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An Approximate Homogenization Scheme for Nonperiodic Materials

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Abstract—Recently in [1], Briane announced a new homogenization method for certain nonperiodic materials in which the H-limit of a highly oscillatory but nonperiodic matrix A^ε is obtained by comparing to a locally-periodic matrix B^ε in domains whose size $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$ but slower than ε. The H-limit of B^ε is a function of every point in the material, and so theoretically, in order to homogenize A^ε, the solution to the usual periodic cell problem must be obtained for every point in the material. Computationally this is not feasible, so we approximate the homogenization method by keeping α fixed. We show that this approximation is $\mathcal{O}(\alpha)$ by proving that the difference of two nearby cell solutions (within a cube of side length α) is $\mathcal{O}(\alpha)$ in the H^1-norm. This result requires that we show a uniform bound exists for the gradients of the periodic cell solutions in L^p. We then apply our approximate homogenization theory to the analysis of certain defects in fiber-reinforced composites. In particular, we show that when unexpected local spreading of the fibers occurs in a *small* region of the material, constituent stress concentrations of nearly three can arise.

Keywords—Homogenization, Nonperiodic, Numerical approximation, Composite material, Pseudodifferential operators.

1. INTRODUCTION

1.1. Objective

We wish to obtain estimates for the numerical approximation to Briane's novel homogenization approach of certain nonperiodic materials (see [1]), and show how this approach can be used to study particular defects in fibrous composites. Mathematically of interest, is the asymptotic behavior of the sequence of equations

$$\begin{gathered} u^\varepsilon \in H_0^1(\Omega), \\ -\operatorname{div}[A^\varepsilon \nabla u^\varepsilon] = f, \qquad f \in H^{-1}(\Omega), \end{gathered} \tag{1}$$

where for each $\varepsilon > 0$, A^ε is a highly-oscillatory but *nonperiodic* uniformly-elliptic matrix-valued function on Ω, an open bounded set in $\mathbb{R}^N$ with sufficiently smooth boundary.

In particular, we consider the nonperiodic material studied by Briane in [1] which is defined as follows: A^ε is equal to the constant matrix A^1 in a nonperiodic distribution of spherical balls of radius ε and to a constant matrix A^2 outside these inclusions; the balls are centered at the points $\theta(k\varepsilon)$, $k \in \mathbb{Z}^N$, where θ is a C^2-diffeomorphism of $\mathbb{R}^N$ such that θ^{-1} is l-Lipschitz continuous with $0 < l < 1$.

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1.2. H-Convergence

The standard homogenization procedure for equation (1) with general coefficient matrices is the method of H-convergence developed by Murat-Tartar (see [2–5]). The idea is to find the limit point A^0 in the topology of H-convergence. This means that

$$\begin{aligned} u^\varepsilon &\rightharpoonup u^0, && \text{in } H_0^1(\Omega) \text{ weakly,} \\ A^\varepsilon \nabla u^\varepsilon &\rightharpoonup A^0 \nabla u^0, && \text{in } L^2(\Omega) \text{ weakly,} \end{aligned} \tag{2}$$

where u^0 in $H_0^1(\Omega)$ solves $-\text{div}[A^0 \nabla u^0] = f$, for any $f \in H^{-1}(\Omega)$, and we denote this by $A^\varepsilon \xrightarrow{H} A^0$. By Rellich's theorem, it is immediate that $u^\varepsilon \to u^0$ in $L^2(\Omega)$ strongly, but in order to obtain strong convergence of ∇u^ε, a *corrector* function $Q^\varepsilon \in L^2_{\text{loc}}(\Omega, \mathbb{R}^{N^2})$ must be introduced so that (see [5])

$$\nabla u^\varepsilon - Q^\varepsilon \nabla u^0 \to 0, \qquad \text{in } L^2_{\text{loc}}(\Omega) \text{ strongly.}$$

Although H-convergence holds for a general class of matrix-valued functions, it is not generally possible to obtain an explicit form for the H-limit. However, when $A^\varepsilon = A(x/e)$ is Y-periodic for some parallelepiped Y in $\mathbb{R}^N$, an explicit form for the corrector does exist and can be traced back to the work of Keller and Babuska (see [6,7]). It is defined by $Q^\varepsilon \eta = \nabla t^\varepsilon_\eta$, where $t^\varepsilon_\eta = \eta \cdot x - \varepsilon w_\eta(x/e)$ and

$$\begin{gathered} w_\eta \in \overline{H}^1_{\text{per}}(Y), \\ \text{div}\,[A(y)\nabla w_\eta] = \text{div}[A(y)\eta] \text{ in } Y. \end{gathered} \tag{3}$$

Then, when $A^\varepsilon \xrightarrow{H} A^0$, the H-limit A^0 can be obtained as

$$A^0 \eta = \frac{1}{m(Y)} \int_Y [A(y)\eta - A(y)\nabla w_\eta(y)]\, dy, \qquad \eta \in \mathbb{R}^N. \tag{4}$$

The following standard result which can be found in [8] or [9] allows us to take advantage of the explicit form of (4), for a certain class of nonperiodic functions A^ε.

THEOREM 1. *Let A^ε and B^ε be sequences of uniformly bounded and coercive matrix-valued functions on Ω and let $B^\varepsilon \xrightarrow{H} B^0$. Then, if $A^\varepsilon - B^\varepsilon \to 0$ in $L^1(\Omega)$, we have that $A^\varepsilon \xrightarrow{H} B^0$.*

1.3. Comparison with Locally-Periodic Functions

In order to use the comparison Theorem 1 to obtain the H-limit of the nonperiodic matrix A^ε defined above, A^ε is compared to periodic matrices in domains for which the size is larger than ε, but still converges to zero with ε. This is accomplished in three steps.

1. *Construct the associated periodic function.* Let $B(x, (x/e))$ be defined such that for each $x \in \Omega$, $B(x, \cdot)$ is Y_x-periodic, $Y_x = T_x(Y)$, $Y = [0,1]^N$, and $T_x = \nabla\theta(\theta^{-1}(x))$. Then, since θ is C^2, T_x is continuous, and Y_x forms a continuous family of parallelepipeds on $\overline{\Omega}$. From this, it follows that $B \in C(\overline{\Omega}, L^1_{\text{loc}}(\mathbb{R}^N))$.
2. *Construct a partition covering for* Ω. Let $\{\Omega_n^{\alpha(\varepsilon)}\}_{n=1}^{n(\alpha(\varepsilon))}$ be a collection of interior disjoint cubes of side length $\alpha(\varepsilon)$ covering Ω, such that $\varepsilon/\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$.
3. *Construct a locally-periodic function.* Define

$$B^\varepsilon(x) = \sum_{n=1}^{n(\alpha(\varepsilon))} B\left(x_n^\varepsilon, \frac{x}{\varepsilon}\right) 1_{\Omega_n^{\alpha(\varepsilon)}}(x),$$

where for each $n \in \{1, \ldots, n(\alpha(\varepsilon))\}$, x_n^ε is some arbitrary point in $\Omega_n^{\alpha(\varepsilon)} \cap \overline{\Omega}$.

This locally-periodic function B^ε satisfies the hypothesis in Theorem 1 [1, Proposition 5.2], and so to obtain the homogenization of the nonperiodic matrix A^ε, it is enough to obtain the H-limit of B^ε. The primary result in [1] is the following theorem.

THEOREM 2. *Let $\alpha(\varepsilon)$ be a function such that $\alpha(\varepsilon) \to 0$ and $\varepsilon^m/\alpha(\varepsilon)^{m+1} \to 0$ for some $m \in \mathbb{N}$. Let $\{Y_x\}_{x\in\overline{\Omega}}$ be a continuous family of parallelepipeds, and let B be the uniformly-elliptic matrix-valued function defined in Step 1. Then the H-limit of the locally-periodic function B^ε defined in Step 3 is equal to the H-limit B^0 of $B(x,(x/e))$ defined by*

$$B^0(x)\eta = \frac{1}{m(Y_x)} \int_{Y_x} \left[B(x,y)\eta - B(x,y)\nabla w^x_\eta(y)\right] dy, \tag{5}$$

where for each $x \in \Omega$, w^x_η is the Y_x-periodic solution to

$$\begin{aligned} w^x_\eta &\in \overline{H}^1_{\text{per}}(Y_x), \\ \operatorname{div}\left[B(x,y)\nabla w^x_\eta\right] &= \operatorname{div}[B(x,y)\eta] \textit{ in } Y_x, \qquad \eta \in \mathbb{R}^N. \end{aligned} \tag{6}$$

1.4. Numerical Implementation

In numerically implementing this homogenization scheme, we cannot actually wait for $\alpha(\varepsilon)$ to get to zero, for this would entail solving the unit cell problem (6) at each point x of Ω; instead, we keep $\alpha > 0$ finite and define the locally-periodic matrix

$$\begin{gathered} B^\alpha\left(x, \frac{x}{\varepsilon}\right) = \sum_{n=1}^{n(\alpha)} B\left(x^\alpha_n, \frac{x}{\varepsilon}\right) 1_{\Omega^\alpha_n}(x), \qquad x^\alpha_n \in \Omega^\alpha_n \cap \overline{\Omega}, \\ B \text{ defined in Step 1,} \end{gathered} \tag{7}$$

and indeed, it is the H-limit $B^{0\alpha}$ of $B^\alpha(x,(x/e))$ that we can actually compute. A fairly standard locality argument can be used to show that $B^{0\alpha} \to B^0$ a.e. in Ω as $\alpha \to 0$. In fact, for each $\alpha > 0$, the matrix $B^\alpha(x,\cdot)$ is $Y_{x^\alpha_n}$-periodic for each $x \in \Omega^\alpha_n$, and thus H-converges to the constant matrix $B^0(x^\alpha_n)$ by the classical periodic homogenization result. The local character of the H-convergence then gives that $B^{0\alpha} = \sum_{n=1}^{n(\alpha)} B(x^\alpha_n)1_{\Omega^\alpha_n}$, and due to the continuity of $x \mapsto B(x,y)$ and T_x, the result is obtained.

Our interest is in the following theorem.

THEOREM 3. *There exists $C > 0$ such that for each $\alpha > 0$,*

$$\left\|B^{0\alpha} - B^0\right\|_{L^\infty(\Omega)} \le \sqrt{N}\alpha C.$$

Moreover, let u^0, $u^{0\alpha} \in H^1_0$ solve $-\operatorname{div}[B^0\nabla u^0] = f$, $-\operatorname{div}[B^{0\alpha}\nabla u^{0\alpha}] = f$. Then there exists $C' > 0$ such that

$$\left\|u^{0\alpha} - u^0\right\|_{H^1_0(\Omega)} \le \sqrt{N}\alpha C'.$$

The proof of this theorem relies on the regularity of the Y_x-periodic cell solutions $w^x_\eta \in \overline{H}^1_{\text{per}}(Y_x)$ satisfying (6). It is a fairly simple exercise to show that w^x_η is locally weakly continuous in the $H^1_{\text{loc}}(\mathbb{R}^N)$-norm, i.e., $w^{x_0} \rightharpoonup w^x_\eta$ in $H^1_{\text{loc}}(\mathbb{R}^N)$ as $x_0 \to x$. It is somewhat more interesting to directly compare two nearby solutions $w^{x_1}_\eta$ and $w^{x_2}_\eta$ (e.g., if $|x_1 - x_2| < \alpha$) by pulling-back equation (6) to the common periodic lattice configuration $\mathbb{Z}^N$. To do so, we consider our family of diffeomorphisms T_x parameterized by points in $\overline{\Omega}$. With T^*_x being the pullback, we analyze the equations

$$\begin{aligned} T^*_x w^x_\eta &\in \overline{H}^1_{\text{per}}(Y), \\ \operatorname{div}_y\left[T_x^{-1}T^*_x B(x,y)T_x^{-1^t}\nabla T^*_x w^x_\eta(y)\right] &= \operatorname{div}_y\left[T_x^{-1}T^*_x B(x,y)\eta\right], \qquad \eta \in \mathbb{R}^N, \end{aligned}$$

for each $x \in \Omega$. By showing that the cell solutions are uniformly bounded in the $W^{1,p}_{\text{per}}$-norm for some $p > 2$, we can obtain the estimate

$$\left\|T^*_{x_1} w^{x_1}_\eta - T^*_{x_2} w^{x_2}_\eta\right\|_{H^1_{\text{per}}(Y)} \le C_\eta \left|x_1 - x_2\right|, \qquad 0 < C_\eta < \infty. \tag{8}$$

With this estimate, Theorem 3 easily follows.

1.5. Application to Defective Fibrous Composites

It is often the case that man-made materials such as composites, which are designed to be periodic, have *defective* regions, wherein the microstructure sharply diverges from its anticipated periodic state. In composites consisting of fiber-reinforced materials, these defects may develop during the manufacturing process and can take the form of local fiber clumping or its antithesis, local fiber spreading, causing stress concentrations to arise and consequently early failure of the material. Such defects usually occur in very *small* regions of the material, for example, the unexpected fiber-spreading may occur in a region whose size is of the order of four or five fiber diameters.

By constructing the appropriate diffeomorphism θ, we show, using our approximate theory of nonperiodic homogenization, that although the global stress deviates (in this small defective region) by only 20% from its value in the uniform portion of the material, the constituent stress can increase by as much as a factor of three. Thus, we are not merely able to capture the large fluctuations in stress, but also to estimate the errors made by using the classical periodic homogenization scheme for defects which are so slight as to escape visual inspection.

1.6. Outline

In Section 2, we define our notation, and prove Theorem 3. In Section 3, we obtain an estimate which is necessary for the the proof of Theorem 3, and show that the periodic solutions to the cell equations have gradients which are uniformly bounded in L^p. The method of proof requires some well known local estimates for elliptic pseudodifferential operators, and relies on a decomposition of the operator developed by Meyers in [10]. Finally, in Section 4, we give an application of this theory to the analysis of certain defects in fibrous composite materials.

2. APPROXIMATE NONPERIODIC HOMOGENIZATION

2.1. Notation

We denote N-dimensional Euclidean space by $\mathbb{R}^N$ and n-tuples of integers by $\mathbb{Z}^N$, and use $|\cdot|$ to denote the standard Euclidean norm when operating on vectors in $\mathbb{R}^N$ or when operating on matrices in $\mathbb{R}^{N^2}$. If $\Omega \subset \mathbb{R}^N$, then $C^\infty(\Omega)$ denotes the vector space of all real-valued infinitely continuously differentiable functions in Ω. Then $C^\infty_0(\Omega)$ is the subspace of Ω consisting of those functions which vanish on the boundary of Ω. Similarly, $C^\infty_{\text{per}}(\mathbb{R}^N)$ is the subspace of $C^\infty(\mathbb{R}^N)$ consisting of periodic functions.

For $1 \le p \le \infty$, we denote by $L^p(\Omega)$ the equivalence class of functions or vector fields in $\mathbb{R}^N$ or matrices in $\mathbb{R}^{N^2}$ which are measurable and have finite $\|\cdot\|_{L^p}$-norm, where

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p\, dx\right)^{1/p} \quad \text{and} \quad \|f\|_{L^\infty(\Omega)} = \underset{x \in \Omega}{\text{ess sup}} |f(x)|.$$

Then $L^p_{\text{loc}}(\mathbb{R}^N)$ is the set of functions which are in $L^p(K)$ for all compact subsets K in $\mathbb{R}^N$. As usual, the Sobolev space $W^{1,p}(\Omega)$ is the completion of $C^\infty(\Omega)$ in the $\|\cdot\|_{W^{1,p}(\Omega)}$-norm, where

$$\|f\|^p_{W^{1,p}(\Omega)} = \|f\|^p_{L^p(\Omega)} + \|\nabla f\|^p_{L^p(\Omega)}.$$

Similarly, the space $W^{1,p}_{\text{per}}(\mathbb{R}^N)$ consisting of Y-periodic functions is the completion of C^{∞}_{per} $(\mathbb{R}^N)$-functions which are Y-periodic in the $\|\cdot\|_{W^{1,p}_{\text{per}}(\mathbb{R}^N)}$-norm. We specify the periodicity by writing $W^{1,p}_{\text{per}}(Y)$ for $W^{1,p}_{\text{per}}(\mathbb{R}^N)$. Since on a bounded domain $L^p \subset L^2$ for $p > 2$, we can also consider the subspace of $W^{1,p}_{\text{per}}(Y)$, for example, consisting of periodic functions whose integral over Y vanishes; we shall denote this class by $\overline{W}^{1,p}(Y)$. Since every closed subspace of $W^{1,p}$ containing only the 0 rigid displacement satisfies a Poincare-type inequality, we have that if $f \in \overline{W}^{1,p}(Y)$, then $\|f\|_{W^{1,p}_{\text{per}}(\mathbb{R}^N)} = \|\nabla f\|_{L^p_{\text{per}}(\mathbb{R}^N)}$. We note that the above also applies to $W^{1,p}_0$. When $p = 2$, we shall write H^1 for $W^{1,2}$.

We shall denote the characteristic function on some subset E of $\mathbb{R}^N$ by 1_E, where $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$, if $x \in E^c$. The ball of radius r in $\mathbb{R}^N$ centered at x will be denoted by $B(x,r)$. By $\|\cdot\|_{\text{op}}$, we shall mean the operator norm, and we will use $(\cdot)^\diamond$ to indicate the adjoint of $(\cdot)$.

DEFINITION 4. *Let $0 < \lambda < \beta$ and let Ω be a bounded open subset of $\mathbb{R}^N$ with sufficiently smooth boundary. We denote by $M(\lambda,\beta;\Omega)$ the bounded subset of $L^\infty(\Omega,\mathbb{R}^{N^2})$ consisting of matrix-valued functions A satisfying*

$$\forall \zeta \in \mathbb{R}^N \ \lambda|\zeta|^2 \leq A(x)\zeta \cdot \zeta \text{ and } \|A\|_{L^\infty(\Omega)} \leq \beta.$$

Before proceeding, let us make a remark on the measurability of functions of the type $B(x, (x/e))$. The condition $B \in L^2(\Omega \times Y)$ is not sufficient to ensure measurability of $B(x,(x/e))$; however, if we require $B \in C(\Omega, L^2(Y))$, for example, then this is equivalent to imposing a Caratheodory type condition on B and this is sufficient to ensure measurability.

2.2. Estimates for the Approximate H-Limit

DEFINITION 5. *Let $Y = [0,1]^N$, and for each $x \in \overline{\Omega}$, let T_x be a C^1-diffeomorphism on $\mathbb{R}^N$ and define*

$$Y_x = T_x(Y). \tag{9}$$

Then $\{Y_x\}_{x\in\overline{\Omega}}$ is a continuous family of parallelepipeds in $\mathbb{R}^N$, in the sense that the vectors spanning each parallelepiped form continuous vector fields. Then for each $x \in \overline{\Omega}$, define T^*_x to be the pullback. Then if $f : Y_x \to \mathbb{R}^d$, we have that

$$T^*_x f = f \circ T_x.$$

For notational convenience, we will often write B_x in place of $B(x,\cdot)$.

DEFINITION 6. *Let B_x be a matrix such that for each $x \in \Omega$ and $1 \leq p < \infty$, there exists a bounded constant $F_p > 0$ depending on p so that*

$$\begin{aligned} &B_x \in M\left(\lambda,\beta;\mathbb{R}^N\right), \qquad Y_x\text{-periodic}, \\ &\left\|T^*_{x_1}B_{x_1} - T^*_{x_2}B_{x_2}\right\|_{L^p_{\text{per}}(Y)} \leq F_p\,|x_1 - x_2|, \end{aligned} \tag{10}$$

i.e., $B_x \circ T_x$ is Lipschitz continuous in the $L^p_{\text{per}}(Y)$-norm.

We will need the following lemma which gives us a uniform bound on the gradients of the solutions w^x_η to (11) in the $L^p_{\text{per}}(Y_x)$-norm. This technical result is established in Section 3.

LEMMA 7. *For each $x \in \Omega$, let B_x satisfy (10) and let w^x_η be the Y_x-periodic solution of*

$$-\text{div}\left[B_x \nabla w^x_\eta\right] = -\text{div}\left[B_x\eta\right], \qquad \eta \in \mathbb{R}^N. \tag{11}$$

Then there exists some $p(N) > 2$ and some $H_\eta > 0$ such that

$$\left\|w^x_\eta\right\|_{W^{1,p}_{\text{per}}(Y_x)} \leq H_\eta. \tag{12}$$

We shall use the notation $|\alpha|$ to mean $\sqrt{N}\alpha$.

THEOREM 8. *For each $x \in \Omega$, let B_x satisfy (10) and w_η^x be the Y_x-periodic solution of (11). Then for all $x \in \Omega_n^\alpha$, there exists some bounded constant $C_\eta(n) > 0$ such that*

$$\left\| T_{x_n^\alpha}^* w_\eta^{x_n^\alpha} - T_x^* w_\eta^x \right\|_{H^1_{\mathrm{per}}(Y)} \le C_\eta |\alpha|. \tag{13}$$

PROOF. Let us denote $\overline{H}^1_{\mathrm{per}}(Y)$ as simply $W(Y)$, and let us define for each $x \in \Omega$ the linear operator L_x mapping $W(Y)$ into its dual $W(Y)'$ by

$$L_x\phi = \operatorname{div}_y \left[T_x^{-1} \left(T_x^* B_x \right) T_x^{-1^t} \nabla_y \phi \right].$$

Using Definition 5 and letting the vector $b_\eta^x = T_x^* B_x \eta$, we may rewrite (11) as

$$\begin{aligned} T_x^* w_\eta^x &\in W(Y), \\ L_x \left(T_x^* w_\eta^x \right) &= \operatorname{div}_y \left[T_x^{-1} b_\eta^x \right]. \end{aligned} \tag{14}$$

Let $f \in W(Y)'$. Then from (14), we have that $|\langle w_\eta^x \circ T_x - w_\eta^{x_n^\alpha} \circ T_{x_n^\alpha}, f \rangle|$ is bounded by

$$\left| \left\langle L_x^{-1} \nabla \left(T_x^{-1} b_\eta^x - T_{x_n^\alpha}^{-1} b_\eta^{x_n^\alpha} \right), f \right\rangle \right| + \left| \left\langle \left(L_{x_n^\alpha}^{-1} - L_x^{-1} \right) \nabla \left(T_{x_n^\alpha}^{-1} b_\eta^x \right), f \right\rangle \right|. \tag{15}$$

As a consequence of (10) and the fact that for each $x \in \Omega$, T_x and T_x^{-1} are both C^1, we have the following estimates:

$$\left\| T_x^{-1} b_\eta^x - T_{x_n^\alpha}^{-1} b_\eta^{x_n^\alpha} \right\|_{L^p_{\mathrm{per}}(Y)} \le J_p^\eta |\alpha|, \tag{16}$$

$$\left\| T_x^{-1} b_\eta^x T_x^{-1^t} - T_{x_n^\alpha}^{-1} b_\eta^{x_n^\alpha} T_{x_n^\alpha}^{-1^t} \right\|_{L^p_{\mathrm{per}}(Y)} \le N_p^\eta |\alpha|, \tag{17}$$

for some bounded positive constants J_p^η and N_p^η depending on p and η.

The first term in (15) is estimated using (16) and is bounded by

$$J_2^\eta |\alpha| \lambda^{-1} \|f\|_{W(Y)'}. \tag{18}$$

The second term requires some algebraic manipulation. We express $T_x^{-1} - T_{x_n^\alpha}^{-1}$ as $T_x^{-1}(T_{x_n^\alpha} - T_x) T_{x_n^\alpha}^{-1}$ and use the properties of the adjoint operator and the Cauchy-Schwartz lemma to bound this term by

$$\left\| \left(T_{x_n^\alpha}^{-1} b_\eta^{x_n^\alpha} T_{x_n^\alpha}^{-1^t} - T_x^{-1} b_\eta^x T_x^{-1^t} \right) \nabla \left(w_\eta^{x_n^\alpha} \circ T_{x_n^\alpha} \right) \right\|_{L^2_{\mathrm{per}}(Y)} \left\| L_{x_n^\alpha}^{-1^\diamond} f \right\|_{W(Y)'}. \tag{19}$$

Lemma 7 gives us the existence of some $p_0 > 2$ and $H_\eta > 0$ such that

$$\left| T_{x_n^\alpha} \right| \left\| \nabla_{y_{x_n^\alpha}} w_\eta^{x_n^\alpha} \right\|_{L^{p_0}_{\mathrm{per}}(Y_{x_n^\alpha})} \le H_\eta. \tag{20}$$

Then, if $p = (1/2)p_0$, then $p > 1$ and so its conjugate $q > 1$, so using Holder's inequality, the term in (19) is bounded by

$$\left\| L_{x_n^\alpha}^{-1} \right\|_{\mathrm{op}} \left| T_{x_n^\alpha} \right| \left\| T_{x_n^\alpha}^{-1} b_\eta^{x_n^\alpha} T_{x_n^\alpha}^{-1^t} - T_x^{-1} b_\eta^x T_x^{-1^t} \right\|_{L^{2q}_{\mathrm{per}}(Y)} \left\| w_\eta^{x_n^\alpha} \right\|_{W^{1,p_0}_{\mathrm{per}}(Y_{x_n^\alpha})} \|f\|_{W(Y)'}. \tag{21}$$

Then, using (17) and (20) and the fact that for each $x \in \Omega$, $B_x \in M(\lambda, \beta; \mathbb{R}^N)$, (21) is bounded by

$$\left[\lambda^{-1} H_\eta \left| T_{x_n^\alpha} \right| N_{2q}^\eta |\alpha| \right] \|f\|_{W(Y)'}, \tag{22}$$

and so by combining this with (18) and taking the supremum over all f in the unit ball of $W(Y)'$, we have the result (13). ∎

DEFINITION 9. *For each $\alpha > 0$, let $\{\Omega_n^\alpha\}_{n=1}^{n(\alpha)}$ be a finite set of interior disjoint cubes of side length α such that each cube is a translate of $(0,\alpha]^N$ and whose union covers $\overline{\Omega}$, and let x_n^α be an arbitrary element of $\Omega_n^\alpha \cap \overline{\Omega}$.*

DEFINITION 10. *For $\alpha > 0$, let $B^\alpha(x,(x/e))$ be defined by*

$$B^\alpha\left(x,\frac{x}{\varepsilon}\right) = \sum_{n=1}^{n(\alpha)} B\left(x_n^\alpha,\frac{x}{\varepsilon}\right) 1_{\Omega_n^\alpha}(x), \qquad x_n^\alpha \in \Omega_n^\alpha \cap \overline{\Omega}. \tag{23}$$

THEOREM 11. *Let $B^\alpha(x,(x/e)) \xrightarrow{H} B^{0\alpha}$ and $B(x,(x/e)) \xrightarrow{H} B^0$. Then there exists $C > 0$ such that for each $\alpha > 0$,*

$$\left\|B^{0\alpha} - B^0\right\|_{L^\infty(\Omega)} \le |\alpha| C. \tag{24}$$

Moreover, let u^0, $u^{0\alpha} \in H_0^1$ solve $-\operatorname{div}[B^0\nabla u^0] = f$, $-\operatorname{div}[B^{0\alpha}\nabla u^{0\alpha}] = f$. Then there exists $C' > 0$ such that

$$\left\|u^{0\alpha} - u^0\right\|_{H_0^1(\Omega)} \le |\alpha| C'. \tag{25}$$

PROOF. We wish to bound

$$\left|\frac{1}{m(Y_x)}\int_{Y_x} B_x \nabla_{y_x} w^x \, dy_x - \frac{1}{m\left(Y_{x_n^\alpha}\right)}\int_{Y_{x_n^\alpha}} B_{x_n^\alpha} \nabla_{y_{x_n^\alpha}} w^{x_n^\alpha} \, dy_{x_n^\alpha}\right|, \tag{26}$$

where for each x, w^x is the vector field on $\mathbb{R}^N$ whose components w_k^x are the Y_x-periodic solutions of (11) with $\eta = e_k$. With (9), we may rewrite (26) as

$$\left|\int_Y \left[T_x^* B_x \nabla_y \left(T_x^* w^x\right) T_x^{-1} - T_{x_n^\alpha}^* B_{x_n^\alpha} \nabla_y \left(T_{x_n^\alpha}^* w^{x_n^\alpha}\right) T_{x_n^\alpha}^{-1}\right] dy\right|. \tag{27}$$

By Definition 5, T_x^{-1} is C^1 so the fundamental theorem of calculus combined with the mean value theorem gives us some $K > 0$ such that

$$\left|T_x^{-1} - T_{x_n^\alpha}^{-1}\right| \le |\alpha| K, \qquad \forall\, x \in \Omega_n^\alpha. \tag{28}$$

Then, since B satisfies conditions (10), we may apply the Cauchy-Schwarz lemma and use Theorem 8 together with (28) to bound (27) by

$$|\alpha| C_n, \qquad \forall\, x \in \Omega_n^\alpha, \tag{29}$$

where C_n is some positive bounded constant depending on $n \in \{1,\dots,n(\alpha)\}$.

By using (10) again, together with the triangle inequality, we may also conclude that

$$\left|\frac{1}{m(Y_x)}\int_{Y_x} B_x e_k \, dy_x - \frac{1}{m\left(Y_{x_n^\alpha}\right)}\int_{Y_{x_n^\alpha}} B_{x_n^\alpha} e_k \, dy_{x_n^\alpha}\right| \le |\alpha| \beta F_1, \qquad \forall\, x \in \Omega_n^\alpha. \tag{30}$$

By noting how B^0 is defined in (5), we need only take the maximum of (29) over the set $\{1,\dots,n(\alpha)\}$ and combine this with (30) to obtain the estimate (24).

Let us now prove (25). For each $\alpha \ge 0$, let us define the operator $G_\alpha \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ by

$$G_\alpha u^{0\alpha} \equiv -\operatorname{div}\left[B^{0\alpha}\nabla u^{0\alpha}\right] = f, \qquad f \in H^{-1}(\Omega) \quad \left(B^0 = B^{00},\ u^0 = u^{00}\right).$$

By the coercivity of $B^{0\alpha}$ for $\alpha \geq 0$, it is clear that

$$\left\|{G_\alpha}^{-1}\right\|_{\text{op}} \leq c \tag{31}$$

for some $c > 0$ independent of f. Next, choose $g \in H^{-1}(\Omega)$ with $\|g\|_{H^{-1}(\Omega)} = 1$. Then

$$\begin{aligned}\left|\langle u^\alpha - u^0, g\rangle\right| &= \left|\left\langle \left({G_\alpha}^{-1} - {G_0}^{-1}\right) f, g\right\rangle\right| = \left|\left\langle {G_0}^{-1}\left(G_\alpha - G_0\right) {G_\alpha}^{-1} f, g\right\rangle\right| \\ &= \left|\left\langle \left(B^{0\alpha} - B^0\right) \nabla {G_\alpha}^{-1} f, \nabla {G_0}^{-1\diamond} g\right\rangle\right| \\ &\leq \left\|B^{0\alpha} - B^0\right\|_{L^\infty(\Omega)} \left\|{G_\alpha}^{-1} f\right\|_{H_0^1(\Omega)} \left\|{G_0}^{-1\diamond} g\right\|_{H_0^1(\Omega)} \\ &\leq c^2 \left\|B^{0\alpha} - B^0\right\|_{L^\infty(\Omega)} \|f\|_{L^2(\Omega)} \|g\|_{H^{-1}(\Omega)},\end{aligned}$$

where we have used (31) for the last inequality, and the fact that on a bounded domain $\|f\|_{H^{-1}(\Omega)} \leq \|f\|_{L^2(\Omega)}$. But $\|u^\alpha - u\|_{H_0^1} = \sup_{\|g\|_{H^{-1}(\Omega)}=1} |\langle u^\alpha - u, g\rangle|$, so with $C_1 = c^2\|f\|_{L^2(\Omega)}$,

$$\left\|u^\alpha - u^0\right\|_{H_0^1(\Omega)} \leq C_1 \left\|B^{0\alpha} - B^0\right\|_{L^\infty(\Omega)}. \tag{32}$$

Combining (32) with (24) gives us the estimate (25).

2.3. A Nonperiodic Distribution of Spheres

We wish to homogenize a problem of potentials in a composite material which possesses a nonperiodic microstructure. In particular, we wish to consider a system of spherical balls of radius ε embedded in a homogeneous material whose centers are not periodically distributed, but, nevertheless, are the image of a periodic lattice under a mapping θ.

Let us consider a bounded open Lipschitz domain Ω in $\mathbb{R}^N$. We study the behavior of

$$\begin{aligned} &u^\varepsilon \in H_0^1(\Omega), \\ &-\operatorname{div}\left[A^\varepsilon \nabla u^\varepsilon\right] = f, \qquad f \in H^{-1}(\Omega), \end{aligned} \tag{33}$$

as $\varepsilon \to 0$, where $A^\varepsilon \in M(a, \beta; \Omega)$ is a rapidly oscillating nonperiodic matrix taking the values A^1 in the spheres and A^2 in the matrix. As in [1], we let θ be a C^2 diffeomorphism of $\mathbb{R}^N$ with θ^{-1} admitting a Lipschitz constant $1/l$, $l > 1$; namely, for all x_1, x_2 in Ω

$$\left|\theta^{-1}(x_1) - \theta^{-1}(x_2)\right| \leq \frac{1}{l}\left|x_1 - x_2\right|. \tag{34}$$

This requirement states that θ "spreads-out" the material. In other words, we ensure that our spherical ball of fixed diameter ε can be strictly contained in each parallelepiped Y_x. Hence, our nonperiodic system of spherical balls is defined by

$$\theta(j\varepsilon) + B\left(0, \frac{\varepsilon}{2}\right), \qquad \forall j \in \mathbb{Z}^N. \tag{35}$$

Let $\Omega_s^\varepsilon = [\sum_{j\in\mathbb{Z}^N} \theta(j\varepsilon) + B(0, \varepsilon/2)] \cap \Omega$ and $\Omega_m^\varepsilon = \Omega/\Omega_s^\varepsilon$. Then the nonperiodic matrix A^ε of (33) is defined by

$$A^\varepsilon(x) = A^1 1_{\Omega_s^\varepsilon}(x) + A^2 1_{\Omega_m^\varepsilon}(x). \tag{36}$$

In order to obtain a homogenization for (33), we form a one-parameter family of periodic matrices. Namely, for each point x in Ω, we associate a periodic matrix, whose fundamental period is the parallelepiped spanned by the push-forward of the standard orthonormal frame at $\theta^{-1}(x)$. Thus, for each $x \in \overline{\Omega}$, we have a periodic system of spherical balls that is defined by

$$\nabla\theta\left(\theta^{-1}(x)\right) \varepsilon j + B\left(x, \frac{\varepsilon}{2}\right), \qquad \forall j \in \mathbb{Z}^N. \tag{37}$$

Hence, with $\Omega^\varepsilon_{s,x} = [\sum_{j\in\mathbf{Z}^N} \nabla\theta(\theta^{-1}(x))\varepsilon j + B(x,\varepsilon/2)] \cap \Omega$ and $\Omega^\varepsilon_{m,x} = \Omega/\Omega^\varepsilon_{s,x}$, we define our pointwise periodic matrix $B^\varepsilon(x) = B(x,(x/e))$ by

$$B(x,y) = A^1 1_{\Omega^1_{s,x}}(y) + A^2 1_{\Omega^1_{m,x}}(y), \tag{38}$$

where for each x, the matrix $B(x,y)$ is $\nabla\theta(\theta^{-1}(x))(Y)$-periodic. Let

$$T_x \equiv \nabla\theta\left(\theta^{-1}(x)\right). \tag{39}$$

If we now wish to use the periodic matrix B to form a locally periodic approximation B^α as in Definition 10, then in order to apply Theorem 11, we must show that $B_x \circ T_x$ is Lipschitz continuous in the $L^p_{\text{per}}(Y)$-norm.

THEOREM 12. *Let Ω^α_n be an arbitrary partitioning cube and let us fix some arbitrary point $x^\alpha_n \in \Omega^\alpha_n$. Let B_x be defined by (38). Then for $1 \le p < \infty$, there exists a bounded constant $F_p > 0$ such that for all $x \in \Omega^\alpha_n$,*

$$\left\|B(x,\cdot)\circ T_x - B\left(x^\alpha_n,\cdot\right)\circ T_{x^\alpha_n}\right\|_{L^p_{\text{per}}(Y)} \le |\alpha| F_p. \tag{40}$$

In the following proof, whenever coordinates are used, they are with respect to the standard orthonormal frame in $\mathbb{R}^N$.

PROOF. Let x be an arbitrary point in Ω^α_n and let γ be the center of Y. Since T_x is linear for all $x \in \overline{\Omega}$, $T_x\gamma$ is the center of Y_x. Let us show that T_x^{-1} maps spheres centered at $T_x\gamma$ into ellipsoids which fit inside Y. From our assumption (34) and with θ being Gateaux differentiable, we have for each $x \in \mathbb{R}^N$,

$$\begin{aligned}\left|\nabla\theta\left(\theta^{-1}(x)\right) z_1 - \nabla\theta\left(\theta^{-1}(x)\right) z_2\right| &= \left|\nabla\theta\left(\theta^{-1}(x)\right)(z_1 - z_2)\right| \\ &= \lim_{\delta\to 0}\frac{1}{\delta}\left|\theta\left(\theta^{-1}(x) + \delta(z_1 - z_2)\right) - \theta\left(\theta^{-1}(x)\right)\right| \\ &\ge l \lim_{\delta\to 0}\frac{1}{\delta}\left|\delta(z_1 - z_2)\right| = l\left|z_1 - z_2\right|.\end{aligned}$$

Hence, with (39) we see that T_x^{-1} also admits a Lipschitz constant $1/l$ and so the length of the major axis of $T_x^{-1}(B(T_x\gamma, 1))$ is less than 1. Thus we can define $B(x,\cdot)$ by the Y_x-periodic extension of

$$\left(A^1 - A^2\right) 1_{B(T_x\gamma,1)} + A^2 1_{Y_x}.$$

Then it follows that $B_x \circ T_x$ is the Y-periodic extension of

$$\left(A^1 - A^2\right) 1_{T_x^{-1}B(T_x\gamma,1)} + A^2 1_Y,$$

so that

$$\int_Y \left|B_x\circ T_x - B_{x^\alpha_n}\circ T_{x^\alpha_n}\right| dy \le \beta \int_Y \left|1_{\gamma+T_x^{-1}B(0,1)} - 1_{\gamma+T_{x^\alpha_n}^{-1}B(0,1)}\right| dy. \tag{41}$$

Without loss of generality, let us consider $\gamma = 0$ and use polar coordinates to integrate (41). Let $d\eta$ be the measure on $\partial B(0,1)$ defined by

$$d\eta = \sin^{N-2}\eta_{N-1}\sin^{N-3}\eta_{N-2}\cdots\sin\eta_2\, d\eta_1\, d\eta_2\cdots,$$

and η a point on $\partial B(0,1)$ corresponding to the $N-1$ angles η_i. Then by the polar decomposition theorem, for each x, $T_x^{-1} = P_x R_x$, where R_x is a unitary matrix. Hence, by the continuity of T_x^{-1}, with Q_x defined by

$$Q_x = R^t_x R_{x^\alpha_n}, \qquad \forall\, x \in \Omega^\alpha_n,$$

Q_x is continuous as a function from $\mathbb{R}^N$ to $\mathbb{R}^{N^2}$ and

$$T_x^{-1} Q_x \eta = k T_{x_n^\alpha}^{-1} \eta, \qquad k \in \mathbb{R}, \forall \eta \text{ on } \partial B(0,1). \tag{42}$$

Then there is a constant c_N depending on N so that

$$\int_Y \left| 1_{T_x^{-1} B(0,1)} - 1_{T_{x_n^\alpha}^{-1} B(0,1)} \right| dy \le c_N \int_{\partial B(0,1)} \left| \left| T_x^{-1} Q_x \eta \right|^N - \left| T_{x_n^\alpha}^{-1} \eta \right|^N \right| d\eta. \tag{43}$$

Let

$$p_x = \frac{1}{2} \left| \left| T_x^{-1} Q_x \eta \right| + \left| T_{x_n^\alpha}^{-1} \eta \right| \right| \quad \text{and} \quad q_x = \frac{1}{2} \left| \left| T_x^{-1} Q_x \eta \right| - \left| T_{x_n^\alpha}^{-1} \eta \right| \right|. \tag{44}$$

Then $p_x \ge q_x \ge 0$ and

$$\left| \left| T_x^{-1} Q_x \eta \right|^N - \left| T_{x_n^\alpha}^{-1} \eta \right|^N \right| = \left| (p_x + q_x)^N - (p_x - q_x)^N \right| \le \left(2^N - 1 + N \right) p_x^{N-1} q_x, \tag{45}$$

where the inequality was furnished by using binomial expansions. Let us estimate q_x. By (42), we can write q_x as

$$\begin{aligned} q_x &= \frac{1}{2} \left| T_x^{-1} Q_x \eta - T_{x_n^\alpha}^{-1} I \eta \right| \\ &\le \frac{1}{2} \left(\left| \left(T_x^{-1} - T_{x_n^\alpha}^{-1} \right) \eta \right| + \left\| T_x^{-1} \right\|_{\text{op}} \left| (Q_x - I) \eta \right| \right). \end{aligned}$$

By the inverse function theorem, θ^{-1} is also a C^2 diffeomorphism and

$$\left[\nabla \theta \left(\theta^{-1}(x) \right) \right]^{-1} = \nabla \theta^{-1}(x), \qquad \forall x \in \Omega_n^\alpha.$$

So, by the fundamental theorem of calculus

$$\left| T_x^{-1}{}_{ij} - T_{x_n^\alpha}^{-1}{}_{ij} \right| \le |\alpha| C'_{ij}, \qquad \forall x \in \Omega_n^\alpha,$$

where

$$C'_{ij} = \left| \int_0^1 \left[\sum_{k=1}^N \frac{\partial^2 \theta_i^{-1}}{\partial x_j \partial x_k} \left(x_n^\alpha + t \left(x - x_n^\alpha \right) \right)^2 \right]^{1/2} dt \right|.$$

Let $\eta' = R_{x_n^\alpha} \eta$. Then $|(Q_x - I)\eta| = |(R_x^t - R_{x_n^\alpha}^t)\eta'|$ and so we can again estimate

$$\left| R_{x\,ij}^t - R_{x_n^\alpha\,ij}^t \right| \le |\alpha| C''_{ij}, \qquad \forall x \in \Omega_n^\alpha,$$

where by setting $z = \theta^{-1}(x)$, we get

$$C''_{ij} = \left| \int_0^1 \left[\sum_{k=1}^N \frac{\partial R_{ij}}{\partial z_k} \left(\theta^{-1} \left(x_n^\alpha + t \left(x - x_n^\alpha \right) \right) \right)^2 \right]^{1/2} \right| \left| \nabla \theta^{-1} \left(x_n^\alpha + t \left(x - x_n^\alpha \right) \right) \right| dt.$$

Then we have that

$$q_x \le \frac{1}{2} |\alpha| \left(|C' \eta| + \left\| T_x^{-1} \right\|_{\text{op}} |C'' \eta'| \right),$$

so that by (43) and (45) and with $f(x, \eta) = (|T_x^{-1} Q_x \eta| + |T_{x_n^\alpha}^{-1} \eta|)^{N-1} (|C' \eta| + \|T_x^{-1}\|_{\text{op}} |C'' \eta'|)$, we have that

$$\int_Y \left| 1_{T_x^{-1} B(0,1)} - 1_{T_{x_n^\alpha}^{-1} B(0,1)} \right| dy \le \frac{1}{4} c_N \left(2^N - 1 + N \right) |\alpha| \int_{\partial B(0,1)} f(x, \eta) \, d\eta.$$

But since θ is a C^2 diffeomorphism, $f(x,\cdot) \in L^p(\partial B(0,1))$, $1 \leq p < \infty$, and since $\overline{\Omega_n^\alpha}$ is compact, $g(\eta) = \max_{x \in \overline{\Omega_n^\alpha}} f(x,\eta)$ exists. Consequently, for all $x \in \Omega_n^\alpha$, and $1 \leq p < \infty$

$$\left\| B_x \circ T_x - B_{x_n^\alpha} \circ T_{x_n^\alpha} \right\|_{L^p_{\mathrm{per}}(Y)} \leq \frac{1}{4} c_N \left(2^n - 1 + N\right) |\alpha| \beta \|g\|_{L^p(\partial B(0,1))} \tag{46}$$

so that with $F_p = (1/4)c_N(2^n - 1 + N)\|g\|_{L^p(\partial B(0,1))}\beta$, we have proven the lemma. ∎

A remark is now in order. Due to the lack of uniformity in the nonperiodic geometry, our locally periodic matrix $B^\alpha(x,(x/e))$ as defined in Definition 10 does not satisfy the hypothesis of the comparison Theorem 1. In fact, as we have mentioned, the novelty of Briane's homogenization scheme requires forming the locally periodic matrix B^ε as defined in Step 3 of Section 1.3, where α is a function of ε such that $\varepsilon^m/\alpha(\varepsilon)^{m+1} \to 0$ for some $m \in \mathbb{N}$. Nevertheless, Theorem 2 guarantees that $B^\varepsilon \xrightarrow{H} B^0$, and we have shown that the H-limit $B^{0\alpha}$ is within $\mathcal{O}(|\alpha|)$ of B^0. In particular, we have the following corollary.

COROLLARY 13. *Let $A^\varepsilon \in M(\lambda,\beta;\Omega)$ H-converge to A^0. Then*

$$\left\| B^{0\alpha} - A^0 \right\|_{L^\infty(\Omega)} \leq |\alpha| C$$

for some bounded $C > 0$ independent of α.

3. A UNIFORM L^p BOUND FOR THE PERIODIC CELL SOLUTIONS

3.1. Discussion and Further Notation

Let us consider the equation

$$Lu \equiv -\mathrm{div}[A(x)\nabla u] = -\mathrm{div} f(x) \text{ in } Y, \qquad u \text{ is } Y\text{-periodic}, \tag{47}$$

where Y is an arbitrary parallelepiped which is the fundamental period of periodic $\mathbb{R}^N$. We assume that A is a Y-periodic measurable function in $M(\lambda,\beta;\mathbb{R}^N)$. Then by the standard theory for $p = 2$, we immediately obtain the estimates

$$\|u\|_{W^{1,2}(Y)} \leq \lambda^{-1}\|f\|_{L^2(Y)} \quad \text{and} \quad \left\|L^{-1}\right\|_{\mathrm{op}} \leq \lambda^{-1}. \tag{48}$$

In order to obtain such estimates when $p > 2$, an algebraic decomposition of the weak form of (47) is made which relates the ellipticity of the coefficient matrix A with the regularity of the Laplacian operator $\triangle$. It turns out that getting an estimate of the form (48) when $p > 2$ reduces to the analysis of the equation

$$-\triangle v = -\mathrm{div} f, \qquad f \in L^p\left(\mathbb{T}^N\right), \tag{49}$$

and in particular, to the determination of when the solution v belongs to $W^{1,p}(\mathbb{T}^N)$ and satisfies

$$\|v\|_{W^{1,p}(\mathbb{T}^N)} \leq K\|f\|_{L^p(\mathbb{T}^N)} \tag{50}$$

for some K independent of f and some p which depends on the dimension N of the torus.

The study of the gradients of the solutions to equations of the form (47) in the spaces L^p, $p > 2$ was made by Meyers in [10] in Euclidean space $\mathbb{R}^N$ for the Dirichlet boundary value problem; therein, problem had the necessity of estimates of the form (50) is established (for $\mathbb{R}^N$ instead of $\mathbb{T}^N$). Nevertheless, such estimates are only assumed as part of the hypothesis of the main theorem and not explicitly proven for any particular scenario. In [5], a similar analysis is performed for the Dirichlet problem in $\mathbb{R}^N$. A bound for the operator norm of the inverse

Laplacian is calculated using a Reisz-Thorin-type interpolation (this is equivalent to finding the constant K in (50)). By virtue of that method, in order to find a bound for a particular value of $p > 2$, it is assumed that such a bound is known for some arbitrarily large $p_o > p$.

We prove the estimate (50) on the torus $\mathbb{T}^N$, by constructing a parametrix for Δ. Namely, we find the inverse of the Laplacian modulo a smoothing operator. This allows us to give an explicit pointwise representation for the solution v and consequently obtain the estimate

$$\|v\|_{L^p(\mathbb{T}^N)} \leq K'\|f\|_{L^p(\mathbb{T}^N)} \tag{51}$$

for some $p > 2$ and some constant $K' > 0$ independent of f. We then use certain local results from the theory of pseudodifferential operators to get (50). We note that our construction of v clearly indicates the dependence of p on the dimension N. This then gives us a bound in $W^{1,p}(\mathbb{T}^N)$ of the solutions u of (47) in the usual way.

With this result, we show that the periodic solutions to the cell problem in homogenization theory have gradients which are uniformly bounded in L^p. In particular, if $Y \equiv [0,1]^N$ and w_η is the Y-periodic solution of

$$-\operatorname{div}\left[B(y)\nabla w_\eta(y)\right] = -\operatorname{div}\left[B_\eta\right],$$

where $B \in M(*\lambda, \beta; \mathbb{R}^N)$ is Y-periodic, and $\eta \in \mathbb{R}^N$, then there exists some $p(N) > 2$ and some bounded constant H_η such that

$$\|w_\eta\|_{W^{1,p}_{\mathrm{per}}(Y)} \leq H_\eta. \tag{52}$$

We will denote by $C_c^\infty(\Omega)$ the subspace of Ω consisting of those functions which are supported on some compact set $K \subset \Omega$. Also, we designate by $\mathcal{S}(\mathbb{R}^N)$, the Schwartz space of real-valued C^∞ functions on $\mathbb{R}^N$ which have rapid decay. For the remainder of this paper, we shall use q to indicate the conjugate exponent of p, i.e., $p^{-1} + q^{-1} = 1$.

3.2. A $W^{1,p}(\mathbb{T}^N)$ Uniform Bound

In order to prove our first result, we will need some well-known results from the theory of pseudodifferential operators. We begin by defining a symbol.

DEFINITION 14. *A function a belongs to S^m and is of order m if $a(x,\xi) \in C^\infty(\mathbb{R}^{2N})$ and satisfies the differential inequalities*

$$\left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}, \qquad \forall\, \alpha, \beta \in \mathbb{Z}_+^N.$$

We associate with each symbol a, the pseudodifferential operator T_a defined by

$$\left(T_a f\right)(x) = \int_{\mathbb{R}^N} a(x,\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}\, d\xi, \qquad \forall\, f \in \mathcal{S}\left(\mathbb{R}^N\right), \tag{53}$$

where

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} f(x)e^{2\pi i x\cdot\xi}\, d\xi$$

is the Fourier transform of f on $\mathbb{R}^N$. Next, we define symbolic calculus on pseudodifferential operators.

LEMMA 15. *Let $a \in S^m$ and $b \in S^{m'}$. Then there exists $c \in S^{m+m'}$ such that*

$$T_c = T_a \circ T_b$$

and

$$c - \sum_{|\alpha|<N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha a \partial_x^\alpha b \in S^{m+m'-N}, \qquad \forall\, N > 0. \tag{54}$$

This in turn implies that if we let $a \in S^2$ be the characteristic polynomial of the Laplacian defined on some open subset Ω of $\mathbb{R}^N$, and if $c(x,\xi)$ is some arbitrary symbol in $S^{m'}$ supported for x in a compact subset of Ω, then there exists $b \in S^{m'-2}$ so that

$$T_b T_a + T_e = T_c, \tag{55}$$

where $e \in S^{-\infty} \equiv \cap_m S^m$. Thus, with $T_c \in S^0$, T_b is the inverse of the Laplacian modulo the infinitely smoothing operator T_e (see [11]). Another result that we shall need is the following lemma (see [12]).

LEMMA 16. *Let T_b be the pseudodifferential operator associated to $b \in S^m$. Then T_b is a bounded mapping from $W^{k,p}(\mathbb{R}^N)$ to $W^{k-m,p}(\mathbb{R}^N)$ for arbitrary real k and m, whenever $1 < p < \infty$.*

We note that this theorem extends (53) from functions in $\mathcal{S}(\mathbb{R}^N)$. We can now prove our result.

THEOREM 17. *Whenever $f \in L^p(\mathbb{T}^N)$ and $v \in \overline{W}^{1,p}(\mathbb{T}^N)$ satisfies*

$$-\triangle v = -\operatorname{div} f, \tag{56}$$

then v verifies the estimate

$$\|v\|_{W^{1,p}(\mathbb{T}^N)} \leq k\|f\|_{L^p(\mathbb{T}^N)} \tag{57}$$

for some k independent of f and $p > 2$ depending on the dimension N of the torus.

PROOF. We first construct a parametrix for the Laplacian on the N-dimensional torus $\mathbb{T}^N$. Let us define

$$d(x,y) = \text{distance from } x \text{ to } y \text{ on } \mathbb{T}^N,$$

and a cutoff function $\phi \in C_c^\infty(\mathbb{R})$ with support on a compact set K such that

$$\begin{aligned} &\text{(i)} \quad K = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ &\text{(ii)} \quad \phi \equiv 1 \text{ on } \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \\ &\text{(iii)} \quad \text{slope of } \phi \text{ does not change sign on } \left(\frac{\pi}{3}, \frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, -\frac{\pi}{3}\right). \end{aligned} \tag{58}$$

Then,

$$\text{for small enough } \varepsilon > 0,\ \nabla_x \phi|_{(\varepsilon,\varepsilon)\cup(-\pi+\varepsilon,\pi-\varepsilon)} = 0. \tag{59}$$

We define

$$G(x,y) = \frac{-\phi(d(x,y))}{4\pi\, d(x,y)}. \tag{60}$$

It is then easy to see that

$$\triangle_x G(x,y) = \delta(x-y) - r(x,y), \tag{61}$$

where

$$r(x,y) = \frac{-\triangle_x \phi(d(x,y))}{4\pi\, d(x,y)} - \frac{\nabla_x \phi(d(x,y)) \nabla_x d(x,y)}{4\pi\, d(x,y)^2}.$$

The function $r(x,y)$ is smooth in (x,y) by (59). Let us write (61) in operator form. We then have

$$\triangle G = I + R, \tag{62}$$

where

$$Gf(x) = \int_{\mathbb{T}^N} G(x,y) f(y)\, dy, \quad \text{and} \quad Rf(x) = \int_{\mathbb{T}^N} r(x,y) f(y)\, dy.$$

Let us define the operator $L = \triangle - P$, where P is the orthogonal projection onto $N(\triangle)$ (null space of the Laplacian), i.e.,

$$Pf(x) = \frac{1}{m\left(\mathbb{T}^N\right)}\left(\int_{\mathbb{T}^N} f(x)\,dx\right)\cdot \mathbf{1}.$$

Then, since $L^{-1} : L^2(\mathbb{T}^N) \to L^2(\mathbb{T}^N)$ is a bounded operator and $\|L^{-1}\|_{\text{op}} \leq 1$, we have that $LG = (\triangle - P)G = I + R - PG$. But

$$(PGf)(x) = \left[m\left(\mathbb{T}^N\right)^{-1}\int_{\mathbb{T}^N} G(z,y)\,dy)\,dz\right]\cdot \mathbf{1} = cPf,$$

where the constant c is independent of y, i.e., $c = \int_{\mathbb{T}^N} G(z,y)\,dy$, so that $LG = I + R - cP$. Let us write L^{-1} in the form $G + S$. Then, $L^{-1} = G + S$ if and only if

$$I = L(G+S) = I + R - cP + LS \quad \text{or} \quad LS = cP - R.$$

Therefore $S = L^{-1}[cP - R]$, but $L^{-1}P = P$ so that $S = cP - L^{-1}R$. It is easy to see by Fourier series arguments that the kernel of S, $s(x,y) = c - L_x^{-1}r(x,y)$ is smooth. Hence, S is a smoothing operator.

We can now consider (56). Since $v \perp 1$, we can write this equation as

$$-Lv = -\text{div} f.$$

Then $v = -L^{-1}(\text{div} f) = -(G+S)\text{div} f$. Integrating by parts and noting that the boundary terms vanish on the torus, we can represent the solution as

$$v = \nabla(G+S)\cdot f.$$

Since the kernels of G and S are smooth, there is no problem in bringing the differentiation under the integral sign so that

$$v(x) = \int_{\mathbb{T}^N} \nabla_y(G(x,y) - s(x,y))f(y)\,dy. \tag{63}$$

By switching to polar coordinates, one can check that $\nabla_y(G-s) \in L^q(\mathbb{T}^N)$ for all $q < N/2$ so that for all such q by Holder's inequality, we have that

$$\|v\|_{L^p(\mathbb{T}^N)} \leq \sup_x \|\nabla_y(G(x,\cdot) - s(x,\cdot))\|_{L^q(\mathbb{T}^N)}\|f\|_{L^p(\mathbb{T}^N)} = C_1\|f\|_{L^p(\mathbb{T}^N)}. \tag{64}$$

It would be nice at this point if we could use (63) to get an estimate of the type (64) for ∇v; however, problems of interest generally are in three-dimensional space and when $N = 3$, we cannot differentiate again under the integral sign, as we lose the bound in L^q. We can, nevertheless, take a different approach to bound v in $W^{1,p}(\mathbb{T}^N)$.

Let Ω be an open subset of $\mathbb{R}^N$ and let us choose cutoff functions ψ_1 and ψ_2 in $C_c^\infty(\Omega)$ such that $\psi_2 \equiv 1$ on $\text{supp}(\psi_1)$. Then if $\Delta v = F$ in $\mathbb{R}^N$ and F is locally in $W^{-1,p}(\mathbb{R}^N)$, then (55) together with Lemma 16 imply that for some bounded constant $c > 0$,

$$\|\psi_1 v\|_{W^{1,p}(\mathbb{R}^N)} \leq c\left[\|\psi_2 f\|_{W^{-1,p}(\mathbb{R}^N)} + \|\psi_2 v\|_{L^p(\mathbb{R}^N)}\right]. \tag{65}$$

We wish to use this result to analyze $\triangle$ on the torus $\mathbb{T}^N$. (In this scenario, the principal symbol of $\triangle$ can be thought of as a nonzero C^∞-section of the cotangent bundle $T^*\mathbb{T}^N$.) Since $\mathbb{T}^N$ is a compact manifold, we choose a locally finite covering by local charts (O_j, χ_j) and a C^∞ partition of unity $\{\rho_j\}$ subordinate to this covering, i.e., we require that

(i) $\text{supp}(\rho_j) \subset O_j$
(ii) $\sum_j \rho_j(p) = 1$, $\forall\, p \in \mathbb{T}^N$, and $\forall\, j\ \rho_j \geq 0$.

Then for each j, we choose $\psi_j \in C_c^\infty(O_j)$ such that $\psi_j = 1$ on the supp(ρ_j). For notational convenience, in each chart we write $\rho_j v \circ \chi_j^{-1}$ as $\rho_j v$. Let $v_j = \rho_j v$, so that v_j has compact support and

$$\triangle v_j = \rho_j F + [\triangle, \rho_j](\psi_j v), \tag{66}$$

where $[\cdot,\cdot]$ is the commutator, whose symbol is congruent modulo a regularization to the Poisson bracket of the symbols of its arguments; hence, $[\triangle, \rho_j]$ has a symbol of order 1 (1 less than the symbol of the Laplacian). Considering (66) and (65), there exists some bounded constant $A > 0$ such that

$$\begin{aligned}\|v_j\|^p_{W^{1,p}(\mathbb{R}^N)} &\le A\left[\|\rho_j F + [\triangle, \rho_j](\chi_j v)\|^p_{W^{-1,p}(\mathbb{R}^N)} + \|v_j\|^p_{L^p(\mathbb{R}^N)}\right] \\ &\le A'\left[\|\rho_j F\|^p_{W^{-1,p}(\mathbb{R}^N)} + \|\chi_j v\|^p_{L^p(\mathbb{R}^N)} + \|v_j\|^p_{L^p(\mathbb{R}^N)}\right],\end{aligned}$$

where the constant A' depends on the support of u_j and the second inequality arises using Lemma 16 and the above mentioned properties of the commutator. Then by changing the constant and noting that we can sum finitely, we have

$$\sum_j \|v_j\|^p_{W^{1,p}(\mathbb{R}^N)} \le A'' \sum_j \left[\|\rho_j F\|^p_{W^{-1,p}(\mathbb{R}^N)} + \|v_j\|^p_{L^p(\mathbb{R}^N)}\right]. \tag{67}$$

By the triangle inequality, we have that

$$\|v\|_{W^{1,p}(\mathbb{T}^N)} \le \sum_j \|v_j\|_{W^{1,p}(\mathbb{R}^N)}, \tag{68}$$

and if we replace F by div$f$ where $f \in L^p(\mathbb{T}^N)$, we see that

$$\|\rho_j F\|_{W^{-1,p}(\mathbb{R}^N)} = \|\mathrm{div}\,(\rho_j f) - [\mathrm{div}, \rho_j]\, f\|_{L^p(\mathbb{R}^N)} \le \|\rho_j f\|_{L^p(\mathbb{R}^N)} + \|f\|_{W^{-1,p}(\mathbb{R}^N)}. \tag{69}$$

But by Holder's inequality

$$\left(\sum_j \|\rho_j f\|^p_{L^p(\mathbb{R}^N)}\right)^{1/p} \le \left(C\left\|\sum_j \rho_j f\right\|^p_{L^p(\mathbb{R}^N)}\right)^{1/p} = C\|f\|_{L^p(\mathbb{R}^N)}. \tag{70}$$

Hence, putting (70), (69), and (68) into (67) asserts that there exists some bounded constant $C_2 > 0$ so that

$$\|u\|_{W^{1,p}(\mathbb{T}^N)} \le C_2\left[\|f\|_{L^p(\mathbb{T}^N)} + \|u\|_{L^p(\mathbb{T}^N)}\right]. \tag{71}$$

Therefore, (71) together with (64) gives us (57).

DEFINITION 18. *Let us define the bilinear functional*

$$D_A(u,\phi) = \int_{\mathbb{T}^N} A(x)\nabla u(x)\nabla\phi(x)\,dx. \tag{72}$$

The following lemma of the existence and boundedness of solutions of the type (47) on $\mathbb{R}^N$ is well known and can be found in [10]. The result also holds on the torus.

THEOREM 19. *Consider the equation*

$$-\mathrm{div}[A(x)\nabla u] = -\mathrm{div} f \text{ on } \mathbb{T}^N. \tag{73}$$

Then, if $f \in L^p(\mathbb{T}^N)$ and

(i) *there exists $\lambda_1 > 0$ such that*

$$\inf_{\|\phi\|_{W^{1,q}(\mathbb{T}^N)}=1}\ \sup_{\|u\|_{W^{1,p}(\mathbb{T}^N)}=1} |D_A(u,\phi)| \ge \lambda_1 > 0 \tag{74}$$

holds for A and A^t, and

(ii) *there exists $k > 0$ such that $\triangle v = \mathrm{div} f$ has a solution in $\overline{W}^{1,p}(\mathbb{T}^N)$ verifying*

$$\|v\|_{W^{1,p}(\mathbb{T}^N)} \le k\|f\|_{L^p(\mathbb{T}^N)}, \tag{75}$$

then

$$\exists \text{ unique } u \in \overline{W}^{1,p}\left(\mathbb{T}^N\right) \text{ satisfying (73)} \tag{76}$$

and

$$\|u\|_{W^{1,p}(\mathbb{T}^N)} \le C\|f\|_{L^p(\mathbb{T}^N)}, \tag{77}$$

for some bounded constant $C > 0$ *independent of* f.

3.3. Application to Cell Solutions

THEOREM 20. *For each* $x \in \Omega$, *let* B^x *satisfy (10) and let* w_η^x *be the* Y_x*-periodic solution of*

$$-\operatorname{div}\left[B^x \nabla w_\eta^x\right] = -\operatorname{div}\left[B^x \eta\right]. \tag{78}$$

Then there exists some $p(N) > 2$ *and some* $H_\eta > 0$ *such that*

$$\left\|w_\eta^x\right\|_{W^{1,p}_{\text{per}}(Y_x)} \le H_\eta. \tag{79}$$

PROOF. Following Definition 18, we define

$$D_B^x(w_\eta^x, \phi) = \int_{Y_x} B^x \nabla w_\eta^x \nabla \phi \, dy_x.$$

Then, since $\mathbb{T}^N \cong \mathbb{R}^N/\mathbb{Z}^N$, with a linear change of variables, Theorem 17 gives us a $p > 2$ which depends on N such that $w_\eta^x \in \overline{W}^{1,p}_{\text{per}}(Y_x)$ satisfies $-\Delta w_\eta^x = \operatorname{div}[B^x\eta]$ in Y_x and $\|w_\eta^x\|_{W^{1,p}_{\text{per}}(Y_x)} \le k\|B^x\eta\|_{L^p_{\text{per}}(Y_x)}$ for some bounded $k > 0$ independent of B^x and η. Thus, by Theorem 19, it suffices to show that there exists $\lambda_1 > 0$ so that

$$\inf_{\|\phi\|_{W^{1,q}_{\text{per}}(Y_x)}=1} \; \sup_{\|w_\eta^x\|_{W^{1,p}_{\text{per}}(Y_x)}=1} \left|D_B^x\left(w_\eta^x, \phi\right)\right| \ge \lambda_1 \tag{80}$$

holds for both B^x and B^{xt}, for we can set H_η to be a constant bounding $C\|B^x\eta\|_{L^p_{\text{per}}(Y_x)}$ for the given p, where C is as in (77).

In order to get the estimate (80), we will make the usual algebraic decomposition (see [5,10]). For each $x \in \Omega$, let

$$B^x = B_1^x + cI - B_2^x - cI, \tag{81}$$

where $B_1^x = (1/2)(B^x + B^{xt})$ and $B_2^x = (1/2)(B^x - B^{xt})$ and for the time being, let $c \in \mathbb{R}_+$ be arbitrary. It is then easy to see that

$$(\lambda + c)|\zeta|^2 \le (B_1^x + cI)\,\zeta \cdot \zeta \le (\beta + c)|\zeta|^2 \tag{82}$$

and that

$$\|B_2^x - cI\|_{\text{op}} \le \left(\beta^2 + c^2\right)^{1/2}.$$

Then, the norm of $B_2^x - cI$ is small when compared to $B_1^x + cI$ as c becomes large. In particular, let $c_m^2 = \min_{c\in\mathbb{R}^+}((\beta^2 + c^2)/(\lambda + c)^2)$ and define θ by

$$1 - \theta = \frac{\left(\beta^2 + c_m^2\right)^{1/2}}{\lambda + c^m}$$

so that $1 - \theta < 1$. Then we can write

$$\|B_2^x - c_m I\|_{\text{op}} \le (1 - \theta)\,(\lambda + c_m). \tag{83}$$

Since $\theta = (\lambda + c_m)^{-1}$ for the Laplacian operator, we see that as $\theta(\lambda + c_m) \to 1$, the more elliptic the matrix B^x becomes.

For each $A \in M(\lambda, \beta; \mathbb{R}^N)$ that is Y_x-periodic, let us define G^x_A to be the operator in $-\text{div}\,[A\nabla w^x_\eta] = -\text{div}[B^x\eta]$ in Y_x taking the vector field $B^x\eta$ to w^x_η, so that $G^x_A : L^p_{\text{per}}(Y_x) \to \overline{W}^{1,p}_{\text{per}}(Y_x)$. We then have that (see [10])

$$\inf_{\|\phi\|_{W^{1,q}_{\text{per}}(Y_x)}=1} \; \sup_{\|w^x_\eta\|_{W^{1,p}_{\text{per}}(Y_x)}=1} \left| D^x_A\left(w^x_\eta, \phi\right)\right| = \|G^x_A\|^{-1}_{\text{op}}. \tag{84}$$

Let us rewrite (78) as $w^x_\eta \in \overline{W}^{1,p}_{\text{per}}(Y_x)$ satisfying

$$\left[D^x_I + D^x_{(B_1+cI)-I} + D^x_{B_2-cI}\right]\left(w^x_\eta, \phi\right) = \int_{Y_x} B^x\eta\nabla\phi\, dy_x, \qquad \forall\, \phi \in \overline{W}^{1,q}_{\text{per}}\left(Y_x\right), \tag{85}$$

where D^x_I is the bilinear form associated with the Laplacian, i.e., $D^x_I(w^x_\eta, \phi) = \int_{Y_x} \nabla w^x_\eta\, \nabla\phi\, dy_x$. Therefore, with (82), (83), and (84), we satisfy the estimate (80), where

$$\lambda_1 = \|G^x_I\|^{-1}_{\text{op}} - \left(1 - \theta\left(\lambda + c_m\right)\right). \tag{86}$$

As discussed in [10], $\|G^x_I\|_{\text{op}}$ is a convex function of p which takes its minimum value of 1 at $p = 2$. Hence, we note that, if λ_1 is not positive with the particular choice of p that we obtained from Theorem 17, then choose some p' such that $2 \le p' \le p$ and use that p' as the appropriate Sobolev exponent for the theorem. Since we can do the same for B^{xt}, we are done. ∎

4. ANALYSIS OF DEFECTIVE FIBROUS COMPOSITES

Let us consider a cross-section of a reinforced composite material, designed to be periodic, but possessing a small defective region, wherein the fibers unexpectedly spread apart as in Figure 1.

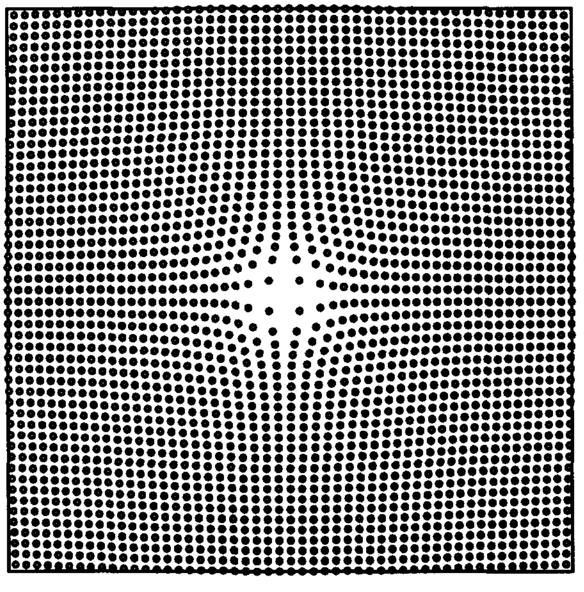

Figure 1. A portion of the square domain Ω with fiber-spreading defect, analyzed for the example problem.

4.1. Construction of θ

Let Ω^p be an open bounded connected subset of the first quadrant of $\mathbb{R}^2$, not containing the origin, and let $\theta : \Omega^p \to \Omega$, where Ω is the subset of $\mathbb{R}^2$ occupied by the fiber-reinforced material with defect. We must choose the diffeomorphism θ such that $\theta(B(\varepsilon j, \varepsilon/2))$, $j \in \mathbb{Z}^2$, describes the geometry of the material.

We examine the simplest case of axisymmetric spreading, and so we need only define a one-dimensional diffeomorphism along each radial direction. For more generality and other types of defects, see [13].

We label points in Ω^p by z and let

$$r\left(z^1, z^2\right) = \left(\left(z^1\right)^2 + \left(z^2\right)^2\right)^{1/2}, \qquad \Phi\left(z^1, z^2\right) = \tan^{-1}\left(\frac{z^2}{z^1}\right),$$

and fix the branch $[0, 2\pi)$. We then define our mapping by

$$\theta\left(z^1, z^2\right) = \left(R\left(z^1, z^2\right)\cos\left(\Phi\left(z^1, z^2\right)\right), R\left(z^1, z^2\right)\sin\left(\Phi\left(z^1, z^2\right)\right)\right), \tag{87}$$

where

$$R\left(z^1, z^2\right) = \left[\left(r\left(z^1, z^2\right) + v + i_0\right)\tanh\left(r\left(z^1, z^2\right) + i_0\right)\right],$$

and where v is y-intercept of the asymptote of $u \tanh(u)$ and i_0 is the inflection point of the function $(u + v)\tanh(u)$ which we include in order to ensure the monotonicity of the derivative of θ in the radial direction. Then, in the usual basis,

$$\nabla\theta(z) = \begin{bmatrix} \cos(\Phi)\dfrac{dR}{dr}\dfrac{z^1}{r} + R\sin(\Phi)\dfrac{z^2}{r^2} & \cos(\Phi)\dfrac{dR}{dr}\dfrac{z^2}{r} - R\sin(\Phi)\dfrac{z^1}{r^2} \\ \sin(\Phi)\dfrac{dR}{dr}\dfrac{z^1}{r} - R\cos(\Phi)\dfrac{z^2}{r^2} & \sin(\Phi)\dfrac{dR}{dr}\dfrac{z^2}{r} + R\cos(\Phi)\dfrac{z^1}{r^2} \end{bmatrix}, \tag{88}$$

where

$$\frac{dR}{dr} = \tanh\left(r + i_0\right) + \left(r + v + i_0\right)\operatorname{sech}^2\left(r + i_0\right). \tag{89}$$

Since

$$\det(\nabla\theta) = \frac{R}{r}\frac{dR}{dr},$$

it is clear that $\nabla\theta$ is nonsingular for each $z \in \Omega^p$, and that for our given branch, $\theta^{-1} \in C^\infty(\Omega, \Omega^p)$. See [13] for further discussion on this choice of diffeomorphism and its satisfaction of all the required conditions.

4.2. An Example

Although for simplicity we developed our theory for conductivity, it is a trivial extension to elasticity. Also, although our global model problem examined Dirichlet boundary conditions, we can also consider the Neumman or mixed problem with minor modifications.

We let Ω be a square region whose side lengths are 400 times that of the fiber diameter, and we position a single defective region in the center of Ω with

$$\text{diam}(\text{defective region}) = 8 \cdot \text{diam}(\text{fiber}),$$

and use the material properties found in Table 1. We note that in order to visually display the fiber cross-sections in Figure 1, we have restricted the domain in the figure to a square whose side lengths are approximately 60 times that of the fiber diameter.

Table 1. Properties used for the fibrous composite.

Properties	Fiber	Matrix
E	60.	.5
ν	.22	.35

Using the symmetry of the problem, we discretize only a quarter of the domain Ω. We partition Ω into $n(\alpha)$ squares Ω_n^α, and choose $x_n^\alpha \in \Omega_n^\alpha \cap \overline{\Omega}$, and use a Newton-Raphson scheme to compute

$$z_n^\alpha = \theta^{-1}(x_n^\alpha), \qquad \forall n \in \{1, \dots, n(\alpha)\},$$

and then solve $n(\alpha)$ periodic cell problems

$$\int_{Y_{x_n^\alpha}} B_{ijkl}(x_n^\alpha, y) \frac{\partial \chi_{kmn}}{\partial y_l}(x_n^\alpha, y) \frac{\partial \phi_{jmn}}{\partial y_i}(y)\, dy = -\int_{Y_{x_n^\alpha}} B_{ijmn}(x_n^\alpha, y) \frac{\partial \phi_{jmn}}{\partial y_i}(y)\, dy, \tag{90}$$

for all $\phi_{jmn} \in \overline{H}^1_{\text{per}}(Y_{x_n^\alpha})$ using the finite element method. From the right-hand side of (90) and the assumed hyperelasticity of the material, it is clear that χ_{kmn} is symmetric in m and n. We need only solve three separate boundary value problems corresponding to the three symmetric pairings of m and n to obtain the six independent components of χ.

To solve our unit cell problems, we use finite elements of class C^0 satisfying the assumptions H1, H2, and H3 [14, p. 132] and rely on [14, Theorem 3.2.3] to obtain

$$\chi^h_{kmn} - \chi_{kmn} \to 0 \quad \text{in } \overline{H}^1_{\text{per}},$$

where χ^h_{kmn} is the finite element solution of (90) corresponding to the discretization whose largest element is contained in a ball of diameter h. For these calculations, h was no larger than .001, and often much smaller for those period cells associated with global positions well inside the defective zone of the material.

The global boundary value problem which we consider is the simplest for which an exact solution is known. Namely, we prescribe a unit traction on the portion of the boundary $\partial\Omega$ normal to the x^1-axis. We employ the finite element method to solve this global boundary value problem as well. The computational domain is a square of side length 200 (using the quarter symmetry), so we choose $\alpha = .025$, and thus, $n(\alpha) = 8000$. Using h once again for the discretization size of the global domain and u_0^h as the finite element solution to (6), we have that $\|u_0 - u_0^h\|_{H_1(\Omega)} = \mathcal{O}(h)$. Thus, to preserve the $\mathcal{O}(.025)$ approximation of our homogenization scheme, we use a graded mesh with $h \le .001$ and take h much smaller in the defective zone to accommodate the small element size used in the finite element solutions of (90) for period cells associated to points in the defective zone.

In the case of a material occupying Ω with a uniformly periodic microstructure, it is well known that the resulting global stress and strain fields are

$$\begin{bmatrix} \overline{\sigma}_{xx} \\ \overline{\sigma}_{yy} \\ \overline{\sigma}_{xy} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \overline{\epsilon}_{xx} \\ \overline{\epsilon}_{yy} \\ \overline{\epsilon}_{xy} \end{bmatrix} = \begin{bmatrix} D_{11} \\ D_{12} \\ D_{13} \end{bmatrix}, \tag{91}$$

where $D = C^{-1}$.

However, the defect present in our material causes strain and stress concentrations to arise. In particular, Table 2 shows a strain concentration factor of 8, and a a global stress reduction of 20%. The following table lists the uniform, minimum, and maximum values of the global strains and stress.

Table 2. Global strains and stresses for spreading defect.

Global $\overline{\epsilon}$ and $\overline{\sigma}$	Minimum	Maximum	Uniform
$\overline{\epsilon}_{xx}$	.15	1.2	.15
$\overline{\epsilon}_{yy}$	−.54	0.0	0.0
$\overline{\epsilon}_{xy}$	−.1	0.1	0.0
$\overline{\sigma}_{xx}$	.80	1.2	1.
$\overline{\sigma}_{yy}$	−0.07	0.2	0.0
$\overline{\sigma}_{xy}$	−0.10	0.0	0.0

Let us define

$$\begin{aligned}\bar{\epsilon}^f(x) &= \frac{1}{m\left(Y_x^f\right)}\int_{Y_x^f}\epsilon(x,y)\,dy, \bar{\epsilon}^m(x) = \frac{1}{m\left(Y_x^m\right)}\int_{Y_x^m}\epsilon(x,y)\,dy,\\ \epsilon_{jl}(x,y) &= \frac{1}{2}\left(\gamma_{jl}+\gamma_{lj}\right),\\ \gamma_{jl} &= \frac{\partial u_i^0}{\partial x_l} + \frac{\partial u_j^0}{\partial x_k}\frac{\partial \chi_{kij}}{\partial y_l}(y),\end{aligned} \tag{92}$$

where v_f and v_m are the local volume (area) fractions of the fiber and matrix constituents, respectively, and Y_x^f and Y_x^m are the subsets of the period cell occupied by the two constituents. Using (92), we may define our cell-averaged constituent stresses as

$$\bar{\sigma}^f(x) = C_f\bar{\epsilon}^f(x), \qquad \bar{\sigma}^m(x) = C_m\bar{\epsilon}^m(x),$$

where C_f and C_m are the elasticity tensors of the fiber and the matrix, respectively. With this definition, Table 3 shows that the matrix stress $\bar{\sigma}^m$ increases by almost a factor of three in the defective region of the material.

Table 3. Matrix strains and stresses for spreading defect.

Global $\bar{\epsilon}^m$ and $\bar{\sigma}^m$	Minimum	Maximum	Uniform
$\bar{\epsilon}^m_{xx}$	.47	1.3	.47
$\bar{\epsilon}^m_{yy}$	−0.58	−0.1	−0.1
$\bar{\epsilon}^m_{xy}$	−0.18	0.1	0.0
$\bar{\sigma}^m_{xx}$	.30	.90	.38
$\bar{\sigma}^m_{yy}$	0.01	0.4	0.1
$\bar{\sigma}^m_{xy}$	~ 0	0.0	0.0

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