $\text{proj-dim}_S(S/M) = 3$, because otherwise M is Cohen–Macaulay and there is nothing to prove. Then there exists a syzygy quadrangle as in Peeva and Sturmfels (1998a, Section 3) for the planar configuration of $n+1$ vectors representing the ideal $I_{\mathcal{L}'}$ from Lemma 3.6. This quadrangle defines a lattice point free polytope as in Hoşten and Thomas (1999, Section 2), and from the explicit primary decomposition given by Hoşten and Thomas (1999, Theorem 4.2) we see that M has an associated prime of codimension 3.□

For an ideal $I \subset S$, it is well known that $\text{proj-dim}_S(S/I) \leq \text{proj-dim}_S(S/\text{in}(I))$. This inequality can be strict even in the codimension 2 toric ideal case. Set $I_{\mathcal{L}} := \langle ac-b^2, ad-bc, bd-c^2\rangle \subset S = k[a,b,c,d]$ be the defining ideal of the twisted cubic curve in $\mathbb{P}^3$. $S/I_{\mathcal{L}}$ is normal and Cohen–Macaulay. The ideal $I_{\mathcal{L}}$ has eight distinct initial ideals, when we consider all possible term orders (see Sturmfels, 1991, Section 4). Four of them are not Cohen–Macaulay and have embedded associated primes of codimension 3.

Remark 3.8. Let $M \subset S$ be a *Borel fixed* monomial ideal (Eisenbud, 1995, Section 15.9). In general, Borel fixed ideals are far from generic. However, it is easy to see that there is an associated prime $P \in \text{Ass}(S/M)$ with $\text{codim}(P) = \text{proj-dim}_S(S/M)$. Hence a Borel fixed ideal M satisfies the conclusion of Conjecture 3.4. Therefore the *generic initial ideal* (Eisenbud, 1995, Section 15) of a homogeneous ideal $I \subset S$ satisfies the conclusion of the conjecture, when $\text{char}(k) = 0$. However, Borel fixed ideals may fail to satisfy the conclusion of Theorem 3.3. For instance, take $M = \langle x^2, xy, xz\rangle = \langle x\rangle \cap \langle x^2, y, z\rangle$.

4. Cogeneric Monomial Ideals

Cogeneric monomial ideals were introduced in Definition 1.3. As with genericity, our definition of cogenericity is slightly different from the original one of Sturmfels (1999). In Theorem 4.6, we shall see that the result of Sturmfels (1999), an explicit description of the minimal free resolution of a cogeneric monomial ideal, is still true here. In fact, Alexander duality for arbitrary monomial ideals (Miller, 1998) allows us to shorten the construction of this resolution and clarify its relation to Theorem 1.5. For the reader's convenience, we recall the definition of Alexander duality. For details see Miller (1998).

Monomials and irreducible monomial ideals may each be specified by a single vector $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{N}^n$, so we write $\mathbf{x^b} = x_1^{b_1} \cdots x_n^{b_n}$ and $\mathfrak{m}^{\mathbf{b}} = \langle x_s^{b_s} \mid b_s \geq 1\rangle$. Given a vector $\mathbf{a} = (a_1, \dots, a_n)$ such that $b_s \leq a_s$ for all s, we define the Alexander dual vector $\mathbf{b^a}$ with respect to $\mathbf{a}$ by setting its sth coordinate to be

$$(\mathbf{b^a})_s = \begin{cases} a_s + 1 - b_s & \text{if } b_s \geq 1 \\ 0 & \text{if } b_s = 0. \end{cases}$$

Whenever we deal with Alexander duality, we assume that we are given a vector $\mathbf{a}$ such that for each s, the integer a_s is larger than or equal to the sth coordinate of any minimal monomial generator of M. This implies that a_s is also larger than or equal to the sth coordinate of any irreducible component of M, and vice versa. The *Alexander dual ideal* $M^{\mathbf{a}}$ of M with respect to $\mathbf{a}$ is defined by

$$\begin{aligned} M^{\mathbf{a}} &= \langle \mathbf{x}^{\mathbf{b^a}} \mid \mathfrak{m}^{\mathbf{b}} \text{ is an irreducible component of } M\rangle \\ &= \bigcap\{\mathfrak{m}^{\mathbf{c^a}} \mid \mathbf{x^c} \text{ is a minimal generator of } M\}. \end{aligned}$$

That these two formulas give the same ideal is not obvious; it is equivalent to $(M^{\mathbf{a}})^{\mathbf{a}} = M$. It follows from these statements that M is generic if and only if $M^{\mathbf{a}}$ is cogeneric.

Example 4.1. The following monomial ideal in $S = k[x,y,z]$ is cogeneric:

$$M = \langle yz^2, xz^2, y^2z, xy^2, x^2\rangle = \langle x, y\rangle \cap \langle x^2, y^2, z^2\rangle \cap \langle x, z\rangle.$$

Its Alexander dual with respect to $\mathbf{a} = (2,2,2)$ is generic:

$$M^{\mathbf{a}} = \langle x^2y^2, xyz, x^2z^2\rangle = \langle y^2, z\rangle \cap \langle x^2, z\rangle \cap \langle y, z^2\rangle \cap \langle x^2, y\rangle \cap \langle x\rangle.$$

Example 4.2. (Miller, 1998, Examples 1.9 and 5.22) If M is the tree ideal of Example 1.2 and $\mathbf{a} = (n, \ldots, n)$, then its Alexander dual $M^{\mathbf{a}}$ is the *permutahedron ideal*:

$$M^{\mathbf{a}} = \langle x_1^{\pi(1)} x_2^{\pi(2)} \cdots x_n^{\pi(n)} \mid \pi \text{ is a permutation of } \{1, 2, \ldots, n\}\rangle.$$

Thus the permutahedron ideal is cogeneric. Its minimal free resolution is the hull resolution supported on a permutahedron; see Bayer and Sturmfels (1998, Example 1.9). The following discussion reinterprets this resolution as a co-Scarf complex.

Definition 4.3. Let $M = \bigcap_{i=1}^{r} M_i$ be a cogeneric monomial ideal. Set $\mathbf{a} = (D-1, \ldots, D-1)$ with D larger than any exponent on any minimal generator of M. The Alexander dual ideal $M^{\mathbf{a}}$ is minimally generated by monomials $m_1, \ldots, m_r$, where $m_i = \mathbf{x}^{\mathbf{b}_i{}^{\mathbf{a}}}$ for $M_i = \mathfrak{m}^{\mathbf{b}_i}$. We define the *co-Scarf complex* $\Delta_M^{\mathbf{a}}$ to be the extended Scarf complex of $M^{\mathbf{a}}$. More precisely, we set $(M^{\mathbf{a}})^* := M^{\mathbf{a}} + \langle x_1^D, \ldots, x_n^D\rangle$ and $\Delta_M^{\mathbf{a}}$ the Scarf complex of $(M^{\mathbf{a}})^*$. Since we index a new monomial x_s^D just by x_s, we see that $\Delta_M^{\mathbf{a}}$ is a simplicial complex on (a subset of) $\{1, \ldots, r, x_1, \ldots, x_n\}$.

Remark 4.4. (a) The co-Scarf complex $\Delta_M^{\mathbf{a}}$ of a cogeneric monomial ideal M is always a regular triangulation of an $(n-1)$-simplex Δ. The vertex set of Δ equals $\{x_1, \ldots, x_n\}$ unless M has a codimension 1 component.

(b) There is nothing special about our choice of $\mathbf{a}$, except that it makes for convenient notation. Everything we do with $\Delta_M^{\mathbf{a}}$ is independent of which sufficiently large $\mathbf{a}$ is chosen. In particular, the regular triangulation of the $(n-1)$-simplex is independent of $\mathbf{a}$, as is the algebraic co-Scarf complex (Definition 4.5) it determines. We therefore set $\mathbf{a} = (D-1, \ldots, D-1)$ for the remainder of this section.

(c) For $\sigma \subseteq \{1, \ldots, r\}$, let M_σ be the irreducible ideal $\sum_{i\in\sigma} M_i$. Then $m_\sigma = \mathbf{x}^{\mathbf{b}^{\mathbf{a}}}$ if $M_\sigma = \mathfrak{m}^{\mathbf{b}}$, and $\Delta_M^{\mathbf{a}} \cap \{1, \ldots, r\} = \{\sigma \subset \{1, \ldots, r\} \mid M_\sigma \neq M_\tau \text{ for all } \tau \neq \sigma\}$ is just the Scarf complex of $M^{\mathbf{a}}$.

A face σ of the co-Scarf complex $\Delta_M^{\mathbf{a}}$ fails to be in the (topological) boundary $\partial\Delta_M^{\mathbf{a}}$ of $\Delta_M^{\mathbf{a}}$ if and only if the monomial m_σ has full support, where m_σ is $\mathrm{lcm}(m_i \mid i \in \sigma)$ under the notation of Definition 4.3. Such a face will be called an *interior face* of $\Delta_M^{\mathbf{a}}$. The set $\mathrm{int}(\Delta_M^{\mathbf{a}})$ of interior faces is closed under taking supersets; that is, $\mathrm{int}(\Delta_M^{\mathbf{a}})$ is a *simplicial cocomplex*. Just as the algebraic Scarf complex is constructed from Δ_M for generic M, we construct a complex of free S-modules from $\mathrm{int}(\Delta_M^{\mathbf{a}})$, but this time we use the coboundary map instead of the boundary map. The following is a special kind of *relative cocellular resolution* (in fact a cohull resolution) (Miller, 1998, Section 5).

Definition 4.5. Let $\mathbf{D} = (D, \ldots, D) \in \mathbb{N}^n$ and $S(\mathbf{a}_\sigma - \mathbf{D})$ be the free S-module with one generator e_σ^* in multidegree $\mathbf{D} - \mathbf{a}_\sigma$. The *algebraic co-Scarf complex* $F^{\Delta_M^{\mathbf{a}}}$ of M is the free S-module

$$\bigoplus_{\sigma\in\mathrm{int}(\Delta_M^{\mathbf{a}})} S(\mathbf{a}_\sigma - \mathbf{D}) \text{ with differential } d^*(e_\sigma^*) = \sum_{\substack{i\notin\sigma \\ \sigma\cup\{i\}\in\mathrm{int}(\Delta_M^{\mathbf{a}})}} \mathrm{sign}(i, \sigma\cup\{i\}) \cdot \frac{m_{\sigma\cup\{i\}}}{m_\sigma} \cdot e_{\sigma\cup\{i\}}^*,$$

where $\mathrm{sign}(i, \sigma\cup\{i\})$ is $(-1)^{j+1}$ if i is the jth element in the ordering of $\sigma\cup\{i\}$. Put the summand $S(\mathbf{a}_\sigma - \mathbf{D})$ in homological degree $n - \#\sigma = n - \dim(\sigma) - 1$.

Theorem 4.6. *If M is a cogeneric monomial ideal, then the algebraic co-Scarf complex*

$F^{\Delta_M^{\mathbf{a}}}$ *equals the minimal free resolution of* M *over* S. *In particular,* M *is minimally generated by the set of monomials* $\{\mathbf{x}^{\mathbf{D}-\mathbf{a}_\sigma} \mid \sigma \text{ is a facet of } \Delta_M^{\mathbf{a}}\}$.

Proof. This follows from Theorem 1.5 and Miller (1998, Theorem 5.8).□

Example 4.1. (continued) For the cogeneric ideal $M = \langle x, y\rangle \cap \langle x^2, y^2, z^2\rangle \cap \langle x, z\rangle$, the interior faces of $\Delta_M^{\mathbf{a}}$ are $\{2\}$, $\{1, 2\}$, $\{2, 3\}$, $\{2, x\}$, $\{2, y\}$, $\{2, z\}$, $\{1, 2, x\}$, $\{1, 2, y\}$, $\{2, 3, x\}$, $\{2, 3, z\}$ and $\{2, y, z\}$. The co-Scarf resolution is $0 \to S \to S^5 \to S^5 \to M \to 0$. The generators of M have exponent vectors $\mathbf{D}-\mathbf{a}_{\{1,2,x\}} = (0, 1, 2)$, $\mathbf{D}-\mathbf{a}_{\{1,2,y\}} = (1, 0, 2)$, $\mathbf{D} - \mathbf{a}_{\{2,3,x\}} = (0, 2, 1)$, $\mathbf{D} - \mathbf{a}_{\{2,3,z\}} = (1, 2, 0)$ and $\mathbf{D} - \mathbf{a}_{\{2,y,z\}} = (2, 0, 0)$.

Remark 4.7. Theorem 4.6 can be strengthened to look pretty much like Theorem 1.5 by taking Alexander duals of each of the conditions there. For instance, hull becomes cohull, irreducible components become generators, etc.

Our next main result, Theorem 4.9, characterizes Cohen–Macaulay cogeneric monomial ideals. First, we give a polyhedral description of depth for cogeneric ideals.

Lemma 4.8. *Let* M *be a cogeneric monomial ideal. Then* $\operatorname{depth}(S/M) \le d$ *if and only if the co-Scarf complex* $\Delta_M^{\mathbf{a}}$ *has an interior face of dimension* d.

Proof. By Theorem 4.6, the shifted augmentation $F^{\Delta_M^{\mathbf{a}}} \to S$ (obtained by including $\operatorname{coker}(F^{\Delta_M^{\mathbf{a}}}) = M$ into S and shifting homological degrees up one) is a minimal free resolution of S/M. The co-Scarf complex $\Delta_M^{\mathbf{a}}$ has an interior face of dimension d if and only if this shifted augmented complex is non-zero in homological degree $n-d$. The lemma now follows from the Auslander–Buchsbaum formula (Eisenbud, 1995, Theorem 19.9).□

Recall that a module N satisfies Serre's condition (S_k) if for every prime $P \subset S$, $\operatorname{depth}(N_P) < k \Rightarrow \operatorname{depth}(N_P) = \dim(N_P)$. Using Bruns and Herzog (1993, Chapter 2.1) and homogeneous localization, it follows that if S/M satisfies (S_k), then

$$\operatorname{depth}((S/M)_{(P)}) < k \Longrightarrow \dim((S/M)_{(P)}) = \operatorname{depth}((S/M)_{(P)}). \tag{4.1}$$

Observe that $M_{(P)}$ is cogeneric if M is, in analogy to Remark 2.1. For condition (d) below, recall the definition of *excess* from before Theorem 2.8.

Theorem 4.9. *Let* $M \subset S$ *be a cogeneric monomial ideal of codimension* c *with the irreducible decomposition* $M = \bigcap_{i=1}^r M_i$. *Then the following conditions are equivalent.*

(a) S/M *is Cohen–Macaulay.*
(b) S/M *satisfies Serre's condition* (S_2).
(c) $\operatorname{codim}(M_i) = c$ *for all* i, *and* $\operatorname{codim}(M_i + M_j) \le c + 1$ *for all edges* $\{i, j\} \in \Delta_M^{\mathbf{a}}$.
(d) Every face of $\Delta_M^{\mathbf{a}}$ *has excess* $< c$.
(e) $\Delta_M^{\mathbf{a}}$ *has no interior faces of dimension* $< n - c$.

Proof. (a) $\Rightarrow$ (b). Cohen–Macaulay $\Leftrightarrow$ (S_k) for all k.

(b) $\Rightarrow$ (c). The initial equality follows from Hartshorne (1962, Remark 2.4.1), so it suffices to prove the inequality. If $c = 1$ this is obvious. We assume $c \ge 2$. Suppose $i \ne j$ with $\{i, j\} \in \Delta_M^{\mathbf{a}}$. Let $P = \operatorname{rad}(M_i + M_j)$, and denote by F the face of $\Delta = 2^{\{x_1,\ldots,x_n\}}$ whose vertices are the variables in P. By Miller (1998, Proposition 4.6), the co-Scarf complex of $M_{(P)}$ is, as a triangulation of the simplex 2^F, the restriction $(\Delta_M^{\mathbf{a}})_F$ of the triangulation $\Delta_M^{\mathbf{a}}$ to 2^F. By our choice of F, $\{i,j\}$ is an interior edge of $(\Delta_M^{\mathbf{a}})_F$, so Lemma 4.8 implies that $\operatorname{depth}((S/M)_{(P)}) \leq 1$, whence (4.1) implies that $\dim((S/M)_{(P)}) \leq 1$. Equivalently, $\operatorname{codim}(M_i + M_j) \leq c+1$.

(c) $\Rightarrow$ (d). The purity of the irreducible components means that all vertices have excess $c-1$ or 0, while the condition on the edges implies that the excess of a non-empty face can only decrease or remain the same upon the addition of a vertex.

(d) $\Rightarrow$ (e). In particular, the interior faces have excess less than c.

(e) $\Rightarrow$ (a). Lemma 4.8.□

Remark 4.10. (a) Hartshorne (1962) proved that a catenary local ring satisfying Serre's condition (S_2) is pure and connected in codimension 1. The converse is not true even for cogeneric monomial ideals. If we take $M = \langle x, y^2\rangle \cap \langle y, z\rangle \cap \langle z^2, w\rangle$ then S/M is pure and connected in codimension 1, but does not satisfy the condition (S_2); in fact, $\operatorname{depth}(S/M) = 1$. On the other hand, $M' = \langle x, y\rangle \cap \langle y^2, z^2\rangle \cap \langle z, w\rangle$ is Cohen–Macaulay, although $\operatorname{Ass}(M) = \operatorname{Ass}(M')$.

(b) Let I be a squarefree monomial ideal and $I^\vee = I^{(1,\ldots,1)}$ its Alexander dual. Eagon and Reiner (1998) proved that S/I is Cohen–Macaulay if and only if $S/I^\vee$ has a linear free resolution. Yanagawa (2000) proved that S/I satisfies the (S_2) condition if and only if all minimal generators of $I^\vee$ have the same degree and all minimal *first* syzygies are linear. So the equivalence between (b) and (c) of Theorem 4.9 seems natural, since an edge $\{i,j\} \in \Delta_M^{\mathbf{a}}$ corresponds to a first syzygy of $M^{\mathbf{a}}$. However, the (S_2) condition is much weaker than Cohen–Macaulayness for squarefree monomial ideals.

The above theorem and remark leads to a natural question.

Problem 4.11. Which Cohen–Macaulay simplicial complexes have Stanley–Reisner ideal $\operatorname{rad}(M)$ for some Cohen–Macaulay cogeneric monomial ideal M?

Recall that the *type* of a Cohen–Macaulay quotient S/M is the non-zero total Betti number of highest homological degree; if M is cogeneric, then this Betti number equals the number of interior faces of minimal dimension in $\Delta_M^{\mathbf{a}}$ by Theorem 4.6.

Theorem 4.12. *Let M be a Cohen–Macaulay cogeneric monomial ideal of codimension ≥ 2. The type of S/M is at least the number of irreducible components of M.*

Recall that S/M is *Gorenstein* if its Cohen–Macaulay type equals 1. This implies:

Corollary 4.13. *Let M be a cogeneric monomial ideal. Then S/M is Gorenstein if and only if M is either a principal ideal or an irreducible ideal.*

Remark 4.14. For the generic monomial ideal case, we have the opposite inequality to the one in Theorem 4.12. More precisely, if M is Cohen–Macaulay and generic, then

$$\begin{aligned} \text{Cohen–Macaulay type of } S/M &= \#\{\text{facets of the Scarf complex } \Delta_M\} \\ \leq \#\{\text{facets of } \Delta_{M^*}\} &= \#\{\text{irreducible components of } M\}, \end{aligned}$$

because the map $\Delta_{M^*} \to \Delta_M, \sigma \mapsto \sigma \cap \{1,\ldots,r\}$, is surjective on facets. Also here, S/M is Gorenstein if and only if it is complete intersection (Yanagawa, 1999, Corollary 2.11).

We present two proofs of Theorem 4.12. The first is algebraic and uses Alexander duality, in particular, the following result. For notation, define $\mathbf{b}\cdot F \in \mathbb{N}^n$, for $F \subseteq \{1,\ldots,n\}$ and $\mathbf{b} \in \mathbb{N}^n$, to have sth coordinate b_s if $s \in F$ and 0 otherwise.

Theorem 4.15. (Miller, 1998, Theorem 4.13) *Let $M \subset S$ be any monomial ideal and let $F \subseteq \{1,\ldots,n\}$. If $\operatorname{supp}(\mathbf{b}) = F$ and $b_s \leq a_s$ for all s, then*

$$\beta_{i,\mathbf{b}^{\mathbf{a}}}(M^{\mathbf{a}}) \leq \sum_{\substack{\mathbf{c}\in\mathbb{N}^n \\ \mathbf{c}\cdot F = \mathbf{b}}} \beta_{\#F-i-1,\mathbf{c}}(M).$$

Proof of Theorem 4.12. Let $\operatorname{Irr}(S/M)$ denote the set of vectors $\mathbf{b} \in \mathbb{N}^n$ for which $\mathfrak{m}^{\mathbf{b}}$ is an irreducible component of M. For any $\mathbf{c} \in \mathbb{N}^n$, we define

$$\gamma_{\mathbf{c}} := \#\{\mathbf{b} \in \operatorname{Irr}(S/M) \mid \mathbf{b} = \mathbf{c}\cdot F \text{ for some } F \subseteq \{1,\ldots,n\}\}.$$

Set $d = \operatorname{codim}(M)$. The first aim is to show that

$$\#\operatorname{Irr}(S/M) \leq \sum_{\mathbf{c}\in\mathbb{N}^n} \gamma_{\mathbf{c}} \cdot \beta_{d-1,\mathbf{c}}(M). \tag{4.2}$$

In fact, this inequality holds even if M is not cogeneric: by the construction of $M^{\mathbf{a}}$,

$$\#\operatorname{Irr}(S/M) = \sum_{\mathbf{b}\in\operatorname{Irr}(S/M)} \beta_{0,\mathbf{b}^{\mathbf{a}}}(M^{\mathbf{a}}) = \sum_{\mathbf{0}\preceq\mathbf{b}\preceq\mathbf{a}} \beta_{0,\mathbf{b}^{\mathbf{a}}}(M^{\mathbf{a}}).$$

Since S/M is Cohen–Macaulay of codimension d, each $\mathbf{b} \in \operatorname{Irr}(S/M)$ has precisely d non-zero coordinates, and $\beta_{i,\mathbf{c}}(M) = 0$ for $i \geq d$. Thus Theorem 4.15 specializes to

$$\beta_{0,\mathbf{b}^{\mathbf{a}}}(M^{\mathbf{a}}) \leq \sum_{\mathbf{c}\cdot F=\mathbf{b}} \beta_{d-1,\mathbf{c}}(M)$$

for fixed $\mathbf{b} = (b_1,\ldots,b_n)$ and $F = \operatorname{supp}(\mathbf{b})$. Summing over all $\mathbf{b}$ proves (4.2).

The Cohen–Macaulay type of S/M is $\sum_{\mathbf{c}\in\mathbb{N}^n} \beta_{d-1,\mathbf{c}}(M)$, so it suffices to prove that if $\beta_{d-1,\mathbf{c}}(M) \neq 0$, then $\gamma_{\mathbf{c}} \leq 1$. Suppose the opposite, that is, $\gamma_{\mathbf{c}} \geq 2$ and $\beta_{d-1,\mathbf{c}}(M) \neq 0$. Then there are sets $F, F' \subseteq \{1,\ldots,n\}$ such that $\mathbf{c}\cdot F, \mathbf{c}\cdot F' \in \operatorname{Irr}(S/M)$ are distinct. Let $M_i = \mathfrak{m}^{\mathbf{c}\cdot F}$ and $M_j = \mathfrak{m}^{\mathbf{c}\cdot F'}$ be the irreducible components of M corresponding to $\mathbf{c}\cdot F$ and $\mathbf{c}\cdot F'$. Since the algebraic co-Scarf complex of M is the minimal free resolution of M and $\beta_{d-1,\mathbf{c}}(M) \neq 0$, there is an interior face σ of the co-Scarf complex $\Delta_M^{\mathbf{a}}$ with $\mathbf{a}_\sigma = \mathbf{D} - \mathbf{c}$. Since $m_i = \mathbf{x}^{(\mathbf{c}\cdot F)^{\mathbf{a}}}$ and $m_j = \mathbf{x}^{(\mathbf{c}\cdot F')^{\mathbf{a}}}$ divide m_σ by construction, σ contains both i and j. In particular, $\{i,j\}$ is an edge of $\Delta_M^{\mathbf{a}}$. Now S/M is Cohen–Macaulay of codimension ≥ 2, so $\operatorname{supp}(m_i) \cap \operatorname{supp}(m_j) \neq \emptyset$ by Theorem 4.9(c). But $\deg_{x_s} m_i = \deg_{x_s} m_j = D - c_s > 0$ for any $s \in \operatorname{supp}(m_i) \cap \operatorname{supp}(m_j)$, contradicting the genericity of $M^{\mathbf{a}}$.$\square$

After we had gotten the above proof, we conjectured the following more general result about arbitrary triangulations of a simplex. Margaret Bayer proved our conjecture for *quasigeometric triangulations*, using local h-vectors (Stanley, 1992). We are grateful for her permission to include her proof in this paper. Since the co-Scarf complex is a quasigeometric triangulation, Theorem 4.16 provides a second proof of Theorem 4.12.

THEOREM 4.16. (M. BAYER, 1999) *Let $p_1, p_2, \ldots, p_r$ be points which lie in the relative interior of $(c-1)$-faces of an $(n-1)$-simplex Δ. Let Γ be a quasigeometric triangulation of Δ having the p_i among its vertices and having no interior $(n-c-1)$-face. Then the number of interior $(n-c)$-faces is at least r.*

PROOF. According to the hypothesis, we have $\sum_{F\in\Delta,\#F=c} f_0(\mathrm{int}(\Gamma_F)) \geq r$, and also $f_i(\mathrm{int}(\Gamma)) = 0$ for all $-1 \leq i \leq n-c-1$. By the decomposition of the h-polynomial of Γ into local h-polynomials and the positivity of local h-vectors (Stanley, 1992, Theorem 4.6), we have

$$h_{c-1}(\Gamma) = \sum_{F\in\Delta} \ell_{c-1}(\Gamma_F) \geq \sum_{\substack{F\in\Delta\\ \#F=c}} \ell_{c-1}(\Gamma_F).$$

On the other hand, we have seen that $\ell_1(\Gamma_F) = f_0(\mathrm{int}(\Gamma_F))$ in the proof of Theorem 2.8. Since a local h-vector is symmetric (Stanley, 1992, Theorem 3.3), we have $\ell_{c-1}(\Gamma_F) = \ell_1(\Gamma_F) = f_0(\mathrm{int}(\Gamma_F))$. So

$$h_{c-1}(\Gamma) \geq \sum_{\substack{F\in\Delta\\ \#F=c}} \ell_{c-1}(\Gamma_F) = \sum_{\substack{F\in\Delta\\ \#F=c}} f_0(\mathrm{int}(\Gamma_F)) \geq r.$$

Since the h-vector of $\mathrm{int}(\Gamma)$ is the reverse of the h-vector of Γ (see the comment preceding Stanley, 1996, Theorem 10.5), we have

$$\begin{aligned} h_{c-1}(\Gamma) &= h_{n+1-c}(\mathrm{int}(\Gamma)) \\ &= \sum_{i=0}^{n-c+1} (-1)^{n+1-c-i}\binom{n-i}{c-1}(f_{i-1}(\mathrm{int}(\Gamma))) \\ &= f_{n-c}(\mathrm{int}(\Gamma)). \end{aligned}$$

Thus, the number of interior $(n-c)$-faces of Γ is at least r.□

Our final results demonstrate the effective translation between generic and cogeneric monomial ideals via Alexander duality.

THEOREM 4.17. *Let M be a cogeneric monomial ideal with r irreducible components, each having the same codimension c. Then M has at least $(c-1)\cdot r+1$ minimal generators. If M has exactly $(c-1)\cdot r+1$ generators, then S/M is Cohen–Macaulay.*

PROOF. The former statement is Alexander dual to Theorem 2.8. To prove the latter statement, we recall the proof of Theorem 2.8. Assume that S/M is not Cohen–Macaulay. Then $\Gamma := \Delta_M^{\mathbf{a}}$ has an edge $\{i,j\}$ whose excess e satisfies $e \geq c$, by Theorem 4.9. Let $W \in \Delta$ be the support of $m_{\{i,j\}}$. Then $\#W = e+2$. By Stanley (1992, Proposition 2.2),

$$\ell_W(\Gamma_W, x) = \ell_2(\Gamma_W)x^2 + \ell_3(\Gamma_W)x^3 + \cdots,$$

where $\ell_2(\Gamma_W)$ is the number of edges of Γ whose supports are W. So we have $f_{n-1}(\Gamma) = h(\Gamma, 1) \geq (c-1)\cdot r + 1 + \ell_2(\Gamma_W) > (c-1)\cdot r+1$ by an argument similar to the proof of Theorem 2.8. Since $f_{n-1}(\Gamma)$ is the number of generators of M, the proof is done.□

EXAMPLE 4.18. (a) The ideal $M = \bigcap_{i=1}^r \langle x_1^i, x_2^i, \ldots, x_{c-1}^i, x_{c-1+i}\rangle$ is cogeneric and has $(c-1)\cdot r+1$ minimal generators. Thus the inequality in Theorem 4.17 is tight.

(b) The converse of the latter statement of Theorem 4.17 is false. For instance, $M = \langle a^4, b, c\rangle \cap \langle a^2, b^4, d\rangle \cap \langle a, b^3, e\rangle \cap \langle a^3, b^2, e^2\rangle \subset k[a, \dots, e]$ is a Cohen–Macaulay cogeneric monomial ideal with four irreducible components, but M needs 12 generators. We also note that the Cohen–Macaulay type of S/M is 7, which is larger than the number of irreducible components.

However, in the codimension 2 case, we can prove the converse.

Proposition 4.19. *Let M be a cogeneric monomial ideal with r irreducible components, all of codimension 2. Then S/M is Cohen–Macaulay if and only if M has exactly $r + 1$ generators.*

Proof. This is Alexander dual to Proposition 2.10, in view of Theorem 4.9.□

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