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# A mathematical modelling of imbibition phenomenon in inclined homogenous porous media during oil recovery process☆

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OHAM

**Summary** The approximate solution of imbibition phenomenon governed by non-linear partial differential equation is discussed in the present paper. Primary oil recovery process due to natural soil pressure, but in the secondary oil recovery process water flooding plays an important role. When water is injected in the injection well for recovering the reaming oil after primary oil recovery process, it comes to contact with the native oil and at that time the imbibition phenomenon occurs due to different viscosity. For the mathematical modelling, we consider the homogeneous porous medium and optimal homotopy analysis method has been used to solve the partial differential equation governed by it. The graphical representation of the solution is given by MATHEMATICA and physically interpreted.


## Introduction

In this research paper, we have discussed the imbibition phenomenon in the horizontal direction with inclined homogeneous porous media (Bourbiaux and Kalaydjian, 1990; Chen, 2007). When porous medium is already filled with some fluid and when it come contact with another fluid then there is a spontaneous flow of wetting fluid into the porous medium and counter flow of native fluid from the porous medium. This process is called imbibition. In secondary oil recovery process during water flooding, instead of regular displacement of the whole front (common interface) the protuberances will occur with irregular fingers in size and shape. Injection of water is the principal form of the secondary oil recovery because supply of water is often plentiful, inexpensive, and it is usually more stable frontal displacement than another form of secondary oil recovery. Due to water injection, oil will displace towards the production well in this way remaining oil can recover in secondary oil recovery process. Cross section shows the distribution of

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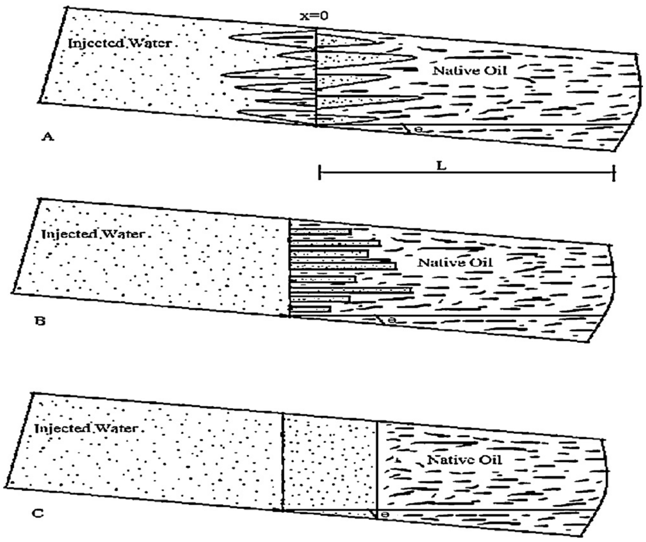


**Figure 1** (A) Representation of imbibition in an inclined homogenous porous matrix. (B) Schematic representation of fingers for imbibition phenomenon in inclined homogeneous porous matrix. (C) Average cross sectional area occupied by small fingers as rectangle for imbibition phenomenon.

oil and water before and after the water has displaced the oil in inclined porous matrix. The stability of water flood depends on the mobility ratio between oil and water, heterogeneity of the porous medium and the interaction of several forces (Chen, 2007). Imbibition phenomenon may occur in both miscible and immiscible processes and originates on the interface between two fluids (oil and water).

## Statement of the problem

For the mathematical modelling, we choose a piece of cylindrical porous matrix from large natural field area and take vertical cross-sectional area of this small finite incline cylindrical porous matrix as a rectangle. It is incline at small angle with negative direction of $x$-axis and its three sides are impermeable expect for one as shown in Fig. 1A. To find the area occupied by water in the form of saturation of water in different irregular fingers, Schidegger and Johnson (Scheidegger, 1958) suggested schematic presentation of fingers by replacing irregular fingers by rectangular fingers for mathematical study as shown in Fig. 1B. But still it is difficult to find the saturation held in schematic fingers. Hence taking average cross-sectional area occupied by schematic finger is considered as saturation of injected water which is rectangular shape as shown by Fig. 1C.

## Mathematical formation

In secondary oil recovery process, during water flooding, we consider injected fluid as water and native fluid as oil. The seepage velocities of the fluids water and oil are given by Darcy's law as followed by (Bear, 1972)

$$V_i = -\frac{k_i}{\mu_i} K \left( \frac{\partial p_i}{\partial x} + \rho_i g \sin\theta \right) \tag{1}$$

$$V_n = -\frac{k_n}{\mu_n} K \left( \frac{\partial p_n}{\partial x} + \rho_n g \sin\theta \right) \tag{2}$$

where $k_i$ and $k_n$ are the relative permeabilities of fluids water and oil respectively, $K$ is permeability of homogenous porous medium, $\mu_i$ and $\mu_n$ are the constant kinematic viscosities of fluids water and oil respectively and $p_i$ and $p_n$ are the pressure of water and oil. In order to prepare the mathematical model for imbibition phenomenon, we assume that the densities of both injected and native fluids are constant. The continuity equation for these two fluids are given as

$$\varepsilon \frac{\partial S_i}{\partial t} + \frac{\partial v_i}{\partial x} = 0 \tag{3}$$

$$\varepsilon \frac{\partial S_n}{\partial t} + \frac{\partial v_n}{\partial x} = 0 \tag{4}$$

where $\varepsilon$ is the porosity and $S_i$ and $S_n$ are the functions of saturations.

Also we consider the porous matrix is fully saturated, i.e. the saturation of injected water and native oil is unity.

$$S_i + S_n = 1 \tag{5}$$

The counter-current imbibitions condition at the common interface,

$$v_i + v_n = 0 \tag{6}$$

The relation between relative permeabilities and phase saturation is given by (Scheidegger, 1958)

$$k_i = S_i, \quad k_n = 1 - \alpha S_i \tag{7}$$

where $\alpha = 1.11$ (Patel et al., 2013)

The relation between capillary pressure and pressure of the native and injected fluids are given by

$$p_c = p_n - p_i \tag{8}$$

Also it is observed the capillary pressure is linearly dependent on the saturation of injected water (Mehta, 1977)

$$p_c(S_i) = -\beta S_i; \quad \beta \text{ is constant} \tag{9}$$

Use (1), (2) into (6) and use (8)

$$\frac{\partial p_i}{\partial x} = \frac{-(k_n/\mu_n)K}{((k_i/\mu_i)K + (k_n/\mu_n)K)} \frac{\partial p_c}{\partial x} - \frac{((k_i/\mu_i)K\rho_i + (k_n/\mu_n)K\rho_n)g\sin\theta}{((k_i/\mu_i)K + (k_n/\mu_n)K)} \tag{10}$$

Use (10) into (1),

$$V_i = K\frac{k_i}{\mu_i}\frac{\partial p_c}{\partial x} + K\frac{k_i}{\mu_i}\rho_n g \sin\theta \tag{11}$$

where $\frac{(k_i/\mu_i)(k_n/\mu_n)}{(k_i/\mu_i)+(k_n/\mu_n)} \cong \frac{k_i}{\mu_i}$ (Scheidegger, 1958)

Putting the value of (11) into (3), we get,

$$\varepsilon \frac{\partial S_i}{\partial t} + \frac{\partial}{\partial x}\left( K\frac{k_i}{\mu_i}\frac{\partial p_c}{\partial x} + K\frac{k_i}{\mu_i}\rho_n g \sin\theta \right) = 0 \tag{12}$$

Doing some simplifications and using Eqs. (7) and (8)

$$\varepsilon \frac{\partial S_i}{\partial t} + \frac{K\beta}{\mu_i}\frac{\partial}{\partial x}\left( S_i \frac{\partial S_i}{\partial x} + S_i \rho_n g \sin\theta \right) = 0 \tag{13}$$

Use dimensionless parameters

$$X = \frac{x}{L}, \qquad T = \frac{K\beta}{\varepsilon\mu_i L^2}t \tag{14}$$

$$\frac{\partial S_i}{\partial T} + \frac{\partial}{\partial X}\left(S_i\frac{\partial S_i}{\partial X}\right) + \gamma\frac{\partial}{\partial X}(S_i) = 0 \tag{15}$$

where $\gamma = L\rho_n g\sin\theta$

The appropriate initial and boundary conditions are given as

$$\begin{aligned} &S_i(X,0) = S_{ic}(1-e^{-X}); \quad 0 \le X \le 1 \\ &S_i(0,T) = S_{ic}; \qquad X = 0\&T > 0 \end{aligned} \tag{16}$$

## Solution by OHAM

According to OHAM first we construct the zeroth order deformation equation as (Liao, 2012; Pathak and Singh, 2015)

$$(1-q)\,£\,[S_i(X,T;q) - S_{i0}(X,T)] = c_0 q\mathcal{N}[S_i(X,T;q)] \tag{17}$$

where $q \in [0, 1]$ is the embedding parameter, $c_0 \neq 0$ is a convergence control parameter. We choose $£ = \partial/\partial T$ (Baxter et al., 2014) is an auxiliary linear operator and $S_{i0}(X, T) = 0.1e^{-X} + 0.5TX$ is an initial guess of $S_i(X, T)$ (Patel, 2014). According to OHAM expanding $S(X, T; q)$ in Maclaurin series with respect to $q$, then the corresponding $m$th order deformation equation is given by

$$£\,[S_{im}(X,T) - \chi_m S_{i(m-1)}(X,T)] = c_0\delta_m[S_{i(m-1)}(X,T)], \tag{18}$$

where

$$\delta_m[S_{i(m-1)}(X,T)] = \frac{\partial}{\partial t}(S_{i(m-1)}) - \sum_{r=0}^{m-1} S_{ir}(S_{i(m-1-r)})_{XX} - (S_{i(m-1)})_X^2 - \gamma(S_{i(m-1)})_X$$

Taking inverse operator

$$S_{im}(X,T) = \chi_m S_{i(m-1)}(X,T) + c_0 £^{-1}\delta_m[S_{i(m-1)}(X,T)], \tag{19}$$

where

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}$$

To, find the optimal value of convergence $c_0$ we use Yabushita's approach (Yabushita et al., 2007) by means of finding the square residual error

$$E_m(c_0) = \frac{1}{(M+1)(N+1)}\sum_{i=0}^{M}\sum_{j=0}^{N}\left\{\mathcal{N}\left[\sum_{n=0}^{m} S_{in}\left(\frac{i}{M},\frac{j}{N}\right)\right]^2\right\} \tag{20}$$

The optimal value of convergence control parameter $c_0 = -0.0018239062890736024$ with minimum square residual error $E_5 = 2.65E-01$ at fifth order approximation, which can be notice by Fig. 2.

Using the optimal value of convergence control parameter we have the following approximations for different

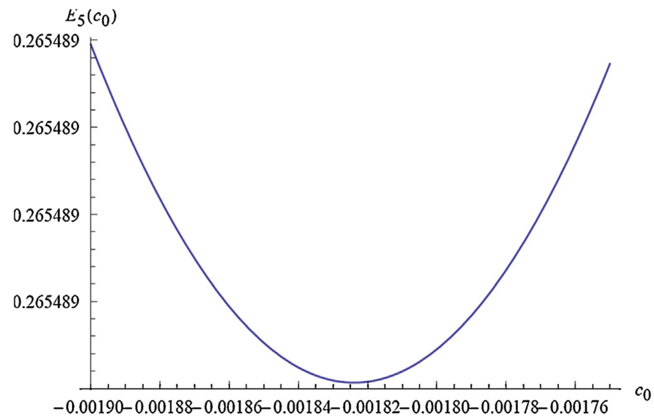


**Figure 2** Square residual error at 5th-order approximation.

values of $m$

$$S_1(X,T) = 0.0017(-0.076e^X T - 0.034e^{2X}T - 0.19T^2 - 0.085e^X T^2 - 0.14T^3 + 0.5TX - 0.0425e^X T^2 X)$$

$$\begin{aligned} S_{i2}(X,T) = {} & 0.0017(-0.076e^X T - 0.034e^{2X}T - 0.19T^2 \\ & - 0.085e^X T^2 - 0.14T^3 + 0.5TX - 0.0425e^X T^2 X) \\ & + 0.0017(-0.0001e^X T - 0.00006e^{2X}T - 0.0007T^2 \\ & - 0.0001e^X T^2 + 0.00007e^{2X}T^2 + 0.00003e^{3X}T^2 \\ & - 0.0002T^3 + 0.00008e^X T^3 + 0.00003e^{2X}T^3 \\ & - 1.7\times 10^{-8}e^{3X}T^3 - 7.9\times 10^{-9}e^{4X}T^3 + 0.00001e^X T^4 \\ & - 2.5\times 10^{-8}e^{2X}T^4 - 2.2\times 10^{-8}e^{3X}T^4 \\ & - 1.7\times 10^{-8}e^{2X}T^5 + 0.0009TX - 0.0001e^X T^2 X \\ & + 0.00006e^X T + 0.00008e^{2X}T^3 X + 0.00006e^X T^4 X \\ & - 8.3\times 10^{-9}e^{2X}T^4 X - 7.4\times 10^{-9}e^{3X}T^4 X \\ & - 1.1\times 10^{-8}e^{2X}T^5 X + 0.00002e^X T^4 X^2 \\ & - 1.9\times 10^{-9}e^{2X}T^5 X^2) \end{aligned}$$

and so on

Adding these approximations up to fifth order approximation including initial guess we get, $S_i(X, T) = 0.1e^{-X} + 0.5TX + 0.0017(-0.076e^X T - 0.034e^{2X}T - 0.19T^2 - 0.085e^X T^2 - 0.14T^3 + 0.5TX - 0.0425e^X T^2 X) + 0.0017(-0.0001e^X T - 0.00006e^{2X}T - 0.0007T^2 - 0.0001e^X T^2 + 0.00007e^{2X}T^2 + 0.00003e^{3X}T^2 - 0.0002T^3 + 0.00008e^X T^3 + 0.00003e^{2X}T^3 - 1.7\times 10^{-8}e^{3X}T^3 - 7.9\times 10^{-9}e^{4X}T^3 + 0.00001e^X T^4 - 2.5\times 10^{-8}e^{2X}T^4 - 2.2\times 10^{-8}e^{3X}T^4 - 1.7\times 10^{-8}e^{2X}T^5 + 0.0009TX - 0.0001e^X T^2 X + 0.00006e^X T + 0.00008e^{2X}T^3 X + 0.00006e^X T^4 X - 8.3\times 10^{-9}e^{2X}T^4 X - 7.4\times 10^{-9}e^{3X}T^4 X - 1.1\times 10^{-8}e^{2X}T^5 X + 0.00002e^X T^4 X^2 - 1.9\times 10^{-9}e^{2X}T^5 X^2) + \cdots$

## Numerical and graphical representation:

Table 1

**Table 1** Numerical result for the imbibition phenomenon in homogeneous porous media with $\gamma = -0.7609$.

| $X$ | $T=0.0$ | $T=0.1$ | $T=0.2$ | $T=0.3$ | $T=0.4$ | $T=0.5$ | $T=0.6$ | $T=0.7$ | $T=0.8$ | $T=0.9$ | $T=1.0$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.1000 | 0.1001 | 0.1002 | 0.1004 | 0.1007 | 0.1010 | 0.1013 | 0.1018 | 0.1023 | 0.1029 | 0.1035 |
| 0.1 | 0.1105 | 0.1156 | 0.1207 | 0.1259 | 0.1311 | 0.1364 | 0.1418 | 0.1472 | 0.1527 | 0.1583 | 0.1639 |
| 0.2 | 0.1221 | 0.1322 | 0.1423 | 0.1525 | 0.1627 | 0.1730 | 0.1833 | 0.1937 | 0.2042 | 0.2148 | 0.2255 |
| 0.3 | 0.1350 | 0.1500 | 0.1651 | 0.1803 | 0.1955 | 0.2108 | 0.2261 | 0.2415 | 0.2570 | 0.2726 | 0.2883 |
| 0.4 | 0.1492 | 0.1692 | 0.1893 | 0.2095 | 0.2297 | 0.2499 | 0.2703 | 0.2907 | 0.3112 | 0.3318 | 0.3525 |
| 0.5 | 0.1649 | 0.1899 | 0.2150 | 0.2401 | 0.2653 | 0.2906 | 0.3159 | 0.3414 | 0.3669 | 0.3925 | 0.4182 |
| 0.6 | 0.1822 | 0.2122 | 0.2423 | 0.2725 | 0.3027 | 0.3329 | 0.3633 | 0.3937 | 0.4243 | 0.4549 | 0.4856 |
| 0.7 | 0.2014 | 0.2364 | 0.2715 | 0.3066 | 0.3418 | 0.3771 | 0.4125 | 0.4480 | 0.4835 | 0.5192 | 0.5549 |
| 0.8 | 0.2226 | 0.2626 | 0.3027 | 0.3428 | 0.3831 | 0.4234 | 0.4638 | 0.5042 | 0.5448 | 0.5855 | 0.6263 |
| 0.9 | 0.2460 | 0.2910 | 0.3361 | 0.3813 | 0.4265 | 0.4719 | 0.5173 | 0.5628 | 0.6084 | 0.6541 | 0.7000 |
| 1 | 0.2718 | 0.3219 | 0.3720 | 0.4222 | 0.4725 | 0.5228 | 0.5733 | 0.6239 | 0.6745 | 0.7253 | 0.7762 |

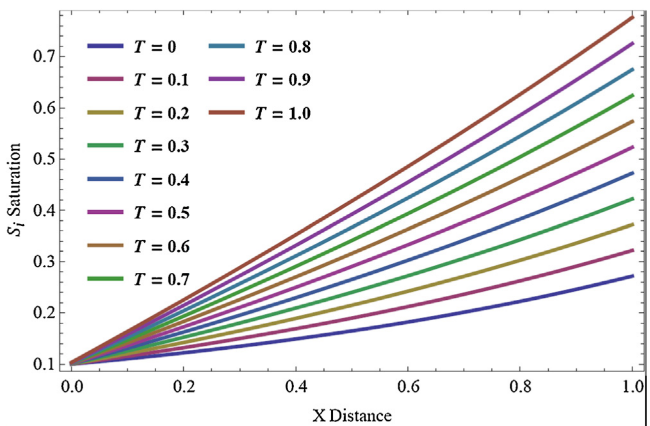


**Figure 3** Saturation of injected water $S_i(X, T)$.

## Conclusion

We have discussed an imbibitions phenomenon in homogeneous porous media under certain assumptions and have obtained the optimal homotopy series solution for the pde governed by the saturation of injected water. The optimal value of convergence control parameter $c_0 = -0.0018239062890736024$ is successfully obtained by reducing the square residual error. From Fig. 3 we conclude that the saturation of injected water is increase with distance and time increases, which is consistent with the physical situation.

## Conflict of interest

The author declares that there is no conflict of interest regarding to the publication.

## References


Baxter, M., Van Gorder, R.A., Vajravelu, K., 2014. On the choice of auxiliary linear operator in the optimal homotopy analysis of the Cahn—Hilliard initial value problem. Numerical Algorithms 66 (2), 269—298.

Bear, J., 1972. Dynamics of Fluids in Porous Media. American Elsevier, New York.

Bourbiaux, B.J., Kalaydjian, F.J., 1990. Experimental study of cocurrent and countercurrent flows in natural porous media. SPE Reserv. Eng. 5 (3), 361—368.

Chen, Z., 2007. Reservoir Simulation: Mathematical Techniques in Oil Recovery, vol. 77. SIAM.

Liao, S., 2012. Homotopy Analysis Method in Nonlinear Differential Equations. Higher Education Press/Springer, Heidelberg/New York/Beijing.

Mehta, M.N., (Ph.D. thesis) 1977. Asymptotic expansion of fluid flow through porous media. South Gujarat University, Surat, India.

Patel, K.K., (Ph.D. thesis) 2014. Approximate solution of nonlinear boundary value problems arising in fluid flow through porous media by homotopy analysis method. S. V. National Institute of Technology, Surat, India.

Patel, K.R., Mehta, M.N., Patel, T.R., 2013. A mathematical model of imbibition phenomenon in heterogeneous porous media during secondary oil recovery process. Appl. Math. Model. 37 (5), 2933—2942.

Pathak, S., Singh, T., 2015. Optimal homotopy analysis methods for solving the linear and nonlinear Fokker—Planck equations. Br. J. Math. Comput. Sci. 7 (3), 209—217, SCIENCEDOMAIN International.

Scheidegger, A.E., 1958. The physics of flow through porous media. Soil Sci. 86 (6), 355.

Yabushita, K., Yamashita, M., Tsuboi, K., 2007. An analytic solution of projectile motion with the quadratic resistance law using the homotopy analysis method. J. Phys. A, 8403—8416.