Embedding paratopological groups into topological products

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ABSTRACT

We show that a Hausdorff paratopological group $G$ admits a topological embedding as a subgroup into a topological product of Hausdorff first-countable (second-countable) paratopological groups if and only if $G$ is $\omega$-balanced (totally $\omega$-narrow) and the Hausdorff number of $G$ is countable, i.e., for every neighbourhood $U$ of the neutral element $e$ of $G$ there exists a countable family $\gamma$ of neighbourhoods of $e$ such that $\bigcap_{V \in \gamma}VV^{-1} \subseteq U$.

Similarly, we prove that a regular paratopological group $G$ can be topologically embedded as a subgroup into a topological product of regular first-countable (second-countable) paratopological groups if and only if $G$ is $\omega$-balanced (totally $\omega$-narrow) and the index of regularity of $G$ is countable.

As a by-product, we show that a regular totally $\omega$-narrow paratopological group with countable index of regularity is Tychonoff.

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1. Introduction

In this article all spaces are assumed to satisfy the Hausdorff axiom of separation. Following N. Bourbaki (see [4]), we say that a group $G$ with a topology is a paratopological group if the multiplication on $G$ is jointly continuous. Given a topological property $P$, we say that a paratopological (or topological) group $G$ is projectively $P$ if for every neighbourhood $U$ of the neutral element in $G$, there exists a continuous homomorphism $p : G \to H$ onto a paratopological (resp., topological) group $H$ with property $P$ such that $p^{-1}(V) \subseteq U$, for some neighbourhood $V$ of the neutral element in $H$.

Therefore, a paratopological group $G$ is projectively Hausdorff first-countable if for every neighbourhood $U$ of the neutral element in $G$, there exists a continuous homomorphism $p : G \to H$ onto a Hausdorff first-countable paratopological group $H$ such that $p^{-1}(V) \subseteq U$, for some neighbourhood $V$ of the neutral element in $H$. Similarly, the class of projectively regular first-countable paratopological groups is introduced. It is worth mentioning that, for topological groups, there is no need in mentioning axioms of separation since $T_0$ is equivalent to $T_{3.5}$ in this case.

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According to Katz’s theorem in [8], a topological group \( G \) is projectively first-countable if and only if \( G \) is \( \omega \)-balanced (equivalently, \( G \) has a quasi-invariant basis). This means that for every neighbourhood \( U \) of the neutral element \( e \) in \( G \), one can find a countable family \( \gamma \) of neighbourhoods of \( e \) such that for each \( x \in G \) there exists \( V \in \gamma \) satisfying \( xVx^{-1} \subseteq U \). Such a family \( \gamma \) is usually called subordinate to \( U \). This definition applies to paratopological and semitopological groups as well. Clearly, every Abelian topological (or paratopological) group is \( \omega \)-balanced. It is also clear that a paratopological group is projectively Hausdorff (regular) first-countable if and only if it is topologically isomorphic to a subgroup of the product of a family of Hausdorff (regular) first-countable paratopological groups.

Unlike topological groups, an \( \omega \)-balanced paratopological group \( G \) can fail to be projectively first-countable, even if \( G \) is completely regular (see Example 2.9). Here we give an internal characterization of projectively Hausdorff first-countable paratopological groups that involves a new cardinal function called the Hausdorff number of a paratopological group (see Theorem 2.7).

The classes of projectively Hausdorff (regular) second-countable paratopological groups are considered as well. Again, a paratopological group is projectively Hausdorff (regular) second-countable if and only if it admits a homeomorphic embedding as a subgroup into a product of Hausdorff (regular) second-countable paratopological groups. Projectively second-countable paratopological groups are completely described by I. Guran in [6]—these are \( \omega \)-narrow topological groups (see also [12, Theorem 3.4]).

Evidently, every projectively Hausdorff (regular) second-countable paratopological group is projectively Hausdorff (regular) first-countable, but not vice versa—the Sorgenfrey line is a counterexample in both cases. We show in Lemma 3.7, (see also [12, Theorem 3.4]).

The case of regular paratopological groups and their embeddings into products of regular first-countable (or second-countable) paratopological groups differs substantially from the Hausdorff case. Again, we present in Theorems 3.6 and 3.8 an internal characterization of projectively regular first-countable and projectively regular second-countable paratopological groups by making use of the index of regularity of a paratopological group. As a by-product, we deduce that a regular totally \( \omega \)-narrow paratopological group with countable index of regularity is completely regular (see Corollary 3.9).

It should be mentioned that Theorems 2.8 and 3.8 generalize several results from [10], where some sufficient conditions were given for a paratopological group to be projectively Hausdorff (regular) second-countable. In fact, the technique applied here is a natural refinement of the methods from [10].

### 1.1. Notation and terminology

An abstract group \( G \) endowed with a topology \( \tau \) is called a semitopological group if, for each \( a \in G \), the left translation \( l_a: G \to G \) and the right translation \( r_a: G \to G \) are continuous. As usual, the translations \( l_a \) and \( r_a \) are defined by \( l_a(x) = ax \) and \( r_a(x) = xa \), for every \( x \in G \). Since \( l_a^{-1} \circ l_b = id_G \) and \( r_a^{-1} \circ r_b = id_G \), the mappings \( l_a \) and \( r_a \) are homeomorphisms. Let \( G \) be an abstract group with multiplication \( m \), where \( m(x, y) = xy \) for \( x, y \in G \). Given a topology \( \tau \) on \( G \), the pair \((G, \tau)\) is called a paratopological group if the mapping \( m: G \times G \to G \) is continuous. Every paratopological group is semitopological, but not vice versa. Unlike topological groups, a Hausdorff paratopological group need not be regular. It is an old problem whether every regular paratopological group is Tychonoff.

A semitopological group \( G \) is called \( \omega \)-narrow if for every neighbourhood \( U \) of the neutral element in \( G \), there exists a countable set \( A \subseteq G \) such that \( AU = G = UA \). Clearly, the Sorgenfrey line is a first-countable \( \omega \)-narrow paratopological group without countable base.

Given a paratopological group \( G \) with topology \( \tau \), one defines the conjugate topology \( \tau^{-1} \) on \( G \) by

\[
\tau^{-1} = \{ U^{-1}: U \in \tau \}.
\]

It is easy to see that \((G, \tau^{-1})\) is again a paratopological group, and the inversion \( ln: (G, \tau) \to (G, \tau^{-1}) \), where \( ln(x) = x^{-1} \) for each \( x \in G \), is a homeomorphism. The upper bound \( \tau^{*} = \tau \vee \tau^{-1} \) of the topologies \( \tau \) and \( \tau^{-1} \) is a topological group topology on \( G \). We say that \( G^{*} = (G, \tau^{*}) \) is a topological group associated to the paratopological group \( G \).

We will call a paratopological group \( G \) totally \( \omega \)-narrow if the associated topological group \( G^{*} \) is \( \omega \)-narrow. For example, the Sorgenfrey line \( S \) is not totally \( \omega \)-narrow, since the associated topological group \( S^{*} \) is discrete and uncountable.

Finally, \( l(X) \) denotes the Lindelöf number of a space \( X \) (see [5]).

### 2. The Hausdorff number of a paratopological group

Let \( G \) be a Hausdorff semitopological group with identity \( e \). Given an arbitrary element \( x \in G \) distinct from \( e \), there exists an open neighbourhood \( V \) of \( e \) in \( G \) such that \( V \cap xV = \emptyset \)—this follows immediately from the continuity of multiplication in \( G \). Hence, \( x \notin VV^{-1} \). In its turn, this implies that \( |e| = \bigcap_{V \in \mathcal{N}(e)} VV^{-1} \), where \( \mathcal{N}(e) \) is the family of all open neighbourhoods of \( e \) in \( G \). Conversely, if the equality \( |e| = \bigcap_{V \in \mathcal{N}(e)} VV^{-1} \) holds true for a semitopological group \( G \), then \( G \) is Hausdorff. This observation gives rise to the following cardinal invariant.
For a Hausdorff semitopological group $G$ with identity $e$, we define the Hausdorff number of $G$, denoted by $Hs(G)$, as the minimum cardinal number $\kappa$ such that for every neighbourhood $U$ of $e$ in $G$, there exists a family $\gamma$ of neighbourhoods of $e$ such that $\bigcap_{\gamma \in \gamma} V^{-1} \subseteq U$ and $|\gamma| \leq \kappa$.

It is clear from the definition that $G$ is a topological group if and only if $Hs(G) = 1$. Evidently, $Hs(G)$ is infinite whenever $G$ fails to be a topological group. For example, the Sorgenfrey line $S$ satisfies $Hs(S) = \omega$. The proofs of the following three simple facts are left to the reader (we recall that all spaces are assumed to be Hausdorff).

**Proposition 2.1.** Every subgroup $K$ of a semitopological group $G$ satisfies $Hs(K) \leq Hs(G)$.

**Proposition 2.2.** Every first-countable semitopological group $G$ satisfies $Hs(G) \leq \omega$.

**Proposition 2.3.** The topological product $G = \prod_{i \in I} G_i$ of any family of semitopological groups satisfying $Hs(G_i) \leq \kappa$ for each $i \in I$ satisfies the same inequality $Hs(G) \leq \kappa$.

It turns out that the Hausdorff number $Hs(G)$ of a paratopological group $G$ is bounded by the Lindelöf number $l(G)$ of the group:

**Proposition 2.4.** Every paratopological group $G$ satisfies $Hs(G) \leq l(G)$.

**Proof.** Let $U$ be an arbitrary neighbourhood of the identity $e$ in $G$. Since $G$ is a Hausdorff paratopological group, for every $x \in G \setminus U$ there exists an open neighbourhood $V_x$ of $e$ such that $V_x \cap xV_x = \emptyset$ or, equivalently, $V_x^{-1} \cap xV_x = \emptyset$. Since the set $G \setminus U$ is closed in $G$ and the family $\{V_x : x \in G \setminus U\}$ covers $G \setminus U$, there exists a set $A \subseteq G \setminus U$ such that $G \setminus U \subseteq \bigcup_{x \in A} xV_x$ and $|A| \leq l(G)$. It follows that the family $\gamma = \{V_x : x \in A\}$ satisfies $\bigcap_{\gamma \in \gamma} V_x^{-1} \subseteq U$ and $|\gamma| \leq l(G)$. This implies the inequality $Hs(G) \leq l(G)$.

The new cardinal invariant $Hs$ makes it possible to characterize the subgroups of topological products of first-countable paratopological groups (see Theorem 2.7 below). Our argument depends on the concept of an $\omega$-good subset of a paratopological group introduced in [10].

A subset $V$ of a paratopological group $H$ is called $\omega$-good if there exists a countable family $\gamma$ of open neighbourhoods of the neutral element in $H$ such that for every $x \in V$, we can find $W \in \gamma$ with $xW \subseteq V$. It is immediate from the definition that every $\omega$-good set is open. It is also clear that the intersection of finitely many $\omega$-good sets is $\omega$-good.

**Lemma 2.5.** ([10, Lemma 3.10]) Every paratopological group $H$ has a local base at the neutral element consisting of $\omega$-good sets.

**Proof.** Let $U$ be an open neighbourhood of the neutral element $e$ in $H$. Choose a sequence $\{U_n : n \in \omega\}$ of open neighbourhoods of $e$ such that $U_0 = U$ and $U^{-1}_{n+1} \subseteq U_n$, for each $n \in \omega$. We put $V_1 = U_1$, $V_2 = U_1U_2$ and, in general, $V_n = U_1U_2\cdots U_n$, for an arbitrary $n \in \omega$. Clearly, the sets $V_n$ are open and $V_{n+1} = V_n \cap V_n^{-1}$, so that $V_n \subseteq V_{n+1}$, for all $n$. A standard argument implies that $U_{k+1}U_{k+2}\cdots U_{k+n+1} \subseteq U_k$, for all $k, n \in \omega$. In particular, $V_n \subseteq U_0 = U$, for each integer $n \geq 1$. We conclude, therefore, that the open set $V = \bigcup_{n=1}^\infty V_n$ is contained in $U$.

Given an element $x \in V$, our definition of $V$ implies that $x \in V_n$, for some $n \geq 1$. Since $V_nV_{n+1} = V_{n+1} \subseteq V$, we have that $xU_{n+1} \subseteq V$. In other words, the family $\{U_n : n \in \omega\}$ witnesses that the set $V$ is $\omega$-good.

The following lemma is an important step towards the proof of Theorem 2.7.

**Lemma 2.6.** Let $N(e)$ be the family of open neighbourhoods of the identity $e$ in a paratopological group $G$. Suppose that a subfamily $\gamma \subseteq N(e)$ satisfies the following conditions:

(a) for every $U \in \gamma$, there exists $V \in \gamma$ such that $V^2 \subseteq U$;
(b) $\gamma$ is closed under finite intersections;
(c) $\gamma$ is subordinated to $U$, for each $U \in \gamma$.

Then the set $P(\gamma) = \bigcap_{V \in \gamma} V^{-1}$ is a closed invariant subgroup of $G$.

**Proof.** Indeed, let $P = P(\gamma)$. It follows from the definition of $P$ that

$$P^{-1} = \left( \bigcap_{V \in \gamma} V^{-1} \right)^{-1} = \bigcap_{V \in \gamma} VV^{-1} = P,$$

that is, the set $P$ is symmetric. Let us verify that $PP \subseteq P$. Take arbitrary points $a, b \in P$. It suffices to show that $ab \in VV^{-1}$, for any $V \in \gamma$. Since $a \in P$, we can find $U \in \gamma$ and elements $u_1, v_1 \in U$ such that $U^2 \subseteq V$ and $a = u_1v_1^{-1}$. By (c) and (b),
there exists \( W \in \gamma \) such that \( v_i^{-1}Wv_1 \subseteq U \) and \( W \subseteq U \). Since \( b \in P \), we can find \( u_2, v_2 \in W \) such that \( b = u_2v_2^{-1} \). It then follows that
\[
ab = u_1v_i^{-1}u_2v_2^{-1} = u_1(v_i^{-1}u_2v_1)v_i^{-1}v_1^{-1} \in UUU^{-1}U^{-1} \subseteq \gamma V^{-1}.
\]
Hence, \( ab \in \bigcap_{V \in \gamma} VV^{-1} = P \). This proves the inclusion \( PP \subseteq P \). Since the set \( P \) is symmetric, we conclude that \( P \) is a subgroup of \( G \).

To show that \( P \) is invariant in \( G \), we take arbitrary \( a \in P \), \( x \in G \), and \( V \in \gamma \). According to (c), there exists \( U \in \gamma \) such that \( xU^{-1} \subseteq V \). Choose \( a, v \in U \) such that \( a = uv^{-1} \). Then
\[
ax^{-1} = xu^{-1}v^{-1} = (xu)(xv^{-1})^{-1} \in VV^{-1}
\]
We have thus proved that \( ax^{-1} \in \bigcap_{V \in \gamma} VV^{-1} = P \). In its turn, this implies that \( xPax^{-1} \subseteq P \). Therefore, \( P \) is an invariant subgroup of \( G \).

It remains to show that \( P \) is closed in \( G \). Take an arbitrary point \( x \in G \setminus P \) and choose \( V \in \gamma \) such that \( x \notin VV^{-1} \) or, equivalently, \( V \cap VX = \emptyset \). We apply (a) to find \( W \in \gamma \) such that \( W^2 \subseteq V \). It then follows that \( xW \cap P = \emptyset \)—otherwise \( x \in PW^{-1} \subseteq WW^{-1} \subseteq VV^{-1} \), a contradiction. We have thus proved that the complement \( G \setminus P \) is open in \( G \), so the set \( P \) is closed.

\[\text{Theorem 2.7.}\]

A Hausdorff paratopological group \( G \) can be topologically embedded as a subgroup into a product of first-countable Hausdorff paratopological groups if and only if \( G \) is \( \omega \)-balanced and the Hausdorff number of \( G \) is countable, i.e., \( Hs(G) \leq \omega \).

\[\text{Proof.}\]
The necessity of the conditions is almost evident. Indeed, suppose that \( G \) is a subgroup of the product \( I = \prod_{i \in I} H_i \) of a family of first-countable Hausdorff paratopological groups. Then the product group \( I \) is \( \omega \)-balanced, since every canonical open set in \( I \) depends on finitely many coordinates and every first-countable group \( H_i \) is \( \omega \)-balanced. Taking into account that the property of being \( \omega \)-balanced is hereditary with respect to taking subgroups, we conclude that \( G \) is \( \omega \)-balanced as well. Further, Propositions 2.2 and 2.3 imply that \( Hs(I) \leq \omega \). Hence, \( Hs(G) \leq Hs(I) \leq \omega \), by Proposition 2.1.

To prove the sufficiency, it suffices to verify that every \( \omega \)-balanced paratopological group \( G \) with \( Hs(G) \leq \omega \) is projectively Hausdorff first-countable. Let \( U_0 \) be a neighbourhood of the neutral element \( e \) in \( G \). We have to define a continuous homomorphism \( p: G \to H \) of \( G \) onto a first-countable Hausdorff paratopological group \( H \) and find a neighbourhood \( V_0 \) of the neutral element \( 0 \) in \( H \) such that \( p^{-1}(V_0) \subseteq U_0 \). To this end, we will construct by induction a countable family \( \gamma \) of open neighbourhoods of \( e \) in \( G \) and take \( p \) to be the natural homomorphism of \( G \) onto the quotient group \( G/N \), where \( N = \bigcap_{V \in \gamma} U_0 \). A local base at the neutral element of \( H = G/N \) will be the family \( \{p(U): U \in \gamma \} \).

Let \( N^*(e) \) be the family of open neighbourhoods of \( e \) in \( G \). Denote by \( N^*(e) \) the subfamily of \( N(e) \) consisting of \( \omega \)-good sets. It follows from Lemma 2.5 that \( N^*(e) \) is a local base for \( G \) at \( e \).

Choose \( U_0 \in N^*(e) \) satisfying \( U_0 \subseteq U_0 \) and put \( \gamma_0 = \{U_0\} \). Suppose that for some \( n \in \omega \) we have defined families \( \gamma_0, \ldots, \gamma_n \) satisfying the following conditions for each \( k \leq n \):

(i) \( \gamma_k \subseteq N^*(e) \) and \( \#(\gamma_k) \leq \omega \);
(ii) \( \gamma_k \subseteq \gamma_{k+1} \);
(iii) \( \gamma_k \) is closed under finite intersections;
(iv) for every \( U \in \gamma_k \), there exists \( V \in \gamma_{k+1} \) such that \( V^2 \subseteq U \);
(v) the family \( \gamma_{k+1} \) is subordinated to \( U \), for every \( U \in \gamma_k \);
(vi) \( \bigcap_{V \in \gamma_k} VV^{-1} \subseteq U \), for each \( U \in \gamma_k \).

Clearly, we assume that \( k + 1 \leq n \) in (ii) and (iv)-(vi). Since \( \gamma_n \) is countable, we can find a countable family \( \lambda_{n,1} \subseteq N^*(e) \) such that each \( U \in \gamma_n \) contains the square of some element \( V \in \lambda_{n,1} \). Further, since the group \( G \) is \( \omega \)-balanced, there exists a countable family \( \lambda_{n,2} \subseteq N^*(e) \) subordinated to each \( U \in \gamma_n \) (this is to guarantee condition (v) at the stage \( n+1 \)). Finally, we use the condition \( Hs(G) \leq \omega \) of the lemma to find a countable family \( \lambda_{n,3} \subseteq N^*(e) \) such that \( \bigcap_{V \in \lambda_{n,3}} VV^{-1} \subseteq U \), for each \( U \in \gamma_n \).

Let \( \gamma_{n+1} \) be the minimal family containing \( \gamma_n \cup \bigcup_{V \in \lambda_{n,1}} \lambda_{n,1} \) and closed under finite intersections. It is clear that \( \gamma_{n+1} \) is countable and that the families \( \gamma_0, \ldots, \gamma_n, \gamma_{n+1} \) satisfy (i)-(vi).

It is easy to see that the family \( \gamma = \bigcup_{n=0}^\infty \gamma_n \) is countable and satisfies conditions (a)-(c) of Lemma 2.6. Therefore, \( N = \bigcap_{V \in \gamma} VV^{-1} \) is a closed invariant subgroup of \( G \). It also follows from (ii) and (vi) that \( N \cap \gamma \neq \emptyset \). Let \( p: G \to G/N \) be the canonical homomorphism. We claim that the family \( \mu = \{p(V): V \in \gamma \} \) is a local base at the neutral element \( e' \) of \( H = G/N \) for a Hausdorff paratopological group topology on \( H \). According to [9, Proposition 2.1], this means exactly that the family \( \mu \) has the following properties:

(1) for all \( U, V \in \mu \), there exists \( W \in \mu \) with \( W \subseteq U \cap V \);
(2) for every \( U \in \mu \) there exists \( V \in \mu \) such that \( V^2 \subseteq U \);
(3) for every \( U \in \mu \) and \( x \in U \), there exists \( V \in \mu \) such that \( xV \subseteq U \) and \( Vx \subseteq U \);
(4) \( \mu \) is subordinated to each \( U \in \mu \);
(5) for every \( x \in H \) distinct from \( e' \), there exists \( U \in \mu \) such that \( U \cap xU = \emptyset \).
Condition (1) holds trivially, since the family \( y \) is closed under finite intersections. Conditions (2)-(4) follow easily from (ii)-(v) of the inductive construction and our definitions of \( y \) and \( \mu \). It is worth noting that the second inclusion \( Vx \subseteq U \) in (3) can be obtained from the first inclusion \( xV \subseteq U \) via (1) and (4). Thus, the family \( \mu \) is a local base for a paratopological group topology \( \tau \) on \( H = G/N \). Hence, it remains to deduce (5). In what follows \( H \) will always carry the topology \( \tau \), that is, \( H = (H, \tau) \). Since the family \( \mu \) is countable, the paratopological group \( H \) is first-countable.

Take an arbitrary point \( x \in H \), \( x \neq e^\prime \). Then \( x = p(y) \) for some \( y \in G \) and, clearly, \( y \neq \emptyset \). Since \( N = \bigcap_{V \in \gamma} VV^{-1} \), there exists \( V \in \gamma \) such that \( y \notin VV^{-1} \), that is, \( V \cap \gamma = \emptyset \). Choose \( W \in \gamma \) such that \( W^2 \subseteq V \). We claim that \( O \cap xO = \emptyset \), where \( O = p(W) \in \mu \). Indeed, otherwise there are \( a, b \in W \) such that \( p(a) = xp(b) \) or, equivalently, \( a^{-1}yb \in N \). Since \( N \subseteq W \) and \( W^2 \subseteq V \), we deduce that

\[
y \in aNb^{-1} \subseteq aWb^{-1} \subseteq W^2W^{-1} \subseteq V V^{-1},
\]

which contradicts the choice of the set \( V \). We have proved that the neutral element \( e^\prime \) and every point \( x \in H \setminus \{e^\prime\} \) can be separated in \( H \) by disjoint open neighbourhoods. Since the paratopological group \( H \) is a homogeneous space, it follows that \( H \) is Hausdorff.

Finally, take an arbitrary \( U \in \gamma \) such that \( U^2 \subseteq U_0^2 \) and put \( V_U = p(U) \). Then \( V_U \) is an open neighbourhood of \( e^\prime \) in \( H \) and \( p^{-1}(V_U) = UN \subseteq U^2 \subseteq U_0^2 \subseteq U_0 \). This finishes the proof of the theorem. \( \Box \)

Making use of the above theorem, we obtain a characterization of projectively Hausdorff second-countable paratopological groups:

**Theorem 2.8.** A Hausdorff paratopological group \( G \) can be topologically embedded as a subgroup into a topological product of second-countable Hausdorff paratopological groups if and only if \( G \) is totally \( \omega \)-narrow and \( HS(G) \leq \omega \).

**Proof.** The necessity of the conditions can be verified similarly to the first part of the proof of Theorem 2.7. Let us verify the sufficiency. According to \([10, \text{Proposition 3.8}]\), every totally \( \omega \)-narrow paratopological group is \( \omega \)-balanced. Therefore, assuming that \( G \) is totally \( \omega \)-narrow and satisfies \( HS(G) \leq \omega \), Theorem 2.7 implies that \( G \) is a subgroup of a product \( \Pi = \prod_{i \in I} H_i \) of first-countable Hausdorff paratopological groups \( H_i \). We can assume without loss of generality that \( p_i(G) = H_i \) for each \( i \in I \), where \( p_i: \Pi \to H_i \) is the natural projection. Hence, each \( H_i \) is totally \( \omega \)-narrow as a continuous homomorphic image of the totally \( \omega \)-narrow group \( G \). However, every first-countable totally \( \omega \)-narrow paratopological group is second-countable (see \([10, \text{Proposition 3.5}]\)). It follows that \( G \) is a subgroup of the product \( \Pi = \prod_{i \in I} H_i \) of second-countable Hausdorff paratopological groups \( H_i \). \( \Box \)

We do not know whether one can drop the condition \( HS(G) \leq \omega \) in Theorem 2.8. In other words, we do not know whether every totally \( \omega \)-narrow paratopological group \( G \) satisfies \( HS(G) \leq \omega \) (see Problem 4.1).

It follows from \([8] \) or \([3, \text{Theorem 3.4.22}]\) that every topological Abelian group is projectively metrizable, i.e., it is topologically isomorphic to a subgroup of a product of metrizable topological groups. It turns out that one cannot extend this result to paratopological groups:

**Example 2.9.** There exists a completely regular Abelian paratopological group \( G \) that fails to be projectively Hausdorff first-countable. In particular, \( \omega \)-balanced paratopological groups need not be projectively Hausdorff first-countable.

Let \( \mathbb{Z} \) be the discrete group of integers and \( \kappa > \omega \) a cardinal. In what follows we will use the additive notation for the binary group operation in \( \mathbb{Z}^\kappa \). For a finite set \( A \subseteq \kappa \), we define a set \( U_A \subseteq \mathbb{Z}^\kappa \) by

\[
U_A = \{ x \in \mathbb{Z}^\kappa : x(\alpha) = 0 \text{ if } \alpha \in A \text{ and } x(\alpha) \geq 0 \text{ if } \alpha \in \kappa \setminus A \}.
\]

It is easy to see that the family \( U = \{ U_A : A \subseteq \kappa, |A| < \omega \} \) is a local base at the neutral element of \( \mathbb{Z}^\kappa \) for a paratopological group topology \( \tau \). It follows from the definition of the topology \( \tau \) that the sets \( U_A \) are clopen in \( G \), so the paratopological group \( G = (\mathbb{Z}^\kappa, \tau) \) is completely regular. Let us show that \( HS(G) = \kappa \), which will imply that \( G \) is not projectively Hausdorff first-countable, by Theorem 2.7.

Denote by \( U^* \) the subset of \( \mathbb{Z}^\kappa \) consisting of all non-negative functions \( x \in \mathbb{Z}^\kappa \), i.e., \( U^* = U_0 \). Suppose that \( \gamma \) is a family of neighbourhoods of the neutral element \( 0 \) in \( G \), where \( |\gamma| < \kappa \). Since \( U \) is a local base at \( 0 \), we can assume that \( \gamma \subseteq U \).

Hence, \( \gamma = \{ U_{A(i)} : i \in I \} \), where \( |I| < \kappa \) and \( A(i) \) is a finite subset of \( \kappa \), for each \( i \in I \). From the definition of the sets \( U_A \) it follows that

\[
O_i = U_{A(i)} - U_{\bar{A}(i)} = \{ x \in \mathbb{Z}^\kappa : x(\alpha) = 0 \text{ for each } \alpha \in A(i) \},
\]

in other words, \( O_i \) is a canonical open neighbourhood of the neutral element in \( \mathbb{Z}^\kappa \) when the latter carries the usual Tychonoff product topology. Let \( F = \bigcup_{i \in I} A(i) \). Then \( F \subseteq \kappa \) and \( |F| < \kappa \). Take an ordinal \( \beta \in \kappa \setminus F \) and consider the element \( y \in \mathbb{Z}^\kappa \) defined by \( y(\alpha) = 0 \) if \( \alpha \neq \beta \) and \( y(\beta) = -1 \). Clearly, \( y \in O_i \) for each \( i \in I \), but \( y \notin U^* \). Therefore, \( y \in \bigcap_{i \in I} (U_{A(i)} - U_{\bar{A}(i)}) \setminus U^* \neq \emptyset \). This implies that the paratopological group \( G \) satisfies \( HS(G) = \kappa > \omega \).
3. The case of regular paratopological groups

Let us define the index of regularity \( Ir(G) \) of a regular semitopological group \( G \) as the minimum cardinal number \( \kappa \) such that for every neighbourhood \( U \) of the identity \( e \) in \( G \), one can find a neighbourhood \( V \) of \( e \) and a family \( \{ \gamma \} \) of neighbourhoods of \( e \) in \( G \) such that \( \bigcap_{\gamma \in \{ \gamma \}} V \gamma^{-1} \subseteq U \) and \( |\{ \gamma \}| \leq \kappa \). It is clear from the definition that \( Hs(G) \leq Ir(G) \), for every regular semitopological group \( G \).

The following three facts are close to Propositions 2.1–2.3. Again, the proofs are omitted.

**Proposition 3.1.** Every subgroup \( K \) of a regular semitopological group \( G \) satisfies \( Ir(K) \leq Ir(G) \).

**Proposition 3.2.** Every first-countable regular semitopological group \( G \) satisfies \( Ir(G) \leq \omega \).

**Proposition 3.3.** The topological product \( G = \prod_{i \in I} G_i \) of a family of regular semitopological groups satisfying \( Ir(G_i) \leq \kappa \) for each \( i \in I \) satisfies the same inequality \( Ir(G) \leq \kappa \).

From Propositions 3.1–3.3, and Theorem 2.7 we deduce the following corollary:

**Corollary 3.4.** Every subgroup \( K \) of the topological product \( \prod_{i \in I} G_i \) of a family of first-countable regular semitopological groups is \( \omega \)-balanced and satisfies \( Ir(K) \leq \omega \).

Similarly to the Hausdorff number, the index of regularity \( Ir(G) \) of a regular paratopological group \( G \) is bounded by the Lindelöf number \( l(G) \) of the group:

**Proposition 3.5.** Every regular paratopological group \( G \) satisfies the inequalities \( Hs(G) \leq Ir(G) \leq l(G) \).

**Proof.** The fact that \( Hs(G) \leq Ir(G) \) is immediate from the definition of the functions \( Hs \) and \( Ir \). Let us verify that \( Ir(G) \leq l(G) \). Take an open neighbourhood \( U \) of the identity \( e \) in \( G \). By the regularity of \( G \), we can find an open neighbourhood \( V \) of \( e \) in \( G \) such that \( V \subseteq U \). Since \( G \) is a paratopological group, for every \( x \in G \setminus U \) there exists an open neighbourhood \( W_x \) of \( e \) such that \( xW_x \cap V = \emptyset \) or, equivalently, \( xW_x \cap V W_x^{-1} = \emptyset \). Since the set \( G \setminus U \) is closed in \( G \) and the family \( \{ xW_x: x \in G \setminus U \} \) covers \( G \setminus U \), there exists a set \( A \subseteq G \setminus U \) such that \( G \setminus U \subseteq \bigcup_{x \in A} xW_x \) and \( |A| \leq l(G) \). It follows that the family \( \{ W_x: x \in A \} \) satisfies \( \bigcap_{x \in A} V W_x^{-1} \subseteq U \) and \( |\{ \gamma \}| \leq l(G) \). This implies the inequality \( Ir(G) \leq l(G) \). 

Here is a characterization of projectively regular first-countable paratopological groups.

**Theorem 3.6.** A regular paratopological group \( G \) can be topologically embedded as a subgroup into a product of first-countable regular paratopological groups if and only if \( G \) is \( \omega \)-balanced and satisfies \( Ir(G) \leq \omega \).

**Proof.** Since our argument is close to the proof of Theorem 2.7, we only sketch it here. The necessity follows from Corollary 3.4, so we have to prove the sufficiency only. Suppose, therefore, that the paratopological group \( G \) is \( \omega \)-balanced and has countable index of regularity. It suffices, for a given neighbourhood \( U_0 \) of the neutral element \( e \) of \( G \), to find a continuous homomorphism \( p: G \to H \) onto a first-countable regular paratopological group \( H \) and a neighbourhood \( V_0 \) of the neutral element in \( H \) such that \( p^{-1}(V_0) \subseteq U_0 \).

Let \( N^+(e) \) be the family of all \( \omega \)-good neighbourhoods of \( e \) in \( G \). By Lemma 2.5, there exists \( U^*_0 \in N^+ \) such that \( U^*_0 \subseteq U_0 \). We put \( \gamma_0 = \{ U^*_n \} \). Making use of the assumptions about \( G \), one constructs a sequence \( \{ \gamma_n: n \in \omega \} \) satisfying the following conditions, for each \( n \in \omega \):

(i) \( \gamma_n \subseteq N^+(e) \) and \( |\gamma_n| \leq \omega \);
(ii) \( \gamma_n \subseteq \gamma_{n+1} \);
(iii) \( \gamma_n \) is closed under finite intersections;
(iv) for every \( U \in \gamma_n \), there exists \( V \in \gamma_{n+1} \) such that \( V^2 \subseteq U \);
(v) \( \gamma_{n+1} \) is subordinated to \( U \), for each \( U \in \gamma_n \);
(vi) for every \( U \in \gamma_n \), there exists \( V \in \gamma_{n+1} \) such that \( \bigcap_{W \in \gamma_{n+1}} V W^{-1} \subseteq U \).

Once the sequence \( \{ \gamma_n: n \in \omega \} \) is constructed, we put \( \gamma = \bigcup_{n \in \omega} \gamma_n \) and \( N = \bigcap_{V \in \gamma} V V^{-1} \). By Lemma 2.6, \( N \) is a closed invariant subgroup of \( G \). It also follows from (vi), combined with (ii) and (iii), that \( N = \bigcap \gamma \). Let \( p: G \to G/N \) be the canonical homomorphism. As in Theorem 2.7, one verifies that the family \( \{ p(U): U \in \gamma \} \) is a local base at the neutral element of \( G/N \) for a paratopological group topology \( \tau \) on \( G/N \).

Let \( H = (G/N, \tau) \). Condition (vi) implies that the paratopological group \( H \) is regular. Indeed, take an arbitrary basic open neighbourhood \( p(U) \) of the neutral element \( e_H \) of \( H \), where \( U \in \gamma \). Then \( U \in \gamma_n \), for some \( n \in \omega \). By (vi), we can
find an element \( V \in \gamma_{n+1} \) such that \( \bigcap \{VW^{-1}; W \in \gamma_{n+1}\} \subseteq U \). Let us verify that \( p(V) \subseteq p(U) \). Take a point \( y \in H \setminus p(U) \) and choose \( x \in G \) such that \( p(x) = y \). Clearly, \( x \notin U \). Hence, there exists \( W \in \gamma_{n+1} \) such that \( x \notin V^{-1}W \), equivalently, \( V \cap xW = \emptyset \). Take an element \( O \in \gamma_{n+2} \) such that \( O^2 \subseteq W \). Then \( V \cap xO^2 = \emptyset \), whence \( V^{-1}O \cap xO = \emptyset \). Since \( N \) is a subgroup of \( G \) and \( N = \bigcap \gamma \), we have that \( N \subseteq O \) and \( N = N^{-1} \subseteq O^{-1} \). In its turn, this implies that \( VN \cap xO = \emptyset \). Since \( N \) is the kernel of the homomorphism \( p \), we conclude that \( p(xO) \cap p(V) = \emptyset \), i.e., \( yp(O) \cap p(V) = \emptyset \). We have thus proved that \( y \notin p(V) \), which implies the inclusion \( p(V) \subseteq p(U) \).

To finish the proof, it suffices to take an element \( V \in \gamma_1 \) with \( V^2 \subseteq U_0^* \) and put \( V_0 = p(V) \). By the definition of \( H \), \( V_0 \) is an open neighbourhood of \( e_H \) in \( H \). Since \( U \subseteq V \), we have that \( p^{-1}(V_0) = VN \subseteq V^2 \subseteq U_0^* \subseteq U_0 \), as required. \( \square \)

To obtain a version of Theorem 3.6 for embeddings into products of regular second-countable paratopological groups, we need an auxiliary fact given below.

**Lemma 3.7.** Every totally \( \omega \)-narrow projectively regular (Hausdorff) first-countable paratopological group \( G \) is projectively regular (Hausdorff) second-countable.

**Proof.** Let \( U \) be a neighbourhood of the neutral element \( e \) in \( G \). By our assumptions, there exists a continuous homomorphism \( p : G \to H \) of \( G \) onto a regular (Hausdorff) first-countable paratopological group \( H \) such that \( p^{-1}(V) \subseteq U \), for some neighbourhood \( V \) of the neutral element of \( H \). Let \( G^* \) and \( H^* \) be the respective topological groups associated to \( G \) and \( H \). Clearly, the homomorphism \( p : G^* \to H^* \) is continuous. By assumptions of the lemma, the group \( G^* \) is \( \omega \)-narrow, and so is the group \( H^* \). It is clear that the group \( H^* \) is first-countable (see, for example, \([10, Corollary 3.3]\)). According to \([6]\) (see also \([12, Lemma 3.5]\)), every \( \omega \)-narrow first-countable topological group has a countable base. Hence, the continuous image \( H \) of \( H^* \) has a countable base. It remains to refer to \([2, Proposition 2.13]\) saying that every first-countable paratopological group with a countable network has a countable base. This finishes the proof. \( \square \)

**Theorem 3.8.** A regular paratopological group \( G \) can be topologically embedded as a subgroup into a product of regular second-countable paratopological groups if and only if \( G \) is totally \( \omega \)-narrow and \( Ir(G) \leq \omega \).

**Proof.** The necessity of the conditions follows from Propositions 3.1–3.3. The sufficiency follows from the combination of Theorem 3.6, Lemma 3.7, and the fact that every totally \( \omega \)-narrow paratopological group is \( \omega \)-balanced (see \([10, Proposition 3.8]\)). \( \square \)

Since every regular second-countable space is completely regular (even normal) and the class of completely regular spaces is productive and closed under taking subspaces, Theorem 3.8 implies the following:

**Corollary 3.9.** Every regular totally \( \omega \)-narrow paratopological group \( G \) satisfying \( Ir(G) \leq \omega \) is completely regular.

4. Open problems

Our first problem is related directly to Theorem 2.8—all known totally \( \omega \)-narrow paratopological groups have countable Hausdorff number.

**Problem 4.1.** Does every totally \( \omega \)-narrow Hausdorff paratopological group \( G \) satisfy the inequality \( HS(G) \leq \omega \)?

It is well known that every Hausdorff (regular, Tychonoff) space with a countable network admits a continuous bijection onto a Hausdorff (regular, Tychonoff) space with a countable base \([5]\). Similarly, many universal topological algebras with a countable network admit a continuous isomorphism onto a similar topological algebra with a countable base (see \([1,11,7]\)). This includes topological fields, topological vector spaces, topological and paratopological groups. However, semitopological groups are not in the above list yet.

**Problem 4.2.** Let \( G \) be a Hausdorff (regular, Tychonoff) semitopological group with a countable network. Does \( G \) admit a continuous isomorphism onto a Hausdorff (regular, Tychonoff) second-countable semitopological group? Is, at least, \( HS(G) \leq \omega \)?

Since a regular (even normal) first-countable paratopological group need not be metrizable, Theorem 2.7 does not help to solve the next problem:

**Problem 4.3.** Characterize the projectively metrizable paratopological groups, i.e., the subgroups of topological products of metrizable paratopological groups.

It is not clear whether Proposition 3.5 can be extended to semitopological groups:
Problem 4.4. Does every regular semitopological group $G$ satisfy the inequality $Ir(G) \leq l(G)$? What if, additionally, the inversion in $G$ is continuous?

We do not know if one can drop “totally $\omega$-narrow” in Corollary 3.9:

Problem 4.5. Let $G$ be a regular paratopological group that satisfies $Ir(G) \leq \omega$. Is $G$ completely regular? What if $G$ is first-countable?

References