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Local Hadamard well-posedness for nonlinear wave equations with supercritical sources and damping ☆

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ABSTRACT

We consider the wave equation with supercritical interior and boundary sources and damping terms. The main result of the paper is local Hadamard well-posedness of finite energy (weak) solutions. The results obtained: (1) extend the existence results previously obtained in the literature (by allowing more singular sources); (2) show that the corresponding solutions satisfy Hadamard well-posedness conditions during the time of existence. This result provides a positive answer to an open question in the area and it allows for the construction of a strongly continuous semigroup representing the dynamics governed by the wave equation with supercritical sources and damping.

1. Introduction

1.1. The model and the description of the problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary Γ. We consider the following model of a semilinear wave equation with double interaction of nonlinear source and monotone damping, both in the interior of the domain and on its boundary.

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$$\begin{cases} u_{tt} + g_0(u_t) = \Delta u + f(u) & \text{in } \Omega \times [0,\infty) \\ \partial_\nu u + u + g(u_t) = h(u) & \text{in } \Gamma \times [0,\infty) \\ u(0) = u_0 \in H^1(\Omega) \quad \text{and} \quad u_t(0) = u_1 \in L_2(\Omega) \end{cases} \tag{1}$$

The functions $g_0(s)$ and $g(s)$ model the interior and boundary dissipations in the equation. The functions $f(s)$ and $h(s)$ represent the modeling of the sources. Sources, by definition, are forcing terms that are typically amplifying (rather than restoring) the energy. In mathematical terms, this means that there is no control on the upper bound of $sf(s)$ (respectively $sh(s)$).

The aim of this paper is to present results on the Hadamard well-posedness of weak solutions generated by the PDE system (1) which is considered on the finite energy space $H = H^1(\Omega) \times L_2(\Omega)$ (the most relevant space from the physical point of view).

Problems related to the well-posedness of solutions to semilinear wave equations are very classical and have attracted a great deal of attention in the literature. In order to focus our presentation, we wish to stress that the analysis in this paper is pertinent to the dynamics defined on a *bounded and sufficiently smooth domain*. This is in contrast with the analysis on $\mathbb{R}^n$, where the literature is abundant, with many results available. However, the nature of propagation of singularities and related regularity is very different for bounded domains. The analysis must take into consideration the role of the boundary and the type of boundary conditions imposed on it. It is thus expected that both the results and the methods should depend on the behavior of solutions near the boundary.

The model under consideration is equipped with the Neumann *nonlinear* boundary conditions. It is known that the Lopatinski condition [23] fails for the Neumann problems, causing the loss of $1/3$ derivative in linear dynamics driven by boundary sources (unless the dimension of Ω is equal to one) [19,20,18,26,27]. It is thus expected that the boundary and boundary conditions will play a prominent role in the analysis. Our study is centered on handling internal sources $f(u)$ and boundary sources $h(u)$ with exponents *exceeding the critical Sobolev's exponents*. While internal sources, up to the critical level, do not pose problems with the treatment of local well-posedness, boundary sources, even mildly nonlinear, are problematic and require much more subtle analysis [16,17]. This is due to the "loss of derivatives" in the linear dynamics. In fact, the map $h \to u(t)$, where

$$u_{tt} = \Delta u, \qquad \partial_\nu u = h, \quad \text{on } \Sigma = \Gamma \times (0,T), \qquad u(0) = u_t(0) = 0, \quad \text{in } \Omega$$

is *not* bounded $L_2(\Sigma) \to H^1(\Omega)$, unless the dimension of $\Omega = 1$. (In the case of general bounded n-dimensional domains, one obtains $H^{2/3}(\Omega)$ regularity *only*. This phenomenon is referred to as "loss of $1/3$ derivative".)

It was already noticed, in the context of boundary control, in [21] and later in [17], that the presence of *boundary damping* plays a critical role in the analysis. Indeed, boundary damping does restore some of the loss of the regularity incurred due to the failure of the Lopatinski condition.

The interaction between boundary damping and source has been further exploited in [28], where local existence of solutions was established for boundary sources of a polynomial structure (with the exclusion of super-supercritical exponents – see Section 1.3 below) interacting with sufficiently high nonlinear damping. The method used in [28] is based on Schauder fixed point, thus depending on compactness. The results obtained in [28] provide existence of finite energy solutions but *without uniqueness* or continuous dependence on the initial data.

The main aim of this paper is to show that the flow associated with (1) is locally Hadamard well-posed. This is to say that solutions exist locally, are unique and they *depend continuously* with respect to the initial data in finite energy topology. This result, in conjunction with global a priori bounds already available in the literature [28,29,4], will allow to construct a dynamical system that is generated by strongly continuous semigroup. As it is well known, the existence of a strongly continuous semigroup is central to the dynamical system theory.

It should be noted that the interaction with the damping is critical not only for global existence (this has been already established in prior literature [12,3,24]), but it is even more critical for local existence, as amply demonstrated in the present paper. To our best knowledge, this is the first result on local Hadamard well-posedness of wave flows generated by supercritical boundary–interior sources and damping terms. In conclusion, the novel contribution of the present work consists of:

(i) Full Hadamard well-posedness of local (in time) flows corresponding to supercritical boundary sources is established. This property, when combined with global existence, provides a definite result in the field asserting existence of a semigroup.
(ii) We are also able to treat *interior* sources which are supercritical. In the supercritical case, the interaction with internal damping plays, again, a major role. This interaction has been already observed in [24,12] in the context of non-existence of global solutions in $\mathbb{R}^n$. Our analysis provides a positive result of existence of local solutions, which is a foundation for applications of the corresponding global existence or non-existence results.
(iii) The methods used in the present paper are very different than the ones used before in the literature [28]. We rely on monotonicity methods combined with suitable truncations-approximations, rather than the compactness used in [28]. This alone allows to extend the range of Sobolev's exponents for which the analysis is applicable (no need for compactness). It is also believed that the method could be used successfully in order to treat unbounded domains [6].

In what follows, precise formulation of the results obtained is presented.

1.2. Assumptions

In this paper we will focus on the most representative case when $n = 3$ (the analysis can be easily adapted to other values of n; $n = 2$ being the least interesting, since the concept of criticality of Sobolev's embedding is much less pronounced).

Assumption 1.1. With reference to system (1), assume

$(A_{g_0,g})$: g, g_0 are monotone increasing and continuous functions such that $g(0) = g_0(0) = 0$. In addition, the following growth conditions at infinity hold: $\exists m_q, M_q, L_m > 0$, $l_m \geqslant 0$ such that for $|s| > 1$:

$$l_m|s|^{m+1} \leqslant g_0(s)s \leqslant L_m|s|^{m+1}, \quad \text{with } m > 0$$

$$m_q|s|^{q+1} \leqslant g(s)s \leqslant M_q|s|^{q+1}, \quad \text{with } q > 0$$

(A_f):
- $f \in C^1(\mathbb{R})$ and the following growth conditions are imposed on $f(s)$ for $|s| > 1$: $|f'(s)| \leqslant C|s|^{p-1}$ where $1 \leqslant p \leqslant 3$.
- If $p > 3$, we additionally assume that $l_m > 0$ and $f \in C^2(\mathbb{R})$, $|f''(s)| \leqslant C_f[1 + |s|^{p-2}]$ with $3 < p \leqslant \frac{6m}{m+1}$.

(A_h): In the case of sublinear damping ($0 < q < 1$), we assume that $h \in C^1(\mathbb{R})$, and $|h'(s)| \leqslant C_h[1 + |s|^{k-1}]$, where $1 \leqslant k \leqslant \frac{4q}{q+1}$.

For superlinear damping ($q \geqslant 1$) we assume that $h \in C^2(\mathbb{R})$ and $|h''(s)| \leqslant C_h[1 + |s|^{k-2}]$, where $2 \leqslant k \leqslant \frac{4q}{q+1}$.

1.3. Classification and properties of sources

1.3.1. Interior source f

In line with the Sobolev's embedding $H^1(\Omega) \to L_6(\Omega)$, we can classify the interior source f as follows:

(1) **Subcritical**: $1 \leqslant p < 3$ and **Critical**: $p = 3$. In these cases, f is locally Lipschitz from $H^1(\Omega)$ into $L_2(\Omega)$.
(2) **Supercritical**: $3 < p < 5$. f is no longer locally Lipschitz. However, the potential well energy associated with f: $\int_\Omega \hat{f}(u)\,dx$, where $\hat{f}$ is the antiderivative of f, is still well defined on the finite energy space.

(3) **Super-supercritical**: $5 \leqslant p < 6$. The potential energy may not be defined on the finite energy space and thus the sources are no longer within the framework of potential well theory.

1.3.2. Boundary source h

Similarly, we can classify the boundary sources with respect to the "criticality" of the Sobolev's embedding $H^{1/2}(\Gamma) \to L_4(\Gamma)$:

(1) **Subcritical**: $1 \leqslant k < 2$ and **Critical**: $k = 2$.
(2) **Supercritical**: $2 < k < 3$.
(3) **Super-supercritical**: $3 \leqslant k < 4$.

Remark 1. Assumption 1.1 allows for both types of supercriticality in the two sources f and h.

Remark 2. Assumption 1.1 guarantees that f is locally Lipschitz $H^1(\Omega) \to L_{\frac{m+1}{m}}(\Omega)$ and h is locally Lipschitz $H^1(\Omega) \to L_{\frac{q+1}{q}}(\Gamma)$.

Remark 3. For $\varepsilon < \min[\frac{1}{2m}, \frac{1}{2q}]$, f is locally Lipschitz $H^{1-\varepsilon}(\Omega) \to L_1(\Omega)$ and h is locally Lipschitz $H^{1-\varepsilon}(\Omega) \to L_1(\Gamma)$.

The results stated in Remarks 2 and 3 above follow from application of Holder inequalities, along with suitable Sobolev's embeddings. The arguments are standard, hence omitted.

Remark 4. The classification of the boundary sources with respect to the "criticality" of the Sobolev's embedding $H^{1/2}(\Gamma) \to L_4(\Gamma)$ has different implications than in the internal case. While critical and subcritical internal sources pose no difficulties at the level of proving local well-posedness (with or without the damping), boundary sources, even subcritical $k \leqslant 2$ with locally Lipschitz $H^1(\Omega) \to L_2(\Gamma)$ continuity, do require boundary damping for local existence. This is due to the loss of 1/3 derivatives for the Neumann – wave map, that does not translate into Lipschitz behavior of the corresponding wave map. So, even in the subcritical case, the analysis is more subtle, requiring special treatment that involves an interaction with the damping (unlike the interior case).

Remark 5. In the case of critical and subcritical interior source f, i.e. $p \leqslant 3$, the presence of the interior damping g_0 is not necessary in order to obtain local existence and uniqueness of solutions. Therefore, in this case, the term $g_0(u_t)$ can be removed from the equation without modifying the results.

Remark 6. The relations describing the interaction between the source and the damping (i.e. between m (resp. q), and p (resp. k)) were introduced in [24] for the interior case and in [28] for the boundary case. However, the range of these parameters considered in this paper is larger (includes super-supercritical values).

1.4. Weak solution

We introduce next the definition of a *weak* solution:

Definition 1.1 *(Weak solution).* By a *weak solution* of (1), defined on some interval $(0, T)$, we mean a function $u \in C_w(0, T; H^1(\Omega))$, $u_t \in C_w(0, T; L_2(\Omega))$ such that

(1) $u_t \in L_{m+1}(0, T; \Omega)$, $u_t|_\Gamma \in L_{q+1}(0, T; \Gamma)$.
(2) For all $\phi \in C(0, T, H^1(\Omega)) \cap C^1(0, T; L_2(\Omega)) \cap L_{m+1}(0, T; \Omega)$, $\phi|_\Gamma \in L_{q+1}(0, T; \Gamma)$

$$\int_0^T\int_\Omega(-u_t\phi_t+\nabla u\nabla\phi)\,d\Omega\,dt+\int_0^T\int_\Gamma \phi u\,d\Gamma\,dt+\int_0^T\int_\Omega g_0(u_t)\phi\,d\Omega\,dt$$
$$=-\int_\Omega u_t\phi\,d\Omega\Big|_0^T+\int_0^T\int_\Gamma h(u)\phi\,d\Gamma\,dt-\int_0^T\int_\Gamma g(u_t)\phi\,d\Gamma\,dt+\int_0^T\int_\Omega f(u)\phi\,d\Omega\,dt \tag{2}$$

(3) $\lim_{t\to 0}(u(t)-u_0,\phi)_{H^1(\Omega)}=0$ and $\lim_{t\to 0}(u_t(t)-u_1,\phi)_{L_2(\Omega)}=0$ for all ϕ as above.

Here $C_w(0,T,Y)$ denotes the space of weakly continuous functions with values in a Banach space Y.

1.5. Notation

In what follows we adopt the following notation:

$$U(t)\equiv\big(u(t),u_t(t)\big),\qquad H\equiv H^1(\Omega)\times L_2(\Omega)$$
$$\|u\|_p\equiv\|u\|_{L_p(\Omega)},\qquad |u|_p\equiv\|u\|_{L_p(\Gamma)},\qquad \|u\|_{s,\Omega}\equiv|u|_{H^s(\Omega)},\qquad |u|_{s,\Gamma}\equiv|u|_{H^s(\Gamma)}$$
$$(u,v)_\Omega\equiv(u,v)_{L_2(\Omega)},\qquad (u,v)_\Gamma\equiv(u,v)_{L_2(\Gamma)},\qquad (u,v)_{1,\Omega}\equiv(u,v)_{H^1(\Omega)}$$
$$Q_T\equiv\Omega\times(0,T),\qquad \Sigma_T\equiv\Gamma\times(0,T)$$

1.6. Main results

Our first theorem is on local existence and uniqueness of (finite energy) *weak* solutions.

Theorem 1.2 *(Local existence). Consider Eq.* (1) *under Assumption* 1.1 *above, $q\geqslant 1$, and with $U(0)\in H$, $u_0\in L_r(\Omega)\cap\tilde{L}_s(\Gamma)$, where $r=\frac{3}{2}(p-1)$ and $s=2(k-1)$ and $\tilde{L}_s(\Gamma)\equiv\{u\in H^1(\Omega), u|_\Gamma\in L_s(\Gamma)\}$. Then there exists* ***unique local in time*** *weak solution $U\in C[(0,T_M),H]$, where the maximal time of existence T_M depends on initial data $|U(0)|_H$, and the constants C_f, C_h, m_q, l_m. In addition, for $t<T_M$,*

$$\big|U(t)\big|_H+|u_t|_{L_{m+1}(0,t;\Omega)}+|u_t|_\Gamma|_{L_{q+1}(0,t;\Gamma)}\leqslant C\big(\big|U(0)\big|_H,C_f,C_h,m_q,l_m\big)$$

Moreover, the following ***energy identity*** *holds for all weak solutions and all $0\leqslant s\leqslant t\leqslant T_M$:*

$$E_u(t)+\int_0^t\int_\Omega g_0(u_t)u_t\,dQ+\int_0^t\int_\Gamma g_1(u_t)u_t\,dQ \tag{3}$$
$$=E_u(s)+\int_s^t\int_\Omega f(u)u_t\,d\Omega\,dz+\int_s^t\int_\Gamma h(u)u_t\,d\Gamma\,dz \tag{4}$$

where

$$E_u(t):=\frac{1}{2}\big\|u_t(t)\big\|_2^2+\frac{1}{2}\big\|\nabla u(t)\big\|_2^2+\frac{1}{2}|u|_2^2 \tag{5}$$

denotes the energy of the system.

Remark 7. Since $\frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2}|u|_2^2$ is an equivalent norm for $H^1(\Omega)$, from now on, we will often use $E_u(t) := \frac{1}{2}\|u_t(t)\|_2^2 + \|u(t)\|_{H^1(\Omega)}^2$.

Remark 8. The additional regularity of the initial data $u_0 \in L_r(\Omega) \cap \tilde{L}_s(\Gamma)$ is redundant for supercritical sources $p \leqslant 5$ and $k \leqslant 3$. The $H^1(\Omega)$ regularity of the initial condition automatically implies the desired integrability, due to Sobolev's embedding.

Remark 9. If $5 < p \leqslant \frac{6m}{m+1}$ and $3 < k \leqslant \frac{4q}{q+1}$, local existence in $H^1(\Omega)$ of weak solutions, along with L_r and L_s regularity of initial data and $L_{m+1}(\Omega)$ (respectively $L_{q+1}(\Gamma)$) regularity of the velocities u_t (respectively $u_t|_\Gamma$), implies the L_r, L_s regularity of the solution. Indeed, the above follows from the following argument: due to the fact that $r < m+1$ and $s < q+1$, we have the following inequalities:

$$\left|u(t)\right|_{L_r(\Omega)} \leqslant \left|u(0)\right|_{L_r(\Omega)} + C\int_0^T \left|u_t(t)\right|_{L_{m+1}(\Omega)}\,dt \quad \text{and}$$

$$\left|u(t)\right|_{L_s(\Gamma)} \leqslant \left|u(0)\right|_{L_s(\Gamma)} + C\int_0^T \left|u_t(t)\right|_{L_{q+1}(\Gamma)}\,dt$$

Remark 10. Note that Theorem 1.2 considers superlinear boundary damping $g(s)$, i.e. $q \geqslant 1$. For the sublinear case $q < 1$, the same result is known under the additional differentiability assumption

$$m_q|s|^{q-1} \leqslant g'(s) \leqslant M_q|s|^{q-1}, \quad \text{for } s \neq 0 \tag{6}$$

Indeed, local existence of weak solutions follows from [28], while uniqueness follows from Theorem 2.2 [5]. Thus, the focus of Theorem 1.2 is on the essential superlinear boundary damping, when $q \geqslant 1$.

Remark 11 *(Global existence and non-existence).* Solutions asserted by Theorem 1.2 are local and they may cease to exist in a finite time. There are various conditions asserting global existence by supplying suitable a priori bounds. However, in the case when the damping is present, damping is responsible not only for local existence, but also may contribute to extend the life-span of solutions. This was observed and proved already in [12]. In fact, a similar argument as in [12] has been extended in [28, 4] in order to prove that solutions exist globally provided that the interaction with the damping is strong enough. This is to say, if $m \geqslant p$, $q \geqslant k$, solutions are global on $[0, T]$ where T is any positive number. In the absence of the above requirements, blow up of the energy originating in a potential well has been demonstrated in [7,8,30]. Since the focus of the present work is on local (in time) well-posedness of solutions, we refer the reader to the literature for information regarding global existence and non-existence.

In general, uniqueness of solutions does not automatically imply continuous dependance with respect to finite energy initial data. However, for supercritical sources ($p < 5$ and $k < 3$), we are able to prove that weak solutions are continuous with respect to the topology induced by the finite energy initial conditions. It should be noted that the energy *identity* plays a critical role in this part of the argument (as in [15]). Our ultimate result is the following:

Theorem 1.3 *(Hadamard (local) well-posedness). We consider system* (1) *with the same assumptions as Theorem* 1.2 *and we take $p < 5$, $k < 3$. In the case $q < 1$, we additionally assume* (6)*. Then, the weak solutions to* (1) *depend continuously on the initial data in finite energy norm, i.e. for all $T < T_{max}$ and all sequences of initial data such that*

$$U_n(0) \to U_0 \quad in\ H$$

the corresponding weak solutions $U_n(t), U(t) \in H$ *satisfy* $U_n \to U$ *in* $C(0, T; H)$.

Remark 12. For super-supercritical sources (when $p \geqslant 5$, or $k \geqslant 3$), we obtain unique weak solution for initial data taken from $H_{r,s} \equiv H^1(\Omega) \cap L_r(\Omega) \cap \tilde{L}_s(\Gamma) \times L_2(\Omega)$, where $r = 3/2(p-1)$, $s = 2(k-1)$. For this range of parameters, we can still prove Hadamard well-posedness, but in a space that is strictly contained in $H_{r,s}$. More specifically, for $p \geqslant 5$, $k \geqslant 3$ the solutions are continuous with respect to initial data taken from the space

$$H_{r,s,\delta} \equiv H^1(\Omega) \cap L_{r+\delta}(\Omega) \cap \tilde{L}_{s+\delta}(\Gamma) \times L_2(\Omega) \subset H_{r,s} \tag{7}$$

where δ can be taken sufficiently small constant.

In comparing our results with those obtained earlier in the literature, we note that local existence (without uniqueness) of solutions to (1) that are driven by boundary data only (with $f = 0$) has been shown in [28] (see also [8]) for $1 \leqslant k < 3$ and $k < \frac{4q}{q+1}$. Thus, the existence results stated in Theorem 1.2 extend the range of the parameter k, and it allows for simultaneous treatment of interior supercritical sources. Most importantly, Theorem 1.2 along with Theorem 1.3 provide *uniqueness and continuous dependence* on initial data of finite energy solutions with supercritical sources.

We conclude this introduction with some open questions:

- An interesting question is whether global weak solutions remain bounded for all time. Such result has been established in [13] for a strongly, linearly damped wave equation (analytic semigroup) with superlinear sources and zero Dirichlet data. [11] shows that the same phenomenon holds in the case of linear damping ($m = 1$) and Dirichlet boundary data. However, when the damping is superlinear and dominating the source $m \geqslant p$, global solutions may be *unbounded*. In fact, such solutions are characterized with respect to the depth of potential well [11].
  It would be interesting to have an answer to the same question in the case of supercritical model considered in (1). However, the nature of the problem and the corresponding technicalities of the analysis are very different. At the time of writing this paper the answer is not known and the problem stated remains an open problem. The importance of such result cannot be overstated in the context of dynamical systems and relevant asymptotic behavior where such questions are critical. In fact, [8] provides conditions under which *bounded* solutions decay to zero at the specified (optimal) rate.
- Another interesting question is to consider the possibility of degenerate damping. One would like to know how much of the damping is needed to guarantee local and global existence of solutions. Problems of this sort for Dirichlet boundary conditions were considered in [3]. However, the situation with Neumann boundary sources is very different and presents a different set of technicalities due to the failure of Lopatinski condition.

The reminder of this paper is devoted to the proofs of Theorem 1.2 and Theorem 1.3. For reader's orientation we describe **the main steps of the proof** which aims at constructing the weak solution to (1) under Assumption 1.1. It may be helpful to note at the outset that the solution is built from a suitable perturbation of a monotone problem that leads to "generalized" solutions, which then are shown to be weak. These weak-variational solutions are then approximated (by approximating the sources and the damping) and shown to produce (in the limit) weak solutions to the original problem, where the latter is neither monotone, nor locally Lipschitz. The steps of this construction are described below.

1. Construct appropriate truncations of nonlinear terms which lead to a globally Lipschitz perturbation of a monotone problem. Use nonlinear semigroup theory, applied to the specified truncation of the problem, in order to establish global existence and uniqueness of the approximation. The solution obtained is "generalized" – in the sense of monotone operator theory – and not necessarily "weak" – in the sense of Definition 2. However, a suitable relaxed version of energy *inequality* is satisfied for the said solutions (see Lemma 2.1). The uniqueness of generalized solutions depends on the strong coercivity of the boundary damping (to be relinquished later on). This step relies on the construction introduced in [17] and expanded later in [9].

2. Extend the existence result obtained in step 1 in order to obtain local existence applicable to (i) sources that are locally Lipschitz $H^1 \to L_2$ and (ii) boundary damping that is still *coercive*. Derive a priori bounds that do not depend on the two properties listed above, but rather on (i) local Lipschitz property $H^1 \to L_1$ and (ii) the growth conditions imposed of the damping (constants m_q, l_m). In particular, the survival time T_m is shown to be independent of the properties required in step 1. The growth conditions imposed on the damping allow to show that the obtained solution (generalized) is in fact *weak*, i.e. it satisfies the variational equality. In addition, the energy inequality obtained for these solutions is expressed in terms of the growth imposed on the damping. This allows for the construction of suitable a priori bounds – see Lemma 2.2.

3. Construct approximations of the original sources that comply with the requirements in step 2. This step is accomplished in Section 3 by using the constructions from [17] (for the damping) and from [22] (for the sources). Passage with the limit on the *weak* variational form with solutions established by Lemma 2.2, but corresponding to approximations of the damping and sources, is the major technical step carried out in Section 3.2. The supercriticality of both sources is a major difficulty, contributing to the lack of compactness. For this step, special estimates involving the so called dissipativity kernels (introduced originally in [10] and later used in [5]) play a critical role. This method, while requiring C^2 regularity of the sources, allows to pass with limits on nonlinear sources without any compactness. The superlinearity of the damping and the associated monotonicity method is critical for this step. The helping ingredient here is the Energy Identity obtained in [5] for general solutions to wave equation with prescribed L_p, L_q regularity of the velocities.

4. Once existence of weak solutions is obtained, the energy identity and uniqueness follow from an appropriate adaptation of the arguments in [5], along with the estimates performed in the context of the existence proof in Section 3.2.

5. Hadamard well-posedness, presented in Section 3, relies on (i) the energy identity of Lemma 3.1 (proved as in [15] using finite time differences [5]), (ii) appropriate L_p, L_q regularity established for weak solutions, and, above all, (iii) the estimates carried out in Section 3.2, including the ones reported in Lemma 3.2 and Lemma 3.3.

2. Preparatory lemmas-generalized "finite energy" solutions

2.1. Strongly monotone boundary dissipation

2.1.1. Globally Lipschitz sources

First we deal with the case when the boundary dissipation is assumed strongly monotone and the sources are globally Lipschitz $f : H^1(\Omega) \to L_2(\Omega)$ and $h : H^1(\Omega) \to L_2(\Gamma)$. In this case, we have the following lemma:

Lemma 2.1. *With reference to model* (1) *we assume that*

(1) *$g(s)$ and $g_0(s)$ are continuous and monotone increasing functions such that $g_0(0) = g(0) = 0$. In addition, the following coercivity condition is imposed on g, i.e.: there exists $m_g > 0$ such that $(g(s) - g(v))(s - v) \geqslant m_g|s - v|^2$.*
(2) *f is globally Lipschitz: $H^1(\Omega) \to L_2(\Omega)$.*
(3) *$\hat{h}(u) \equiv h(u|_\Gamma)$ is globally Lipschitz: $H^1(\Omega) \to L_2(\Gamma)$.*

Then the system (1) *generates a ω-contraction nonlinear semigroup $S(t)$ on H. Hence, there exists a unique global in time solution $U \in C([0, T]; H = H^1(\Omega) \times L_2(\Omega))$ such that $U(t) = S(t)U(0)$, $0 < t \leqslant T$, where T is* *arbitrary. The solution obtained is "generalized" (i.e. strong limit of strong solutions) and enjoys the following additional properties*:

- $u_t|_\Gamma \in L_2(\Sigma_T)$,
- *the following relaxed energy inequality holds*

$$E_u(t) + \int_s^t \int_\Omega g_0\big(u_t(s)\big)u_t(s)\,dx\,d\tau + \int_s^t \int_\Gamma \varphi_g\big(u_t|_\Gamma(s)\big)\,dx\,d\tau$$

$$\leqslant E_u(s) + \left| \int_s^t \int_\Omega f(u)u_t\,dx\,d\tau + \int_s^t \int_\Gamma h(u|_\Gamma)u_t|_\Gamma\,dx\,d\tau \right| \tag{8}$$

where $\varphi_g(s)$ *denotes a proper convex lower semicontinuous function generated by its subgradient* $g(s)$, *i.e.* $\varphi_g(s) = \int_0^s g(\tau)\,d\tau$.

Remark 13. "Finite energy" solutions referred to in Lemma 2.1 correspond to generalized solutions associated with ω-accretive operators. In line with monotone operator theory, these are defined as H limits of strong solutions evolving in the domain of the generator. In the absence of additional hypotheses (growth conditions) imposed on the damping, such solutions may not be weak – in the sense of variational Definition 2. A sufficient additional condition imposed on the damping that guarantees that generalized solutions (in the context of Lemma 2.1) are weak and that energy *identity* holds is that there exist pivot spaces $V \subset L_2(Q_T) \subset V'$, $U \subset L_2(Q_T) \subset U'$ such that

$$\begin{aligned} &u_t \in V \subset L_2(Q_T), \qquad g_0(u_t) \in V' \\ &u_t|_\Gamma \in U \in L_2(\Sigma_T), \qquad g(u_t|_\Gamma) \in U' \end{aligned} \tag{9}$$

Indeed, the above observation follows from standard arguments in monotone operator theory [1].

The proof of Lemma 2.1 is based on an extended monotonicity method developed in [17], and used later in [9]. Several adjustments of the arguments are required due to the presence of two competing unbounded dissipative mechanisms in the equation.

The boundary value problem is formulated as a Lipschitz perturbation of a maximal monotone problem. This is accomplished by a suitable use of semigroup theory, allowing the representation of boundary conditions via a singular variation of parameter formula [21,17,16]. Maximal monotone operator theory is then extended in order to incorporate Lipschitz perturbations.

Proof. The proof follows the method of [17]. Our first goal is an operator theoretic formulation of the problem (1) involving a maximal monotone operator. For that sake we introduce the Laplacian with Robin boundary condition, namely $\Delta_R : D(\Delta_R) \subset L_2(\Omega) \to L_2(\Omega)$,

$$\Delta_R u = -\Delta u, \quad \text{with domain } D(\Delta_R) = \big\{u \in H^2(\Omega) \,\big|\, \partial_\nu u + u = 0 \text{ on } \Gamma\big\}$$

The operator Δ_R is symmetric and maximal monotone on $L_2(\Omega)$, and thus positive self-adjoint. Hence we can define fractional powers of Δ_R and we can identify $D(\Delta_R^{1/2}) \equiv H^1(\Omega)$ [14]. Moreover, Δ_R is coercive on $H^1(\Omega) \equiv D(\Delta_R^{1/2})$.

Let R (Robin map) be the harmonic extension of the boundary data defined as

$$f = Rg \quad \Leftrightarrow \quad \begin{cases} \Delta f = 0 & \text{in } \Omega \\ \partial_\nu f + f = g & \text{on } \Gamma \end{cases}$$

where R is uniquely defined: R continuous: $H^s(\Gamma) \to H^{s+3/2}(\Omega)$, $\forall s \in \mathbb{R}$.

In particular, for $s=0$, we have $R: L_2(\Gamma) \to H^{3/2}(\Omega) \subset H^{\frac{3}{2}-2\varepsilon}(\Omega) \equiv D(\Delta_R^{\frac{3}{4}-\varepsilon}) \Leftrightarrow \Delta_R^{\frac{3}{4}-\varepsilon} R: L_2(\Gamma) \to L_2(\Omega)$ continuous.

The adjoint of the Robin map satisfies:

$$R^*\Delta_R^* h = h|_\Gamma, \quad \text{for all } h \in D\big(\Delta_R^{1/2}\big) \equiv H^1(\Omega) \tag{10}$$

This can be seen as follows. Given $h \in D(\Delta_R)$ and $g \in L_2(\Gamma)$ we have

$$\begin{aligned}
\big(R^*\Delta_R^* h, g\big)_{L_2(\Gamma)} &= \big(\Delta_R^* h, Rg\big)_{L_2(\Omega)} = (-\Delta h, Rg)_{L_2(\Omega)} \\
&= (-h, \Delta Rg)_{L_2(\Omega)} - \int_\Gamma \frac{\partial}{\partial \nu} h Rg\, d\Gamma + \int_\Gamma h \frac{\partial}{\partial \nu} Rg\, d\Gamma \\
&= \int_\Gamma hRg\, d\Gamma + \int_\Gamma h(g - Rg)\, d\Gamma = \int_\Gamma hg\, d\Gamma
\end{aligned} \tag{11}$$

Then the validity of the trace result (11) can be extended to all $h \in H^1(\Omega) \equiv D(\Delta_R^{1/2})$, as $D(\Delta_R)$ is dense in $D(\Delta_R^{1/2})$.

Using the operators introduced above, we can write system (1) as an abstract equation:

$$u_{tt} + g_0(u_t) = -\Delta_R\big(u + Rg\big(R^*\Delta_R u_t\big) - Rh\big(R^*\Delta_R u\big)\big) + f(u) \tag{12}$$

Note that $\Delta_R : D(\Delta_R) \subset L_2(\Omega) \to L_2(\Omega)$ can be extended by transposition as $\Delta_R^{ext} : L_2(\Omega) \to [D(\Delta_R^*)]' = [D(\Delta_R)]'$, where $[D(\Delta_R)]'$ is the dual of $D(\Delta_R)$ with respect to $L_2(\Omega)$. Convention: from now on, by Δ_R we mean Δ_R^{ext}. Thus (12) becomes

$$u_{tt} = -\Delta_R u + \Delta_R R\big(h(u) - g(u_t)\big) + f(u) - g_0(u_t) \tag{13}$$

The abstract second order equation (13) will be considered (see [9]) as an evolution on $H = D(\Delta_R^{1/2}) \times L_2(\Omega)$, where the norm generated by $\Delta_R^{1/2}$ is equivalent to the $H^1(\Omega)$ norm.

For this we introduce the nonlinear operator $A : D(A) \subset H \to H$ by

$$A = \begin{pmatrix} 0 & -I \\ \Delta_R - \Delta_R Rh - f & \Delta_R Rg + g_0 \end{pmatrix}$$

where

$$D(A) = \big\{(u, v) \in H^1(\Omega) \times H^1(\Omega) \;\big|\; u + Rg(v|_\Gamma) - Rh(u|_\Gamma) + \Delta_R^{-1} g_0(v) \in D(\Delta_R)\big\}$$

The definition of $D(A)$, along with the assumptions imposed on f, g, g_0, h imply that for all $(u, v) \in D(A)$ the following holds

$$\begin{aligned}
&\big|\big(g_0(v), \psi\big)_\Omega + \big(g(v|_\Gamma), \psi|_\Gamma\big)_\Gamma\big| \leqslant C_v |\psi|_{1,\Omega}, \quad \forall \psi \in D\big(\Delta_R^{1/2}\big) \\
&g_0(v) \in H^{-1}(\Omega) \\
&g_0(v)v \in L_1(\Omega), \qquad g(v|_\Gamma)v|_\Gamma \in L_1(\Gamma)
\end{aligned} \tag{14}$$

Then system (12) with $(u, v = u_t) \in D(A)$ is equivalent to

$$\partial_t \begin{pmatrix} u \\ u_t \end{pmatrix} + A \begin{pmatrix} u \\ u_t \end{pmatrix} = 0$$

In order to prove our Lemma 2.1 it suffices to show that the operator A is w m-accretive, for some positive w (Theorem IV.4.1 in [25]). We endow H with the standard inner product $D(\Delta_R^{1/2}) \times L_2(\Omega)$.

Step 1: Proof of A is w-accretive. Let $U = (u, v) \in D(A)$, $\hat{U} = (\hat{u}, \hat{v}) \in D(A)$. We want to find $w \geqslant 0$ such that $((A + wI)(U - \hat{U}), U - \hat{U})_H \geqslant 0$.

Let $(\cdot,\cdot)_\Omega = (\cdot,\cdot)_{L_2(\Omega)}$, $(\cdot,\cdot)_\Gamma = (\cdot,\cdot)_{L_2(\Gamma)}$, and $(u, \hat{u})_{1,\Omega} = (\Delta_R^{1/2} u, \Delta_R^{1/2} \hat{u})_{L_2(\Omega)}$. Also, to shorten the notation, let $\tilde{\Delta} = \Delta_R^{1/2}$ and $h(u) \equiv h(u|_\Gamma)$, $g(v) \equiv g(v|_\Gamma)$.

Using the regularity properties of elements in $D(A)$ (see (14)) and assumptions on the damping terms and the sources, combined with the properties of the operators introduced above, we have

$$\begin{aligned}
&\big((A + wI)(U - \hat{U}), U - \hat{U}\big)_H \\
&\quad = \big(A(U) - A(\hat{U}), U - \hat{U}\big)_H + w|U - \hat{U}|_H^2 \\
&\quad = \big(\hat{v} - v, \Delta_R(u - \hat{u})\big)_\Omega + \big(\Delta_R(u - \hat{u}), v - \hat{v}\big)_\Omega - \big(h(u) - h(\hat{u}), (v - \hat{v})\big)_\Gamma \\
&\qquad - \big(f(u) - f(\hat{u}), v - \hat{v}\big)_\Omega + \big(g(v) - g(\hat{v}), (v - \hat{v})\big)_\Gamma + \big(g_0(v) - g_0(\hat{v}), v - \hat{v}\big)_\Omega \\
&\qquad + w|u - \hat{u}|_{1,\Omega}^2 + w|v - \hat{v}|_\Omega^2 \\
&\quad \geqslant -\big|h(u) - h(\hat{u})\big|_\Gamma |v - \hat{v}|_\Gamma - \big|f(u) - f(\hat{u})\big|_\Omega |v - \hat{v}|_\Omega \\
&\qquad + \big(g(v) - g(\hat{v}), (v - \hat{v})\big)_\Gamma + \big(g_0(v) - g_0(\hat{v}), v - \hat{v}\big)_\Omega + w|u - \hat{u}|_{1,\Omega}^2 + w|v - \hat{v}|_\Omega^2 \\
&\quad \geqslant -L_h|u - \hat{u}|_{1,\Omega}|v - \hat{v}|_\Gamma - L_f|u - \hat{u}|_{1,\Omega}|v - \hat{v}|_\Omega + m_g|v - \hat{v}|_\Gamma^2 + w|u - \hat{u}|_{1,\Omega}^2 \\
&\qquad + w|v - \hat{v}|_\Omega^2 \\
&\quad \geqslant -\varepsilon|v - \hat{v}|_\Gamma^2 - L_h^2 C_\varepsilon |u - \hat{u}|_{1,\Omega}^2 |v - \hat{v}|_\Gamma - \frac{L_f}{2}\big[|u - \hat{u}|_{1,\Omega}^2 + |v - \hat{v}|_\Omega^2\big] \\
&\qquad + m_g|v - \hat{v}|^2{}_\Gamma + w|u - \hat{u}|_{1,\Omega}^2 + w|v - \hat{v}|_\Omega^2 \\
&\quad \geqslant (m_g - \varepsilon)|v - \hat{v}|^2{}_\Gamma + \big(-L_h^2 C_\varepsilon - L_f/2 + w\big)|u - \hat{u}|_{1,\Omega}^2 + (w - L_f)|v - \hat{v}|_\Omega^2
\end{aligned}$$

where L_f and L_h are the Lipschitz constants of the source f, and h, respectively.

Thus if we choose $\varepsilon < m_g$ and $w > L_h^2 C_\varepsilon + L_f$ then $((A + wI)(U - \hat{U}), U - \hat{U})_H \geqslant 0$, which shows that A is w-accretive.

Step 2: Proof for $A + wI$ is m-accretive. We need to show that the operator $A + wI + \lambda I$ is onto H for some $\lambda > 0$. Without loss of generality we can take $w = 0$ (otherwise we adjust λ).

Let $a \in H^1(\Omega)$ and $b \in L_2(\Omega)$. We have to show that there exist $u \in H^1(\Omega)$ and $v \in L_2(\Omega)$ such that $\begin{cases} -v + \lambda u = a, \\ \Delta_R u - \Delta_R R h(u) - f(u) + \Delta_R R g(v) + g_0(v) + \lambda v = b. \end{cases}$

This is equivalent to

$$\begin{aligned}
&\Delta_R\left(\frac{a + v}{\lambda}\right) - \Delta_R R h\left(\frac{a + v}{\lambda}\right) + \Delta_R R g(v) - f\left(\frac{a + v}{\lambda}\right) + g_0(v) + \lambda v = b \\
&\quad \Leftrightarrow \quad \frac{1}{\lambda}\Delta_R(v) - \Delta_R R h\left(\frac{a + v}{\lambda}\right) + \Delta_R R g(v) - f\left(\frac{a + v}{\lambda}\right) + g_0(v) + \lambda v = b - \frac{1}{\lambda}\Delta_R(a)
\end{aligned}$$

Let $V = D(\Delta_R^{1/2}) \approx H^1(\Omega)$ and let $\hat{b} = b - \frac{1}{\lambda}\Delta_R(a) \in V'$. Thus the issue reduces to proving surjectivity of the operator $Tv = \frac{1}{\lambda}\Delta_R(v) - \Delta_R Rh(\frac{a+v}{\lambda}) + \Delta_R Rg(v) - f(\frac{a+v}{\lambda}) + g_0(v) + \lambda v$ from V into V'.

To prove that T is surjective, it is enough to show that T is m-accretive and coercive (Corollary 1.2 (Section 2.1) in [2]).

Let

$$T_1(v) = \Delta_R Rg\big(R^*\Delta_R v\big), \qquad T_3(v) = \frac{1}{2\lambda}\Delta_R(v) \quad \text{and}$$

$$T_2(v) = \frac{1}{2\lambda}\Delta_R(v) - \Delta_R Rh\left(\frac{a+v}{\lambda}\right) - f\left(\frac{a+v}{\lambda}\right) + g_0(v) + \lambda v$$

Since T_1 as $T_1 = \Delta_R Rg(R^*\Delta_R) = \Delta_R Rg(\gamma)$, where $\gamma u = u|_\Gamma$ is surjective $H^1(\Omega) \to H^{1/2}(\Gamma)$ and g is monotone increasing, we can write $T_1 = \partial(\Phi \circ \gamma)$ ([9,17], Proposition II.7.8 [25]), where

$$\Phi(\nu) = \begin{cases} \int_\Gamma \phi(\nu), & \text{if } \phi(\nu) \in L_1(\Gamma) \\ \infty, & \text{otherwise} \end{cases} \quad \text{and} \quad \phi(s) = \int_0^s g(\tau)\, d\tau$$

The functional Φ is convex and lower semi-continuous, and so is $\Phi \circ \gamma$. And by Theorem 2.1 (Section 2.2) in [2], we can conclude that T_1 is maximal monotone $V \to V'$.

We already showed that T_3 is maximal monotone. So what is left to do is that $T_2 : V \to V'$ is maximal monotone. Then using Theorem 1.5 [2], we will conclude that T is maximal monotone in $V \times V'$.

First, we will show that T_2 is monotone. Let v and $\hat{v} \in V$, then

$$\begin{aligned}
&(T_2 v - T_2\hat{v}, v - \hat{v})_\Omega \\
&\quad = \frac{1}{2\lambda}\big(\Delta_R(v - \hat{v}), v - \hat{v}\big)_\Omega - \left(f\left(\frac{a+v}{\lambda}\right) - f\left(\frac{a+\hat{v}}{\lambda}\right), v - \hat{v}\right)_\Omega \\
&\qquad - \left(\Delta_R R\left[h\left(\frac{a+v}{\lambda}\right) - h\left(\frac{a+\hat{v}}{\lambda}\right)\right], v - \hat{v}\right)_\Omega + \big(g_0(v) - g_0(\hat{v}), v - \hat{v}\big)_\Omega + \lambda|v - \hat{v}|_\Omega^2 \\
&\quad \geqslant \frac{1}{2\lambda}|v - \hat{v}|_{1,\Omega}^2 + (\lambda - 1)|v - \hat{v}|_\Omega^2 - \frac{L_h^2 C_\varepsilon}{\lambda^2}|v - \hat{v}|_{1,\Omega}^2 - \varepsilon|v - \hat{v}|_\Gamma^2 - \frac{L_f^2}{\lambda^2}|v - \hat{v}|_{1,\Omega}^2 \\
&\quad \geqslant \left[\frac{1}{2\lambda} - \frac{L_h^2 C_\varepsilon}{\lambda^2} - \frac{L_f^2}{\lambda^2} - \varepsilon C\right]|v - \hat{v}|_{1,\Omega}^2 + (\lambda - 1)|v - \hat{v}|_\Omega^2
\end{aligned}$$

Choosing $\lambda > 1$ big enough and ε small enough such that $\frac{1}{2\lambda} - \frac{L_h^2 C_\varepsilon}{\lambda^2} - \frac{L_f^2}{\lambda^2} - \varepsilon C \geqslant 0$, we obtain that T_2 is monotone on V.

Now we know that g_0 is continuous, f is Lipschitz, and since we are working with VV' framework and $\Delta_R^{-1/2}\Delta_R R : L_2(\Gamma) \to L_2(\Omega)$ is bounded, the operator $-\Delta_R Rh(\frac{a+v}{\lambda})$ is Lipschitz $V \to V'$, with a Lipschitz constant proportional to $\frac{L_h}{\lambda}$. Thus T_2 is monotone and continuous, and thus maximal monotone (see [1], p. 46).

Moreover, T_3 is coercive, and thus T is coercive. Thus $Rg(T) = V'$ and therefore we have proved existence of v in $V = H^1(\Omega)$. Thus $u = \frac{v+a}{\lambda} \in V$. In addition, one can easily show that the pair (u, v) is also in $D(A)$. Indeed, we have

$$\Delta_R\big(u - Rh(u) + Rg(v)\big) = -\lambda v + b + f(u) - g_0(v) \in L_2(\Omega)$$

hence $(u - Rh(u) + Rg(v)) \in D(\Delta_R)$ as desired. The proof of maximal accretivity is thus completed. This implies that the system (1) generates a strongly continuous semigroup $S(t)U(0) \equiv U(t) \in C(0, T; H)$, where $U(t)$ is a strong (in H) limit of strong solutions evolving in $D(A)$. Strong solutions – say $U_n(t)$ – corresponding to initial data in $D(A)$ evolve in $D(A)$, posses the regularity listed in (14) and satisfy the energy inequality

$$E_{u_n}(t) + \int_s^t \int_\Omega g_0\big(u_{nt}(s)\big)u_{nt}(s)\,dx\,d\tau + \int_s^t \int_\Gamma g\big(u_{nt}|_\Gamma(s)\big)u_{nt}|_\Gamma\,dx\,d\tau$$
$$\leqslant E_{u_n}(s) + \int_s^t \int_\Omega f(u_n)u_{nt}\,dx\,d\tau + \int_s^t \int_\Gamma h(u_n|_\Gamma)u_{nt}|_\Gamma\,dx\,d\tau \tag{15}$$

Lipschitz continuity of f and h, along with coercivity of $g(s)$ easily imply the bound

$$\int_0^T |u_{nt}|_\Gamma^2|_2\,d\tau \leqslant C_{m_g}\big(T, E_u(0)\big)$$

Thus, on a subsequence (denoted by the same symbol) we have

$$u_{nt}|_\Gamma \rightharpoonup u_t|_\Gamma, \quad \text{weakly in } L_2(\Sigma_T) \tag{16}$$

This gives the boundary regularity claimed in Lemma 2.1.

Standard weak lower semicontinuity – convexity argument along with (16) implies

$$\int_0^T \int_\Gamma \varphi_g(u_t|_\Gamma)\,dx\,dt \leqslant \liminf_n \int_0^T \int_\Gamma g(u_{nt}|_\Gamma)u_{nt}|_\Gamma\,dx\,dt \tag{17}$$

Since $u_{nt} \to u_t$ in $L_2(\Omega)$ and $g_0(s)s \geqslant 0$, Fatou's lemma implies

$$\int_0^T \int_\Omega g_0(u_t)u_t\,dx\,dt \leqslant \liminf_n \int_0^T \int_\Omega g_0(u_{nt})u_{nt}\,dx\,dt \tag{18}$$

This concludes the proof of Lemma 2.1. Indeed, strong convergence of $U_n(t) \to U(t)$, in H, Lipschitz continuity of f and h along with (16), (17), (18), allow us to pass with the limit on (15) and yield the conclusions in Lemma 2.1. □

2.1.2. Locally Lipschitz sources and uniformity of the bounds

Note that Lemma 2.1 assumes global Lipschitz conditions on the sources f and h. The next step towards the proof of Theorem 1.2 is to relinquish the requirement of globality. This is accomplished in the following lemma:

Lemma 2.2. *We assume that*

(1) *$g(s)$ and $g_0(s)$ are continuous and monotone increasing functions with $g_0(0) = g(0) = 0$. We assume the following coercivity condition is imposed on g: there exists $m_g > 0$ such that $(g(s) - g(v))(s - v) \geqslant m_g|s - v|^2$.*

(2) *f is locally Lipschitz: $H^1(\Omega) \to L_2(\Omega)$ and $\hat{h}(u) \equiv h(u|_\Gamma)$ is locally Lipschitz: $H^1(\Omega) \to L_2(\Gamma)$.*
(3) *g_0 and g satisfy growth conditions $(A_{g_0,g})$ in Assumption* 1.1.
(4) *f is locally Lipschitz: $H^1(\Omega) \to L_{\frac{m+1}{m}}(\Omega)$ with Lipschitz constant $L_f(R)$, for $|u|_{1,\Omega} \leqslant R$.*
(5) *h is locally Lipschitz: $H^1(\Omega) \to L_{\frac{q+1}{q}}(\Gamma)$ with Lipschitz constant $L_h(R)$, for $|u|_{1,\Omega} \leqslant R$.*

Then for any $R > 0$, and any $U(0) \in H$, $|U(0)|_H \leqslant R$, there exists $T_M > 0$ and a local unique weak solution $U \in C[(0, T_M), H]$ such that $|U(t)|_H \leqslant 2R$, $t \in [0, T_M)$. Here T_M depends on R, m_q, l_q, $L_f(R)$, $L_h(R)$, but does not depend on m_g. In addition, solution $U(t)$ enjoys the following additional properties:

- $\int_0^T \|u_t\|_{m+1}^{m+1}\,dt \leqslant C(l_m, T, E(0))$, $\int_0^T |u_t|_{q+1}^{q+1}\,dt \leqslant C(m_q, T, E(0))$.
- *The following energy inequality holds*

$$\begin{aligned} E_u(t) + l_m \int_s^t \int_\Omega |u_t|^{m+1}\,dx\,d\tau + m_q \int_s^t \int_\Gamma |u_t|_\Gamma|^{q+1}\,dx\,d\tau \\ \leqslant E_u(s) + \left| \int_s^t \int_\Omega f(u)u_t\,dx\,d\tau + \int_s^t \int_\Gamma h(u|_\Gamma)u_t|_\Gamma\,dx\,d\tau \right| \end{aligned} \tag{19}$$

Remark 14. We note that the quantitative description of the parameters – in particular the survival time T_M – depends neither on the $H^1 \to L_2$ Lipschitz property of functions f and h, nor on the coercivity constant m_g. This fact will be critical for future arguments.

Remark 15. The solutions $U(t)$ obtained in Lemma 2.2 are "generalized", which in addition are "weak". The energy inequality (19) is shown to be satisfied for all weak solutions. However, for initial data in the domain of the accretive operator A, the corresponding solutions are strong and one obtains energy *equality*, as predicted by monotone operator theory.

Proof. The proof follows a truncation of sources idea presented in [9]. On the strength of Hypothesis 4, f and h are locally Lipschitz functions $H^1(\Omega) \to L_2$, so that the truncation of sources introduced in [9] produces globally Lipschitz approximations. More specifically, for $K > 0$, consider

$$f_K(u) = \begin{cases} f(u) & \text{if } |u|_{1,\Omega} \leqslant K \\ f\left(\frac{Ku}{|u|_{1,\Omega}}\right) & \text{if } |u|_{1,\Omega} \geqslant K \end{cases} \quad \text{and} \quad h_K(u) = \begin{cases} h(u) & \text{if } |u|_{1,\Omega} \leqslant K \\ h\left(\frac{Ku}{|u|_{1,\Omega}}\right) & \text{if } |u|_{1,\Omega} \geqslant K \end{cases}$$

With the truncated f_K and h_K, we consider the following (K) problem, where $K = 2R$.

$$(K) \quad \begin{cases} u_{tt} + g_0(u_t) = \Delta u + f_K(u) & \text{in } Q = [0,\infty) \times \Omega \\ \partial_\nu u + u + g(u_t) = h_K(u) & \text{in } \Sigma = [0,\infty) \times \Gamma \\ u(0) = u_0 \in H^1(\Omega) \quad \text{and} & u_t(0) = u_1 \in L_2(\Omega) \end{cases}$$

For each K, f_K and h_K are globally Lipschitz with Lipschitz constants bounded by $L(K)$ (see [9], p. 1946). In addition, the damping $g(s)$ satisfies the coercivity condition with constant m_g. This allows us to use the previous Lemma 2.1 for the (K) problem and obtain a unique global solution $U_K(\cdot) \in C([O, T], H)$ for any $T > 0$. In the following calculations, by $u(t)$ we mean solution $u_K(t)$.

From now on, we will use the following notation: $\int_Q = \int_0^T \int_\Omega$, $dQ = d\Omega\,dt$, $\int_\Sigma = \int_0^T \int_\Gamma$, and $d\Sigma = d\Gamma\,dt$. Also, $\|\cdot\|_\Omega$ represents $|\cdot|_{L_2(\Omega)}$ and $|\cdot|_\Gamma$ represents $|\cdot|_{L_2(\Gamma)}$. For $|\cdot|_{H^1(\Omega)}$ we will use $\|\cdot\|_{1,\Omega}$.

The energy inequality (8) in Lemma 2.1 along with

$$m_q|s|^{q+1} \leqslant \varphi_g(s)$$

is a starting point for the estimate:

$$\frac{1}{2}\big(\|u_t(T)\|_\Omega^2 + \|u(T)\|_{1,\Omega}^2\big) + \int_Q g_0\big(u_t(t)\big)u_t(t)\,dQ + m_q\int_\Sigma |u_t|^{q+1}\,d\Sigma$$
$$\leqslant \frac{1}{2}\big(\|u_t(0)\|_\Omega^2 + \|u(0)\|_{1,\Omega}^2\big) + \left|\int_Q f_K\big(u(t)\big)u_t(t)\,dQ\right| + \left|\int_\Sigma h_K\big(u(t)\big)u_t(t)\,d\Sigma\right| \tag{20}$$

We proceed by estimating the terms on the right side of the energy equality (20). The terms involving the sources f and h will be estimated by using Holder's Inequality with $\tilde{m} = (m+1)/m$ and $m+1$ and $\tilde{q} = (q+1)/q$ and $q+1$, respectively, followed by Young's Inequality with the corresponding exponents.

$$\left|\int_Q f_K\big(u(t)\big)u_t(t)\,dQ\right| \leqslant \int_0^T \big\|f_K\big(u(t)\big)\big\|_{\tilde{m}} \cdot \big\|u_t(t)\big\|_{m+1}\,dt$$
$$\leqslant \varepsilon_1\int_0^T \big\|u_t(t)\big\|_{m+1}^{m+1}\,dt + C_{\varepsilon_1}\int_0^T \big\|f_K(u)\big\|_{\tilde{m}}^{\tilde{m}}\,dt \tag{21}$$

We denote: $A_u = \{t \in [0,T],\ \|u(t)\|_{1,\Omega} \leqslant K\}$ and B_u the complement of A_u in $[0,T]$. Recalling the definition of the truncated function f_K, and using the locally Lipschitz estimate along with the fact that the arguments of the function f in the above integrals are always in $B_{H^1}(0,K)$, we obtain:

$$\int_0^T \big\|f_K\big(u(t)\big)\big\|_{\tilde{m}}^{\tilde{m}} \leqslant \int_{A_u}\int_\Omega \big|f\big(u(t,x)\big)\big|^{\tilde{m}}\,dx\,dt + \int_{B_u}\left\|f\left(\frac{Ku}{\|u\|_{1,\Omega}}\right)\right\|_{\tilde{m}}^{\tilde{m}}\,dt$$
$$\leqslant C\int_{A_u}\big[L_f^{\tilde{m}}(K)\big\|u(t)\big\|_{1,\Omega}^{\tilde{m}} + \big|f(0)\big|^{\tilde{m}}\big]\,dt + \int_{B_u} L_f^{\tilde{m}}(K)K^{\tilde{m}}\,dt$$
$$\leqslant C\int_{A_u} L_f^{\tilde{m}}(K)\big\|u(t)\big\|_{1,\Omega}^{\tilde{m}} + \big|f(0)\big|^{\tilde{m}}\,dt + \int_{B_u} L_f^{\tilde{m}}(K)K^{\tilde{m}-2}\big\|u(t)\big\|_{1,\Omega}^2\,dt$$
$$\leqslant C\big[L_f^{\tilde{m}}(K) + L_f^{\tilde{m}}(K)K^{\tilde{m}-2}\big]\int_0^T \big\|u(t)\big\|_{1,\Omega}^2\,dt + C_f T \tag{22}$$

Combining (21) with (22), we obtain our final estimate for the term involving the truncated source f_K:

$$\left|\int_Q f_K\big(u(t)\big)u_t(t)\,dQ\right| \leqslant \varepsilon_1\int_0^T \big\|u_t(t)\big\|_{m+1}^{m+1}\,dt$$

$$+ C_{\varepsilon_1}\big[L_f^{\tilde m}(K) + L_f^{\tilde m}(K)K^{\tilde m-2}\big]\int_0^T \big\|u(t)\big\|_{1,\Omega}^2\,dt + C_{\varepsilon_1}C_f T \tag{23}$$

We apply similar arguments in order to estimate the term in (20) involving the truncated source $h_k(u)$:

$$\begin{aligned}
&\left|\int_\Sigma h_K\big(u(t)\big)u_t(t)\,d\Sigma\right| \\
&\leqslant \int_0^T \big|h_K\big(u(t)\big)\big|_{\tilde q}\cdot\big|u_t(t)\big|_{q+1}\,dt \\
&\leqslant \varepsilon_2\int_0^T \big|u_t(t)\big|_{q+1}^{q+1}\,dt + C_{\varepsilon_2}\int_0^T \big|h_K(u)\big|_{\tilde q}^{\tilde q}\,dt \\
&\leqslant \varepsilon_2\int_0^T \big|u_t(t)\big|_{q+1}^{q+1}\,dt + C_{\varepsilon_2}\big[L_h^{\tilde q}(K) + L_h(K)^{\tilde q}K^{\tilde q-2}\big]\int_0^T \big\|u(t)\big\|_{1,\Omega}^2\,dt + C_{\varepsilon_2}C_h T
\end{aligned} \tag{24}$$

In (23) and (24), the constants C_{ε_1} and C_{ε_2} are generic constants, independent of the functions f and h and $C_f = C_{f(0)}$, $C_h = C_{h(0)}$.

Combining (20) with (23) and (24), and using the growth conditions imposed on g_0 and g, we obtain:

$$\begin{aligned}
&\frac{1}{2}\big(\big\|u_t(T)\big\|_\Omega^2 + \big\|u(T)\big\|_{1,\Omega}^2\big) + l_m\int_0^T \big\|u_t(t)\big\|_{m+1}^{m+1}\,dt + m_q\int_0^T \big|u_t(t)\big|_{q+1}^{q+1}\,dt - C_{g_0,g,f,h}T \\
&\leqslant \frac{1}{2}\big(\big\|u_t(0)\big\|_\Omega^2 + \big\|u(0)\big\|_{1,\Omega}^2\big) + \varepsilon_1\int_0^T \big\|u_t(t)\big\|_{m+1}^{m+1}\,dt + \varepsilon_2\int_0^T \big|u_t(t)\big|_{q+1}^{q+1}\,dt \\
&\quad + C_{\varepsilon_1}\big[L_f^{\tilde m}(K) + L_f^{\tilde m}(K)K^{\tilde m-2}\big]\int_0^T \big\|u(t)\big\|_{1,\Omega}^2\,dt \\
&\quad + C_{\varepsilon_2}\big[L_h^{\tilde q}(K) + L_h^{\tilde q}(K)K^{q-2}\big]\int_0^T \big\|u(t)\big\|_{1,\Omega}^2\,dt
\end{aligned} \tag{25}$$

Choosing $\varepsilon_1 < \frac{l_m}{2}$ and $\varepsilon_2 < \frac{m_q}{2}$ we obtain:

$$\big\|u_t(T)\big\|_\Omega^2 + \big\|u(T)\big\|_{1,\Omega}^2 \leqslant \big[\big\|u_t(0)\big\|_\Omega^2 + \big\|u(0)\big\|_{1,\Omega}^2 + \bar C T\big]\cdot e^{C(l_m,m_q,K,L_f(K),L_h(K))T} \tag{26}$$

Recalling $|U(0)|_H \leqslant R \leqslant 1/2K$, and selecting T such that

$$T \leqslant \frac{K^2}{4\bar{C}}$$
$$e^{C(l_m, m_q, K, L_f(K), L_h(K))T} \leqslant 2$$

leads to

$$T_K = \min\left\{\frac{K^2}{4\bar{C}}, C^{-1}\big(l_m, m_q, K, L_f(K), L_h(K)\big) \cdot \ln 2\right\}$$

Thus, for $0 \leqslant t < T_K$ we have

$$\left|U_K(t)\right|_H \leqslant K$$

Thus for $t < T_K$, $f_K(u) = f(u)$ and $h_K(u) = h(u)$. Because of the uniqueness of solutions for the K problem (since g is coercive) the solution to the truncated problem (K) coincides with the solution to the original, un-truncated equation (1). By reiterating the procedure, with $u(T_K)$ as initial value and with a larger K, we obtain maximal time of existence T_M, which does depend on m_q, l_m, L_f, L_h (but it does not depend on m_g and does not depend on the Lipschitz constant between H^1 and L_2). Thus we have proved local existence and uniqueness of solutions of the problem with g coercive and locally Lipschitz f and h.

The L_{m+1} regularity of u_t and L_{q+1} regularity of $u_t|_\Gamma$ follow from (25).

The energy inequality in Lemma 2.2 follows from the one given in Lemma 2.1, after accounting for the lower bounds imposed on g and g_0.

In order to complete the proof, we need to show that the obtained generalized-semigroup solutions are actually weak (i.e. they satisfy the variational equality). However, this follows from the monotonicity argument and the obtained a priori bounds. Indeed, in order to pass with the limit on strong solutions evolving on $D(A)$ (guaranteed by Lemma 2.1), the only issue is to pass with the limit on the nonlinear damping terms. However, uniform in n bounds

$$\begin{aligned} &\left|g_0(u_{nt})\right|_{L_{\frac{m+1}{m}}(Q_T)} \leqslant M, \qquad |u_{nt}|_{L_{m+1}(Q_T)} \leqslant M \\ &\left|g(u_{nt})\right|_{L_{\frac{q+1}{q}}(\Sigma_T)} \leqslant M, \qquad \left|u_{nt}|_\Gamma|_{L_{q+1}(\Sigma_T)}\right| \leqslant M \end{aligned} \tag{27}$$

and the lim sup condition

$$\int_0^T \big(g_0(u_{nt}) - g_0(u_{mt}), u_{nt} - u_{mt}\big)_\Omega + \big(g(u_{nt}) - g(u_{mt}), u_{nt} - u_{mt}\big)_\Gamma \to 0$$

(easily obtainable from the energy inequality applied to the difference of two strong solutions) allows to reconstruct the weak limits and pass with the limit (via Lemma 1.3 [1]) in the variational form of *strong* solutions. This implies that the "finite energy" solutions constructed in Lemma 2.2 are also *weak* solutions (see Remark 13 applied with $V = L_{\frac{m+1}{m}}(Q_T)$, $V' = L_{\frac{m+1}{m}}(Q_T)$, $U = L_{q+1}(\Sigma_T)$, $U' = L_{\frac{q+1}{q}}(\Sigma_T)$). $\square$

3. Proofs of Theorem 1.2 and Theorem 1.3

In order to establish the results in both theorems, one needs to relinquish the requirements of coercivity of $g(s)$ and local Lipschitz condition $H^1 \to L_2$ imposed on both f and h. This is accomplished by suitable approximation procedures which are introduced below.

3.1. Approximations of the sources and of the damping

The result of Lemma 2.2 provides local solutions $U(t)$ with the length of maximal time T_M depending on the constants m_q, l_m, L_f, L_h but *neither* on the strong monotonicity constant m_g, nor on the Lipschitz constant between $H^1 \to L_2$. This is essential, since otherwise we will not be able to treat supercritical sources or boundary damping that is not coercive.

Proceeding with the proof of Theorem 1.2 we aim to relinquish (i) the condition that g is coercive, (ii) the condition that f is locally Lipschitz $H^1(\Omega) \to L_2(\Omega)$ and (iii) the condition that h is locally Lipschitz $H^1(\Omega) \to L_2(\Gamma)$. For this step we shall use the constructions inspired by [17] and [22].

Approximation of g, f and h: We consider the following approximation of Eq. (1.1). With $n \to \infty$ as the parameter of approximation,

$$\begin{cases} u_{tt}^n + g_0(u_t) = \Delta u^n + f_n(u) \quad \text{in } \Omega \times [0,\infty) \\ \partial_\nu u^n + u^n + g_n\big(u_t^n\big) = h_n\big(u^n\big) \quad \text{in } \Gamma \times [0,\infty) \\ u^n(0) = u_0 \in H^1(\Omega) \quad \text{and} \quad u_t^n(0) = u_1 \in L_2(\Omega) \end{cases} \tag{28}$$

where we constructed the approximating functions as follows.

Let $g_n(s) = g(s) + \frac{1}{n}s$, $n \to \infty$. Notice that g_n is coercive with constant $m_g = \frac{1}{n} > 0$: $(g_n(s) - g_n(t), s-t) = (g(s) - g(t) + \frac{1}{n}s - \frac{1}{n}t, s-t) = (g(s) - g(t), s-t) + \frac{1}{n}|s-t|^2 \geqslant \frac{1}{n}|s-t|^2$, since g is monotone.

For the construction of the Lipschitz approximations for the sources, we use the cutoff function introduced in [22]. Let η be a cutoff smooth function such that (1) $0 \leqslant \eta \leqslant 1$, (2) $\eta(u) = 1$, if $|u| \leqslant n$, (3) $\eta(u) = 0$, if $|u| > 2n$ and (4) $|\eta'(u)| \leqslant C/n$.

Now construct $f_n : H^1(\Omega) \to L_{\frac{m+1}{m}}(\Omega)$, $f_n(u) := f(u)\eta(u)$ and $h_n : H^1(\Omega) \to L_{\frac{q+1}{q}}(\Gamma)$, $h_n(u) := h(u)\eta(u)$. This means that

$$f_n(u) = \begin{cases} f(u), & |u| \leqslant n \\ f(u)\eta(u), & n < |u| < 2n \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad h_n(u) = \begin{cases} h(u), & |u| \leqslant n \\ h(u)\eta(u), & n < |u| < 2n \\ 0, & \text{otherwise} \end{cases}$$

The interior approximation f_n satisfies the following properties:

Claim 1. *f_n is locally Lipschitz $H^1(\Omega) \to L_{\frac{m+1}{m}}(\Omega)$ (uniformly in n).*

Claim 2. *For each n, f_n is locally Lipschitz $H^1(\Omega) \to L_2(\Omega)$.*

Claim 3. *f_n is locally Lipschitz $H^{1-\varepsilon}(\Omega) \to L_1(\Omega)$, for ε chosen such that $2m\varepsilon \leqslant 1$.*

Claim 4. *$|f_n(u) - f(u)|_{L_{\frac{m+1}{m}}(\Omega)} \to 0$ as $n \to \infty$ for all $u \in H^1(\Omega)$.*

Proof of Claims 1, 2, and 3.

Proof. We will combine the proofs for the first three claims, because for each one of them, we need to consider the following four cases:

Case 1. $|u|, |v| \leqslant n$.

(1) Claim 1: Recall that $\tilde{m} = \frac{m+1}{m}$. Then $\|f_n(u) - f_n(v)\|_{\tilde{m}} = \|f(u) - f(v)\|_{\tilde{m}}$ and we already know that f is locally Lipschitz $H^1(\Omega) \to L_{\tilde{m}}(\Omega)$.

(2) Claim 2:

$$\begin{aligned}\|f_n(u)-f_n(v)\|_{\Omega} &= \|f(u)-f(v)\|_{\Omega}\\ &\leqslant \left(\int_{\Omega} C|u-v|^2\big[|u|^{p-1}+|v|^{p-1}+1\big]^2\,d\Omega\right)^{1/2}\end{aligned} \tag{29}$$

Using Holder's Inequality with 3 and 3/2, the fact that $|u|\leqslant n$ and $|v|\leqslant n$ and Sobolev's Imbedding $H^1(\Omega)\to L_6(\Omega)$, (29) becomes

$$\|f_n(u)-f_n(v)\|_{\Omega}\leqslant C_n\|u-v\|_{1,\Omega} \tag{30}$$

(3) Claim 3: f_n equals f, and we already know that f is locally Lipschitz $H^{1-\varepsilon}(\Omega)\to L_1(\Omega)$, for ε chosen such that $2m\varepsilon\leqslant 1$.

Case 2. $n\leqslant|u|,\ |v|\leqslant 2n$.

(1) Claim 1:

$$\begin{aligned}\|f_n(u)-f_n(v)\|_{\tilde m} &= \|f(u)\eta(u)-f(v)\eta(v)\|_{\tilde m}\\ &\leqslant \|f(u)\eta(u)-f(v)\eta(u)+f(v)\eta(u)-f(v)\eta(v)\|_{\tilde m}\\ &\leqslant \|f(u)-f(v)\|_{\tilde m}+\left(\int_{\Omega}\big[|f(v)||\eta(u)-\eta(v)|\big]^{\tilde m}\,d\Omega\right)^{m/(m+1)}\\ &\leqslant \|f(u)-f(v)\|_{\tilde m}+\left(\int_{\Omega}\big[|v|^{p-1}|v|\max|\eta'(\xi)||u-v|\big]^{\tilde m}\,d\Omega\right)^{m/(m+1)}\end{aligned} \tag{31}$$

Now using the definition of the cutoff function η and the fact that $|v|\leqslant 2n$, we can see that $|v|\max|\eta'(\xi)|\leqslant C$ and thus (31) becomes

$$\|f_n(u)-f_n(v)\|_{\tilde m}\leqslant |f(u)-f(v)|_{\tilde m}+\left(\int_{\Omega}|v|^{(p-1)\tilde m}|u-v|^{\tilde m}\,d\Omega\right)^{m/(m+1)} \tag{32}$$

For the second term on the right side of (32), we use Holder's Inequality with p and $p/(p-1)$, the fact that $p(m+1)/m\leqslant 6$, and Sobolev's Imbedding $H^1(\Omega)\to L_6(\Omega)$ and obtain

$$\begin{aligned}\|f_n(u)-f_n(v)\|_{\tilde m} &\leqslant \|f(u)-f(v)\|_{\tilde m}+C|v|^{p-1}_{L_6(\Omega)}|u-v|_{L_6(\Omega)}\\ &\leqslant \|f(u)-f(v)\|_{\tilde m}+C|v|^{p-1}_{H^1(\Omega)}|u-v|_{H^1(\Omega)}\end{aligned} \tag{33}$$

which proves that f_n is locally Lipschitz $H^1(\Omega)\to L_{\frac{m+1}{m}}(\Omega)$.

(2) Claim 2: We use the calculations performed in Case 2 of Claim 1 and obtain

$$\left\| f_n(u) - f_n(v) \right\|_{\Omega} \leqslant \left\| f(u) - f(v) \right\|_{\Omega} + \left(\int_{\Omega} C|v|^{2(p-1)}|u-v|^2 \, d\Omega \right)^{1/2} \tag{34}$$

Now reiterating the strategy used in Case 1 (Claim 2), we obtain the desired result.

(3) Claim 3: We have

$$\begin{aligned}
\left\| f_n(u) - f_n(v) \right\|_{L_1(\Omega)} &= \left\| f(u)\eta(u) - f(v)\eta(v) \right\|_{L_1(\Omega)} \\
&\leqslant \left\| f(u)\eta(u) - f(v)\eta(u) + f(v)\eta(u) - f(v)\eta(v) \right\|_{L_1(\Omega)} \\
&\leqslant \left\| f(u) - f(v) \right\|_{L_1(\Omega)} + \int_{\Omega} \left|f(v)\right| \left|\eta(u) - \eta(v)\right| d\Omega \\
&\leqslant \left\| f(u) - f(v) \right\|_{L_1(\Omega)} + \int_{\Omega} |v|^{p-1}|v| \max\left|\eta'(\xi)\right| |u-v| \, d\Omega \\
&\leqslant \left\| f(u) - f(v) \right\|_{L_1(\Omega)} + \int_{\Omega} |v|^{p-1}|u-v| \, d\Omega
\end{aligned} \tag{35}$$

Now using Holder's Inequality with $\frac{6}{5-2\varepsilon}$ and $\frac{6}{1+2\varepsilon}$, the fact that $\frac{6(p-1)}{5-2\varepsilon} \leqslant \frac{6}{1+2\varepsilon}$ and Sobolev's Imbedding $H^{1-\varepsilon}(\Omega) \to L_{\frac{6}{1+2\varepsilon}}(\Omega)$, we obtain

$$\left\| f_n(u) - f_n(v) \right\|_{L_1(\Omega)} \leqslant \left\| f(u) - f(v) \right\|_{L_1(\Omega)} + C_S |v|_{H^{1-\varepsilon}(\Omega)} |u-v|_{H^{1-\varepsilon}(\Omega)}$$

which proves that f_n is locally Lipschitz $H^{1-\varepsilon}(\Omega) \to L_1(\Omega)$ in this case.

Case 3. If $|u| \leqslant n$ and $n < |v| \leqslant 2n$,

(1) Claim 1: We have

$$\begin{aligned}
\left\| f_n(u) - f_n(v) \right\|_{\tilde{m}} &= \left\| f(u) - f(v)\eta(v) \right\|_{\tilde{m}} \\
&\leqslant \left\| f(u) - f(v) \right\|_{\tilde{m}} + \left(\int_{\Omega} \left|f(v)\right| \left|1 - \eta(v)\right| d\Omega \right)^{m/(m+1)}
\end{aligned} \tag{36}$$

In (36), we can replace $1 = \eta(u)$, since $|u| \leqslant n$ and then the calculations follow exactly as in Case 2, Claim 1.

(2) Claim 2: As before, the case when $|u| \leqslant n$ and $n < |v| \leqslant 2n$ reduces to Case 2.

(3) Claim 3: We have

$$\begin{aligned}
\left\| f_n(u) - f_n(v) \right\|_{L_1(\Omega)} &= \left\| f(u) - f(v)\eta(v) \right\|_{L_1(\Omega)} \\
&\leqslant \left\| f(u) - f(v) \right\|_{L_1(\Omega)} + \int_{\Omega} \left|f(v)\right| \left|1 - \eta(v)\right| d\Omega
\end{aligned} \tag{37}$$

In (37), since $|u| \leqslant n$, we can replace 1 by $\eta(u)$ and then the calculations follow exactly as in Case 2.

Case 4. If $|u| \leqslant n$ and $|v| \geqslant 2n$, then we have

(1) Claim 1:

$$\begin{aligned}
\|f_n(u) - f_n(v)\|_{\tilde{m}} &= \|f(u)\eta(u) - f(v)\eta(v)\|_{\tilde{m}} \\
&\leqslant \|f(u)\eta(u) - f(u)\eta(v)\|_{\tilde{m}} + \|f(u)\eta(v) - f(v)\eta(v)\|_{\tilde{m}} \\
&= \left(\int_\Omega \left[|u|^{p-1}|u| \max\left|\eta'(\xi)\right||u-v|\right]^{\tilde{m}} d\Omega\right)^{m/(m+1)} \\
&\leqslant C_\eta \left(\int_\Omega |u|^{(p-1)\tilde{m}}|u-v|^{\tilde{m}} d\Omega\right)^{m/(m+1)} \\
&\leqslant C_\eta |u|^{p-1}_{H^1(\Omega)}|u-v|_{H^1(\Omega)} \qquad (38)
\end{aligned}$$

(2) Claim 2:

$$\begin{aligned}
\|f_n(u) - f_n(v)\|_\Omega &= \|f(u)\eta(u) - f(v)\eta(v)\|_\Omega \leqslant \|f(u)\eta(u) - f(u)\eta(v)\|_\Omega \\
&= \left(\int_\Omega \left[|u|^{p-1}|u| \max\left|\eta'(\xi)\right||u-v|\right]^2 d\Omega\right)^{1/2} \\
&\leqslant C_\eta \left(\int_\Omega |u|^{(2(p-1)}|u-v|^2 d\Omega\right)^{1/2} \\
&\leqslant C_{n,\eta}|u-v|_{H^1(\Omega)}
\end{aligned}$$

(3) Claim 3:

$$\begin{aligned}
\|f_n(u) - f_n(v)\|_\Omega &= \|f(u)\eta(u) - f(v)\eta(v)\|_{L_1(\Omega)} \leqslant \|f(u)\eta(u) - f(u)\eta(v)\|_\Omega \\
&= \int_\Omega |u|^{p-1}|u| \max\left|\eta'(\xi)\right||u-v| \, d\Omega \leqslant C_S |u|_{H^{1-\varepsilon}(\Omega)} \cdot |u-v|_{H^{1-\varepsilon}(\Omega)}
\end{aligned}$$

The case $n < |u| < 2n$ and $|v| \geqslant 2n$ is very similar to Case 4 and thus omitted. □

Proof for Claim 4.

Proof. We will show that $|f_n(u) - f(u)|_{L_{\frac{m+1}{m}}(\Omega)} \to 0$ as $n \to \infty$ for all $u \in H^1(\Omega)$. This can be seen as follows: the fact that $|f_n(u) - f(u)| = |f(u)||\eta(u) - 1|$ shows that $f_n(u) \to f(u)$ a.e. (because f is continuous and $\eta \to 1$ as $n \to \infty$). Then we also have that $|f_n(u) - f|^{\frac{m+1}{m}} \leqslant 2^{\frac{m+1}{m}}|f(u)|^{\frac{m+1}{m}}$ and $f(u) \in L_{\frac{m+1}{m}}(\Omega)$, for $u \in H^1(\Omega)$. Thus by Lebesgue Dominated Convergence Theorem, $f_n \to f$ in $L_{\frac{m+1}{m}}(\Omega)$. □

Similarly, the boundary approximation h_n satisfies the following properties:

Claim 1. *h_n is locally Lipschitz $H^1(\Omega) \to L_{\frac{q+1}{q}}(\Gamma)$ (uniformly in n).*

Claim 2. *For each n, h_n is locally Lipschitz $H^1(\Omega) \to L_2(\Gamma)$.*

Claim 3. *h_n is locally Lipschitz $H^{1-\varepsilon}(\Omega) \to L_1(\Gamma)$ (uniformly in n), for ε chosen such that $2q\varepsilon \leqslant 1$.*

Claim 4. *$|h_n(u) - h(u)|_{L_{\frac{q+1}{q}}(\Gamma)} \to 0$, as $n \to \infty$, for all $u \in H^1(\Omega)$.*

The proofs for the above claims are similar to the ones performed for the interior approximation f_n (with the adjusted Sobolev's embedding corresponding to the boundary), and thus the details will be omitted.

3.2. Proof of Theorem 1.3 – Final passage with the limit

For each n, g_n, f_n and h_n satisfy the assumptions of Lemma 2.2. Thus the result of Lemma 2.2 holds true for each n with $T_M(|U_0|_H, l_m, m_q)$ (with T_M uniform in n), i.e. for each n, there exists a solution $U^n(t) \in C(0, T_M; H)$ to the "approximated" problem (28). Thus, on the strength of Lemma 2.2, $u^n(t)$ satisfies the following variational equality: for any $\phi \in C(0, T; H^1(\Omega)) \cap C^1(0, T; L_2(\Omega)) \cap L_{m+1}(0, T; \Omega)$, with $\phi|_\Gamma \in L_{q+1}(0, T; \Gamma)$, we have:

$$\int_Q \left(-u_t^n \phi_t + \nabla u^n \nabla \phi\right) dQ + \int_\Sigma u^n \phi \, d\Sigma + \int_\Omega u_t^n \phi \, d\Omega \Big|_0^T + \int_Q g_0\left(u_t^n\right)\phi \, dQ + \int_\Sigma g^n\left(u_t^n\right)\phi \, d\Sigma = \int_Q f_n\left(u^n\right)\phi \, dQ + \int_\Sigma h_n\left(u^n\right)\phi \, d\Sigma \tag{39}$$

We will prove that this sequence of solutions U^n has, on a subsequence, an appropriate limit which is a solution to the original problem (1.1).

From Lemma 2.2, we know that u^n satisfies the following energy inequality

$$E_{u^n}(t) + l_m \int_s^t \int_\Omega \left|u_t^n\right|^{m+1} dx \, d\tau + m_q \int_s^t \int_\Gamma \left|u_t^n\right|_\Gamma|^{q+1} dx \, d\tau \leqslant E_{u^n}(s) + \left| \int_s^t \int_\Omega f\left(u^n\right) u_t^n \, dx \, d\tau + \int_s^t \int_\Gamma h\left(u^n|_\Gamma\right) u_t^n|_\Gamma \, dx \, d\tau \right| \tag{40}$$

and thus from (26) we obtain that for all $T < T_M$, we have

$$\left\|u_t^n(T)\right\|_\Omega^2 + \left\|u^n(T)\right\|_{1,\Omega}^2 \leqslant \left[\left\|u_t^n(0)\right\|_\Omega^2 + \left\|u^n(0)\right\|_{1,\Omega}^2 + \bar{C} T_M\right] \cdot e^{C_{l_m, m_q} T_M} \tag{41}$$

and

$$\int_0^T \left[\left\|u_t^n(t)\right\|_{m+1}^{m+1} + \left|u_t^n(t)\right|_{q+1}^{q+1}\right] dt \leqslant C_{\|u_0\|_{1,\Omega}, \|u_1\|_\Omega, T_M} \tag{42}$$

From (42), combined with the growth assumptions imposed on the damping terms g_0 and g, we also obtain that

$$\int_Q \left|g_0\left(u_t^n(t)\right)\right|^{\tilde{m}} dQ \leqslant \int_Q L_m^{\tilde{m}} \left|u_t^n(t)\right|^{m+1} dQ$$

$$= L_m^{\tilde{m}} \int_0^T \left\|u_t^n(t)\right\|_{m+1}^{m+1} dt \leqslant C_{\|u_0\|_{1,\Omega}, \|u_1\|_\Omega, T_{max}} \tag{43}$$

and

$$\int_\Sigma \left|g\left(u_t^n(t)\right)\right|^{\tilde{q}} d\Sigma \leqslant \int_\Sigma M_q^{\tilde{q}} \left|u_t^n(t)\right|^{q+1} d\Sigma$$

$$= M_q^{\tilde{q}} \int_0^T \left|u_t^n(t)\right|_{q+1}^{q+1} \leqslant C_{\|u_0\|_{1,\Omega}, \|u_1\|_\Omega, T_{max}} \tag{44}$$

Therefore, for any $T < T_{max}$, on a subsequence we have

$$U^n \to U \text{ weakly}^* \text{ in } L^\infty(0, T; H)$$
$$u_t^n \to u_t \text{ weakly in } L_{m+1}(0, T; \Omega)$$
$$u_t^n|_\Gamma \to u_t|_\Gamma \text{ weakly in } L_{q+1}(0, T; \Gamma)$$
$$g_0\left(u_t^n\right) \to g^* \text{ weakly in } L_{\frac{m+1}{m}}(0, T; \Omega), \text{ for some } g^* \in L_{\frac{m+1}{m}}(0, T; \Omega)$$
$$g\left(u_t^n\right) \to g^{**} \text{ weakly in } L_{\frac{q+1}{q}}(0, T; \Gamma), \text{ for some } g^{**} \in L_{\frac{q+1}{q}}(\Sigma)$$

Using (42) and weak lowersemicontinuity, we obtain similar estimates for the limits:

$$\int_0^T \left[\left\|u_t(t)\right\|_{m+1}^{m+1} + \left|u_t(t)\right|_{q+1}^{q+1}\right] dt \leqslant C_{\|u_0\|_{1,\Omega}, \|u_1\|_\Omega, T_{max}} \tag{45}$$

In order to finish the proof, we need to show that $g^* = g_0(u_t)$ and $g^{**} = g(u_t)$.

First, consider the variational equality (39) for smooth test functions $\phi \in C(0, T; H^2(\Omega)) \cap C^1(0, T; L_2(\Omega)) \cap L_{m+1}(0, T; \Omega)$, $\phi|_\Gamma \in L_{q+1}(0, T; \Gamma)$ and let $n \to \infty$. We obtain:

$$\int_Q (-u_t\phi_t + \nabla u \nabla \phi)\, dQ + \int_\Sigma \phi u\, d\Sigma + \int_\Omega u_t \phi\, d\Omega\Big|_0^T$$
$$+ \int_Q g^* \phi\, dQ + \int_\Sigma g^{**} \phi\, d\Sigma = \int_Q f(u)\phi\, dQ + \int_\Sigma h(u)\phi\, d\Sigma \tag{46}$$

To pass with the limit in the source terms and obtain (46), we used the facts that f_n is locally Lipschitz $H^{1-\varepsilon}(\Omega) \to L_1(\Omega)$ and $f_n(u) \to f(u)$ in $L_{\tilde{m}}(\Omega)$ [and h_n is locally Lipschitz $H^{1-\varepsilon}(\Omega) \to L_1(\Gamma)$ and $h_n(u) \to h(u)$ in $L_{\tilde{q}}(\Gamma)$, respectively].

Now we want to extend the variational form (46) to all $\phi \in C(0, T; H^1(\Omega)) \cap C^1(0, T; L_2(\Omega)) \cap L_{m+1}(0, T; \Omega)$, $\phi|_\Gamma \in L_{q+1}(0, T; \Gamma)$. In order to do this, we use the density of $H^2(\Omega)$ into $H^1(\Omega)$. Let $\phi \in C(0, T; H^1(\Omega)) \cap C^1(0, T; L_2(\Omega)) \cap L_{m+1}(0, T; \Omega)$, $\phi|_\Gamma \in L_{q+1}(0, T; \Gamma)$. Since $H^2(\Omega)$ is dense in

$H^1(\Omega)$, then there exists $\phi^n \in C(0,T;H^2(\Omega)) \cap C^1(0,T;L_2(\Omega)) \cap L_{m+1}(0,T;\Omega)$, $\phi^n|_\Gamma \in L_{q+1}(0,T;\Gamma)$ such that $\phi^n \to \phi$ in $H^1(\Omega)$. So ϕ^n satisfies (46):

$$\int_Q \big(-u_t\phi_t^n + \nabla u \nabla \phi^n\big)\,dQ + \int_\Sigma \phi^n u\,d\Sigma + \int_\Omega u_t\phi^n\,d\Omega\Big|_0^T + \int_Q g^*\phi^n\,dQ + \int_\Sigma g^{**}\phi^n\,d\Sigma = \int_Q f(u)\phi^n\,dQ + \int_\Sigma h(u)\phi^n\,d\Sigma \tag{47}$$

We pass with the limit as $n \to \infty$, using the bounds for $f(u)$ and $h(u)$ along with the strong convergence of $\phi_n \to \phi$ in $H^1(\Omega)$. We obtain that for any $\phi \in C(0,T;H^1(\Omega)) \cap C^1(0,T;L_2(\Omega)) \cap L_{m+1}(0,T;\Omega)$, $\phi|_\Gamma \in L_{q+1}(0,T;\Gamma)$ we have

$$\int_Q (-u_t\phi_t + \nabla u \nabla \phi)\,dQ + \int_\Sigma \phi u\,d\Sigma + \int_\Omega u_t\phi\,d\Omega\big|_0^T + \int_Q g^*\phi\,dQ + \int_\Sigma g^{**}\phi\,d\Sigma = \int_Q f(u)\phi\,dQ + \int_\Sigma h(u)\phi\,d\Sigma \tag{48}$$

In what follows, the following result proved in [5] will be used.

Lemma 3.1. *Let $U = (u,u_t) \in C_w(0,T;H)$ and $u_t \in L_{m+1}(Q_T)$, $u_t|_\Gamma \in L_{q+1}(\Sigma_T)$ be a solution to the following variational form*

$$-\int_0^T (u_t,\phi_t)_\Omega + \int_0^T (\nabla u,\nabla\phi)_\Omega + \int_0^T (u,\phi)_\Gamma = -(u_t,\phi)_\Omega\big|_0^T + \int_0^T (F,\phi)_\Omega + \langle H,\phi\rangle_\Gamma \tag{49}$$

$\forall \phi \in C(0,T;H^1(\Omega)) \cap C^1(0,T;L_2(\Omega)) \cap L_{m+1}(Q_T), \phi|_\Gamma \in L_{q+1}(\Sigma_T)$, and with $F \in L_{\frac{m}{m+1}}(Q_T)$ and $H \in L_{\frac{q}{q+1}}(\Sigma_T)$. Then, such solution satisfies the energy equality:

$$E_u(t) = E_u(0) + \int_0^T (F,u_t)_\Omega + \langle H, u_t|_\Gamma\rangle\,dt \tag{50}$$

This lemma follows from arguments that are identical to those given in Lemma 3.1 in [5]. Indeed, it suffices to use finite difference approximations of time derivatives and apply the variational form with $\phi = D_h u$. Manipulations with the limit on $h \to 0$ (as in [15]) lead to the final form specified in Lemma 3.1.

Now for $\phi \in C(0,T;H^1(\Omega)) \cap C^1(0,T;L_2(\Omega)) \cap L_{m+1}(0,T;\Omega)$, $\phi|_\Gamma \in L_{q+1}(0,T;\Gamma)$, u^n and u satisfy the variational equalities (39) and (48), respectively. Thus if we set $\tilde u = u_n - u$, then it follows that $\tilde u$ also satisfies the corresponding variational equality (49) with

$$F \equiv g_0\big(u_t^n\big) - g^* + f_n\big(u^n\big) - f(u) \quad \text{and}$$
$$H \equiv g^n\big(u_t^n\big) - g^{**} + h_n\big(u^n\big) - h(u)$$

Both F and H have enough regularity (see Lemma 2.2) in order to apply Lemma 3.1 and obtain the following energy identity for $\tilde u$:

$$\tilde{E}(T) + \int_Q \left[g_0\left(u_t^n(t)\right) - g^*\right]\tilde{u}_t(t)\,dQ + \int_\Sigma \left[g^n\left(u_t^n(t)\right) - g^{**}\right]\tilde{u}_t(t)\,d\Sigma$$
$$= \int_Q \left[f_n\left(u^n(t)\right) - f(u)\right]\tilde{u}_t(t)\,dQ + \int_\Sigma \left[h_n\left(u^n(t)\right) - h(u)\right]\tilde{u}_t(t)\,d\Sigma \tag{51}$$

where $\tilde{E}(t) = \frac{1}{2}(\|\tilde{u}_t(t)\|_\Omega^2 + \|\tilde{u}(t)\|_{1,\Omega}^2)$.

In order to estimate the right-hand side of (51), we shall appeal to two technical lemmas established in [5] and based on a "compensated compactness method" involving dissipativity kernels [10].

Lemma 3.2. *Let u and v be any two weak solutions of* (1) *under the set of hypotheses stated in Assumption* 1.1, *and such that for $t < T$*

$$\left|U(t)\right|_H + \left|V(t)\right|_H + |u_0|_{L_r(\Omega)} + |v_0|_{L_r(\Omega)} \leqslant R$$

Then $\forall\varepsilon > 0$, $\forall\eta > 0$, and $\forall u_0 \in L_r(\Omega)$, $v_0 \in L_r(\Omega)$ there exist constants $0 < C_\varepsilon < \infty$, $0 < C_{\eta,u_0,v_0} < \infty$ such that

$$\int_0^T \int_\Omega \big(f(u) - f(v)\big)(u_t - v_t)\,dx\,dt$$
$$\leqslant C(R)C_\varepsilon C_{\eta,u_0,v_0} \int_0^T \left(\left|u_t(t)\right|_{L_{m+1}(\Omega)} + \left|v_t(t)\right|_{L_{m+1}(\Omega)} + 1\right)\left|U(t) - V(t)\right|_H dt$$
$$+ \left|U(T) - V(T)\right|_H \left(C(R)T^\lambda + \eta + \varepsilon C_{\eta,u_0,v_0}\right) \tag{52}$$

where $\lambda \equiv \frac{m(p-1)}{m+1} > 0$.

Lemma 3.3. *Let u and v be any two weak solutions of* (1) *under the set of hypotheses stated in Assumption* 1.1 *and such that for $t < T$*

$$\left|U(t)\right|_H + \left|V(t)\right|_H + |u_0|_{L_s(\Gamma)} + |v_0|_{L_s(\Gamma)} \leqslant R$$

Then $\forall\varepsilon > 0$, $\forall\eta > 0$ and $\forall u_0, v_0 \in \tilde{L}_s(\Gamma)$, $\exists$ constants $0 < C_\varepsilon < \infty$, $0 < C_{\eta,u_0,v_0} < \infty$ such that

$$\int_0^T \int_\Gamma \big(h(u) - h(v)\big)(u_t|_\Gamma - u_t|_\Gamma)\,d\Gamma\,dt$$
$$\leqslant C(R)C_\varepsilon C_{\eta,u_0,v_0} \int_0^T \left(\left|u_t(t)\right|_{L_{q+1}(\Gamma)} + \left|v_t(t)\right|_{L_{q+1}(\Gamma)} + 1\right)\left|U(t) - V(t)\right|_H dt$$
$$+ \left|U(T) - V(T)\right|_H \left(C(R)T^\gamma + \eta + \varepsilon C_{\eta,u_0,v_0}\right) \tag{53}$$

where $\gamma = \frac{q(k-1)}{q+1} > 0$.

Remark 16. We note that the C^2 continuity of f and h are critical for the validity of Lemma 3.2 and Lemma 3.3. Whether these hypotheses (not used in other parts of the proof) are critical, is an open question.

Returning to (51), we focus on the estimate for its right-hand side. We start with the first term, namely $|\int_Q [f_n(u^n(t)) - f(u)]\tilde{u}_t(t)\,dQ|$. First, triangle inequality gives

$$\left|\int_Q \left[f_n\left(u^n(t)\right) - f(u)\right]\tilde{u}_t(t)\,dQ\right| \leqslant \left|\int_Q \left(f_n\left(u^n(t)\right) - f_n(u)\right)\tilde{u}_t(t)\,dQ\right| + \left|\int_Q \left(f_n(u) - f(u)\right)\tilde{u}_t(t)\,dQ\right| \tag{54}$$

For the first term on the right side of (54), we use Lemma 3.2, applied to each f_n, to obtain that $\forall\varepsilon > 0$, $\forall\eta > 0$ and $\forall u_0 \in L_r(\Omega)$, there exist constants $0 < C_\varepsilon < \infty$, $0 < C_{\eta,u_0} < \infty$ such that

$$\left|\int_Q \left[f_n\left(u^n(t)\right) - f_n(u)\right]\tilde{u}_t(t)\,dQ\right| \leqslant C(R)C_\varepsilon C_{\eta,u_0}\int_0^T \left(\left|u_t^n(t)\right|_{L_{m+1}(\Omega)} + \left|u_t(t)\right|_{L_{m+1}(\Omega)} + 1\right)\tilde{E}(t)\,dt + \tilde{E}(T)\left(C(R)T^\lambda + \eta + \varepsilon C_{\eta,u_0}\right) \tag{55}$$

where $\lambda \equiv \frac{m(p-1)}{m+1} > 0$ and R denotes an upper bound for the a priori regularity assumed to hold for weak solutions. We want to remark here that the constants involved in (55) are uniform in n, due to Claim 1 for f_n.

Similarly, using the triangle inequality for the second term on the right side of (54), we obtain

$$\left|\int_\Sigma \left[h_n\left(u^n(t)\right) - h(u)\right]\tilde{u}_t(t)\,d\Sigma\right| \leqslant \left|\int_\Sigma \left[h_n\left(u^n(t)\right) - h_n(u)\right]\cdot\tilde{u}_t(t)\,d\Sigma\right| + \left|\int_Q \left[h_n(u) - h(u)\right]\cdot\tilde{u}_t(t)\,d\Sigma\right| \tag{56}$$

For the first term on the right side of (56), since $q \geqslant 1$, we use Lemma 3.3, applied to each h_n and obtain that $\forall\varepsilon > 0$, $\forall\eta > 0$ and $\forall u_0 \in \tilde{L}_s(\Gamma)$, $\exists$ constants $0 < C_\varepsilon < \infty$, $0 < C_{\eta,u_0} < \infty$ such that

$$\left|\int_\Sigma \left[h_n\left(u^n(t)\right) - h_n(u)\right]\cdot\tilde{u}_t(t)\,d\Sigma\right| \leqslant C(R)C_\varepsilon C_{\eta,u_0}\int_0^T \left(\left|u_t^n(t)\right|_{L_{q+1}(\Gamma)} + \left|u_t(t)\right|_{L_{q+1}(\Gamma)} + 1\right)\tilde{E}(t)\,dt + \tilde{E}(T)\left(C(R)T^\gamma + \eta + \varepsilon C_{\eta,u_0}\right) \tag{57}$$

where $\gamma = \frac{q(k-1)}{q+1} > 0$. Again, we want to point out that the constants involved in (57) do not depend on n, due to Claim 1 for h_n.

In order to shorten the exposition, let $D_i = \int_Q [g_0(u_t^n(t)) - g^*]\tilde{u}_t(t)\,dQ$ and $D_b = \int_\Sigma [g^n(u_t^n(t)) - g^{**}]\tilde{u}_t(t)\,d\Sigma$.

Combining (51) with (55) and (57) we obtain that $\forall\varepsilon > 0$, $\forall\eta > 0$ and $\forall u_0, v_0 \in L_r(\Omega) \cap \tilde{L}_s(\Gamma)$, $\exists$ constants $0 < C_\varepsilon < \infty$, $0 < C_{\eta,u_0,v_0} < \infty$ such that

$$\begin{aligned}
&\tilde{E}(t) + \int_Q \big[g_0\big(u_t^n(t)\big) - g_0(u_t)\big]\tilde{u}_t(t)\,dQ + \int_\Sigma \big[g\big(u_t^n(t)\big) - g(u_t)\big]\tilde{u}_t(t)\,d\Sigma \\
&\quad \leqslant 2\big(C(R)\big(T^\lambda + T^\gamma\big) + \eta + \varepsilon C_{\eta,u_0,v_0}\big)\tilde{E}(t) \\
&\qquad + \int_Q \big|f_n(u) - f(u)\big| \cdot \big|\tilde{u}_t(t)\big|\,dQ + \int_\Sigma \big|h_n(u) - h(u)\big| \cdot \big|\tilde{u}_t(t)\big|\,d\Sigma \\
&\qquad - \int_Q \big[g_0\big(u_t(t)\big) - g^*\big]\tilde{u}_t(t)\,dQ - \int_\Sigma \big[g\big(u_t(t)\big) - g^{**}\big]\tilde{u}_t(t)\,d\Sigma - \int_\Sigma \frac{1}{n}u_t^n(t)\tilde{u}_t(t)\,d\Sigma \\
&\qquad + C(R)C_\varepsilon C_{\eta,u_0,v_0}\int_0^T \big(\big|u_t(t)\big|_{L_{m+1}(\Omega)} + \big|u_t^n(t)\big|_{L_{m+1}(\Omega)} \\
&\qquad + \big|u_t(t)\big|_{L_{q+1}(\Gamma)} + \big|u_t^n(t)\big|_{L_{q+1}(\Gamma)} + 2\big)\tilde{E}(t)\,dt
\end{aligned} \tag{58}$$

where $\lambda = \frac{m(p-1)}{m+1} > 0$ and $\gamma = \frac{q(k-1)}{q+1} > 0$. $C(R)$ in (58) denotes a function bounded for bounded arguments of R.

In (58), we choose ε, η and T such that $C(R)(T^\lambda + T^\gamma) + \eta + \varepsilon C_{\eta,u_0,v_0} < 1$ (it is enough to look at a small interval for T, since the process can be reiterated), we use the fact that g_0 and g are monotone, we apply Gronwall's Inequality with L_1 kernel ($u_t, v_t \in L_{m+1}(0,T;\Omega) \cap L_{q+1}(0,T;\Gamma)$, $u_t^n, v_t^n \in L_{m+1}(0,T;\Omega) \cap L_{q+1}(0,T;\Gamma)$) and obtain that

$$\begin{aligned}
\tilde{E}(t) &\leqslant \bigg(\int_Q \big|f_n(u) - f(u)\big| \cdot \big|\tilde{u}_t(t)\big|\,dQ + \int_\Sigma \big|h_n(u) - h(u)\big| \cdot \big|\tilde{u}_t(t)\big|\,d\Sigma \\
&\quad - \int_Q \big[g_0\big(u_t(t)\big) - g^*\big]\tilde{u}_t(t)\,dQ - \int_\Sigma \big[g\big(u_t(t)\big) - g^{**}\big]\tilde{u}_t(t)\,d\Sigma - \int_\Sigma \frac{1}{n}u_t^n(t)\tilde{u}_t(t)\,d\Sigma\bigg) \\
&\quad \times e^{(K\int_0^T (|u_t(t)|_{L_{m+1}(\Omega)} + |u_t^n(t)|_{L_{m+1}(\Omega)} + |u_t(t)|_{L_{q+1}(\Gamma)} + |u_t^n(t)|_{L_{q+1}(\Gamma)} + 2)\,dt)}
\end{aligned} \tag{59}$$

Now we take $\lim_{n\to\infty}$ on both sides of (59) and note that:

- From Claim 4, we know that $f_n(u) \to f(u)$ in $L_{\frac{m+1}{m}}(\Omega)$, as $n \to \infty$, for all $u \in H^1(\Omega)$.
- Similarly, we have that $h_n(u) \to h(u)$ in $L_{\frac{q+1}{q}}(\Gamma)$, as $n \to \infty$, for all $u \in H^1(\Omega)$.
- $\tilde{u}_t \to 0$ weakly in $L_{m+1}(0,T;\Omega)$ and $\tilde{u}_t|_\Gamma \to 0$ weakly in $L_{q+1}(0,T;\Gamma)$.
- $u_t^n(t)\tilde{u}_t(t) \in L_1(\Gamma)$ for $q \geqslant 1$ (since $u_t(t)$ and $u_t^n(t) \in L_{q+1}(\Gamma) \subset L_{\frac{q+1}{q}}(\Gamma)$). Thus $\int_\Sigma \frac{1}{n}u_t^n(t)\tilde{u}_t(t)\,d\Sigma \to 0$ as $n \to \infty$.

Thus we obtain:

$$\limsup_{n\to\infty} D_i = 0 \quad \text{and} \quad \limsup_{n\to\infty} D_b = 0$$

Now we are in position to use Barbu's lemma (Lemma 1.3 in [2]) and conclude that $g^* = g_0(u_t)$ and $g^{**} = g(u_t)$. This ends the proof for local existence of solutions.

Uniqueness of solution, along with energy *identity*, follows from substituting the difference of solutions back into Lemma 3.1 and from the estimates reported in Lemma 3.2 and Lemma 3.3. The arguments are similar to the ones given above with the complete details reported in [5].

3.3. Proof of Theorem 1.3 – Hadamard well-posedness

The ground work for the proof of Theorem 1.3 has been laid down in the estimates of the previous section. Indeed, the proof follows from a mild modification of the estimates that we obtained in Theorem 1.2. Let $U(t)$ and $U^n(t)$ be the weak solutions corresponding to the initial data U_0 and U_0^n, respectively. From Theorem 1.2, we have that

$$\begin{aligned} u, u^n &\in C\big(0, T_{max}; H^1(\Omega)\big), \qquad u_t, u_t^n \in C\big(0, T_{max}; L_2(\Omega)\big) \\ u_t, u_t^n &\in L_{m+1}(0, T_{max}; \Omega), \qquad u_t, u_t^n \in L_{q+1}(0, T_{max}; \Gamma) \end{aligned} \tag{60}$$

where the bounds on the respective norms are bounded by some constant R uniformly in n. Our goal is to prove that $U_n \to U$ in $C(0, T; H)$. In order to do that, we use the energy identity applied to the difference of the two solutions $\tilde{u}(t) = u^n(t) - u(t)$, where each solution is a *weak solution* corresponding to different initial data. By using the energy identity in Lemma 3.1, additional L_p, L_q regularity and the fact that the damping terms are monotone increasing, we obtain:

$$\begin{aligned} &\tilde{E}(T) + \int_Q \big[g_0\big(u_t^n(t)\big) - g_0\big(u_t(t)\big)\big]\tilde{u}_t(t)\,dQ + \int_\Sigma \big[g\big(u_t^n(t)\big) - g\big(u_t(t)\big)\big]\tilde{u}_t(t)\,d\Sigma \\ &\quad = \tilde{E}(0) + \int_Q \big[f\big(u^n(t)\big) - f(u)\big]\tilde{u}_t(t)\,dQ + \int_\Sigma \big[h\big(u^n(t)\big) - h(u)\big]\tilde{u}_t(t)\,d\Sigma \\ &\quad \Rightarrow \tilde{E}(T) \leqslant \tilde{E}(0) + \int_Q \big[f\big(u^n(t)\big) - f(u)\big]\tilde{u}_t(t)\,dQ + \int_\Sigma \big[h\big(u^n(t)\big) - h(u)\big]\tilde{u}_t(t)\,d\Sigma \end{aligned} \tag{61}$$

Now we need to estimate the terms on the right side that involve the sources. We start with $\int_Q [f(u^n(t)) - f(u)]\tilde{u}_t(t)\,dQ$. By using Lemma 3.2, and accounting for the fact that $p < 5$ we obtain

$$\begin{aligned} &\left|\int_Q \big[f\big(u^n(t)\big) - f(u)\big]\tilde{u}_t(t)\,dQ\right| \\ &\quad \leqslant \varepsilon \tilde{E}(t) + C_1(\varepsilon, R, T)\int_0^T \tilde{E}(t)\big[\big\|u_t(t)\big\|_{L_{m+1}} + \big\|u_t^n(t)\big\|_{L_{m+1}} + 1\big]\,dt \end{aligned} \tag{62}$$

Similarly, using a modification of (57) due to $k < 3$ and applying Lemma 3.3 (with $q \geqslant 1$), we obtain our estimate for $\int_\Sigma [h(u^n(t)) - h(u)]\tilde{u}_t(t)\,d\Sigma$:

$$\begin{aligned} &\left|\int_Q \big[h\big(u^n(t)\big) - h(u)\big]\tilde{u}_t(t)\,dQ\right| \\ &\quad \leqslant \varepsilon \tilde{E}(t) + C_2(\varepsilon, R, T)\int_0^T \tilde{E}(t)\big[\big|u_t(t)\big|_{L_{q+1}} + \big|u_t^n(t)\big|_{L_{q+1}} + 1\big]\,dt \end{aligned} \tag{63}$$

For the sublinear boundary damping ($q < 1$) we use the following estimate proved in Lemma 4.3 [5]:

Lemma 3.4. *Let (A_h) in Assumption* 1.1 *be satisfied and let $q < 1$, $1 \leqslant k \leqslant \frac{4q}{q+1}$, $g'(s) \geqslant m_q|s|^{q-1}$. Then for every $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that*

$$\left|\int_Q \left[h\left(u^n(t)\right) - h(u)\right]\tilde{u}_t(t)\right| \leqslant C_\epsilon \int_0^T \left(\left|u_t^n(t)\right|_{L_{q+1}(\Gamma)} + \left|u_t(t)\right|_{L_{q+1}(\Gamma)} + 1\right)\tilde{E}(t)\,dt$$

$$+ \epsilon \int_0^T \int_\Gamma \left[g\left(u_t^n\right) - g(u_t), u_t^n(t) - u_t(t)\right) d\Gamma\,dt \tag{64}$$

Combining (61) with (62) and (63) (for $q \geqslant 1$), and Lemma 3.4 (for $q < 1$), we obtain

$$\tilde{E}(T) \leqslant \tilde{E}(0) + \varepsilon \tilde{E}(t)$$

$$+ C_\epsilon(R, T) \int_0^T \tilde{E}(t)\left[\left\|u_t(t)\right\|_{L_{m+1}} + \left\|u_t^n(t)\right\|_{L_{m+1}} + \left|u_t(t)\right|_{L_{q+1}} + \left|u_t^n(t)\right|_{L_{q+1}} + 1\right] dt \tag{65}$$

If we let $s(t) \equiv [\|u_t(t)\|_{m+1} + \|u_t^n(t)\|_{m+1} + |u_t(t)|_{q+1} + |u_t^n(t)|_{q+1} + 1] \in L_1(R)$ and we choose ε small enough, then Gronwall's inequality with L_1-kernel yields:

$$\tilde{E}(T) \leqslant C(R)\tilde{E}(0)e^{\int_0^T s(t)\,dt} \leqslant C(T, R)\tilde{E}(0), \quad T \leqslant T_{max}$$

This ends the proof of Hadamard well-posedness for supercritical sources ($p < 5$ and $k < 3$). Moreover, this also shows uniqueness of solution in the supercritical case. For super-supercritical sources, the proof of uniqueness is more involved and we refer the reader to [5].

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