View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 343 (2008) 621–635
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Blow-up properties for a degenerate parabolic system with nonlinear localized sources ☆

Minxing Wang *, Yunfeng Wei

Department of Mathematics, Southeast University, Nanjing 210018, PR China

Received 25 July 2007
Available online 2 February 2008
Submitted by P. Sacks

Abstract

This paper deals with blow-up properties for a degenerate parabolic system with nonlinear localized sources subject to the homogeneous Dirichlet boundary conditions. The main aim of this paper is to study the blow-up rate estimate and the uniform blow-up profile of the blow-up solution. Our conclusions extend the results of [L.L. Du, Blow-up for a degenerate reaction–diffusion system with nonlinear localized sources, J. Math. Anal. Appl. 324 (2006) 304–320]. At the end, the blow-up set and blow up rate with respect to the radial variable is considered when the domain Ω is a ball.
© 2008 Elsevier Inc. All rights reserved.

Keywords: Degenerate parabolic system; Blow-up rate; Uniform blow-up profile

1. Introduction

In this paper, we consider the following degenerate parabolic system with nonlinear localized sources

$$\begin{cases} u_t = u^{\alpha}\big(\Delta u + u^{p}(x,t)v^{q}(x_0,t)\big), & (x,t) \in \Omega \times (0,T), \\ v_t = v^{\beta}\big(\Delta v + v^{m}(x,t)u^{n}(x_0,t)\big), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), & x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

where parameters $q, n > 0$, $p, m \geqslant 0$, $\alpha, \beta \in (0,1)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and $x_0 \in \Omega$ is a fixed point. The initial data u_0, v_0 satisfies the following conditions:

(H1) $u_0, v_0 \in C^{2+\tilde{\alpha}}(\Omega) \cap C^1(\bar{\Omega})$ for some $\tilde{\alpha} \in (0,1)$, $u_0, v_0 > 0$ in Ω, and $u_0 = v_0 = 0$, $\frac{\partial u_0}{\partial \nu} < 0$, $\frac{\partial v_0}{\partial \nu} < 0$ on $\partial\Omega$, where ν is the unit outward normal vector on $\partial\Omega$;

☆ This work was supported by PRC Grants NSFC 10471022, 10771032, and the Natural Science Foundation of Jiangsu province (BK2006088).
* Corresponding author.
E-mail address: mxwang@seu.edu.cn (M. Wang).

0022-247X/$ – see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2008.01.073

(H2) $\Delta u_0 + u_0^p v_0^q(x_0) \geqslant 0$, $\Delta v_0 + v_0^m u_0^n(x_0) \geqslant 0$ in Ω, and $\Delta u_0 = 0 = \Delta v_0$ on $\partial\Omega$;
(H3) $\Delta u_0 + u_0^p v_0^q(x_0) \geqslant \eta u_0^{1/\rho+1-\alpha}$, $\Delta v_0 + v_0^m u_0^n(x_0) \geqslant \eta v_0^{1/\theta+1-\beta}$, where positive constants ρ and θ are given in (2.1), and η is given in (2.2).

Set $Q_T = \Omega \times (0,T)$, $\Gamma_T = \partial\Omega \times (0,T)$ with $0 < T < \infty$.

Theorem 1.1. *Assume that* (H1)–(H2) *hold. Then the problem* (1.1) *has a unique positive classical solution* $(u,v) \in [C_{\mathrm{loc}}^{2+\hat\alpha,\,1+\hat\alpha/2}(\Omega \times (0,T]) \cap C(\overline{\Omega} \times [0,T])]^2$ *for some* $\hat\alpha$: $0 < \hat\alpha < 1$, *and* $u_t \geqslant 0$, $v_t \geqslant 0$. *Moreover, if* $T < \infty$ *then*

$$\lim_{t\to T}\left(\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty\right) = \infty.$$

Proof. Under the condition (H1), by the standard perturbation methods of [2, Theorem 2.5] for the single equation with a localized source and [17, Theorem 1] for the systems with two components, we can prove that the problem (1.1) has at least one positive classical solution $(u,v) \in [C_{\mathrm{loc}}^{2+\hat\alpha,1+\hat\alpha/2}(\Omega \times (0,T]) \cap C(\overline{\Omega} \times [0,T])]^2$ for some $\hat\alpha$: $0 < \hat\alpha < 1$. Thanks to the condition (H2), similar to Steps 1 and 2 in the proof of [17, Lemma 3], it can be proved that the positive classical solution is unique. The details was omitted here. □

Recently, the parabolic equations and systems with localized sources and local terms have attracted and been discussed by many authors, see [1–10,12–14,16,18,19]. Particularly, in the paper [4], Du proved that if $p < 1$, $m < 1$ and $qn < (1-p)(1-m)$, then every solution (u,v) of (1.1) is global; if $p > 1$ or $m > 1$ or $qn > (1-p)(1-m)$, then the solution (u,v) of (1.1) blows up in finite time for the large initial data and exists globally for the small initial data. Moreover, Du also studied the blow-up rates and uniform blow-up profiles of blow-up solutions for some special cases.

Theorem A. *(See [4].) Let conditions* (H1)–(H3) *hold and* (u,v) *be a solution of* (1.1) *which blows up in finite time* T.

(i) *If* $p = 0$ *or* $p > 1$, $m = 0$ *or* $m > 1$ *and satisfy* $q > \max\{1, m+\beta-1\}$, $n > \max\{1, p+\alpha-1\}$, *then there exist positive constants* C_i $(i = 1,2,3,4)$ *such that*

$$C_1 \leqslant \max_{x\in\bar\Omega} u(x,t)(T-t)^{(q+1-m-\beta)/[nq-(p+\alpha-1)(m+\beta-1)]} \leqslant C_2, \quad \forall 0 < t < T,$$
$$C_3 \leqslant \max_{x\in\bar\Omega} v(x,t)(T-t)^{(n+1-p-\alpha)/[nq-(p+\alpha-1)(m+\beta-1)]} \leqslant C_4, \quad \forall 0 < t < T.$$

(ii) *If* $p = m = 0$, *and* $n > 1$, $q > 1$, *then*

$$\lim_{t\to T}(T-t)^{(q+1-\beta)/\mu} u(x,t) = \mu^{-(q+1-\beta)/\mu}(n+1-\alpha)^{q/\mu}(q+1-\beta)^{(1-\beta)/\mu},$$
$$\lim_{t\to T}(T-t)^{(n+1-\alpha)/\mu} v(x,t) = \mu^{-(n+1-\alpha)/\mu}(q+1-\beta)^{n/\mu}(n+1-\alpha)^{(1-\alpha)/\mu}$$

uniformly on any compact subset of Ω, *where* $\mu = qn - (1-\alpha)(1-\beta)$.

The main purpose of the present paper is to study the blow-up rate estimate and uniform blow-up profile of the blow-up solution. Our results extend Theorem A. Moreover, we will discuss blow up set and blow-up rate with respect to the radial variable when the domain Ω is a ball.

This paper is organized as follows. In Sections 2 and 3, we estimate the blow-up rate and the uniform blow-up profile for the blow-up solution by modifying Souplet's method. In the final section, we will study the blow-up set and the blow-up rate in space with respect to the radial variable of blow-up solution when the domain Ω is a ball. Throughout this paper, we always assume that the solution (u,v) blows up in finite time T.

2. Estimate of the blow-up rate

Throughout this section we assume that

$$q > m+\beta-1, \qquad n > p+\alpha-1, \qquad nq > (p+\alpha-1)(m+\beta-1).$$

To simplify the notations, we set

$$1-p-\alpha=h, \qquad 1-m-\beta=k, \qquad \rho=\frac{q+k}{nq-hk}, \qquad \theta=\frac{n+h}{nq-hk}. \tag{2.1}$$

Then $\rho,\theta>0$ by our assumption. Denote

$$\eta_1=\frac{1}{\theta\alpha}\left(\frac{\theta(\rho+1)}{\rho(q\theta+1)}\right)^{q\theta+1}, \qquad \eta_2=\frac{1}{\beta\rho}\left(\frac{\rho(\theta+1)}{\theta(n\rho+1)}\right)^{n\rho+1},$$
$$\eta=\max\left\{\eta_1,\eta_2,\rho,\theta,\rho\left(2^{-1}c_0\right)^{\frac{-1}{\rho(n+h)}},\theta\left(2^{-1}c_0\right)^{\frac{-1}{\theta(q+k)}}\right\}, \tag{2.2}$$

where c_0 is given by (2.5).

The main result of this section is the following:

Theorem 2.1. *Assume that* (H1)–(H3) *hold. Then we have the following estimates*:

$$\left(\frac{c_0}{2}\right)^{1/(n+h)}(T-t)^{-\rho}\leqslant\max_{\bar\Omega}u(x,t)\leqslant\eta^{-\rho}\rho^{\rho}(T-t)^{-\rho},$$
$$\left(\frac{c_0}{2}\right)^{1/(q+k)}(T-t)^{-\theta}\leqslant\max_{\bar\Omega}v(x,t)\leqslant\eta^{-\theta}\theta^{\theta}(T-t)^{-\theta}.$$

To prove Theorem 2.1, we first prove two lemmas.

Lemma 2.1. *Assume that* (H1)–(H2) *hold. Let* $M_1(t)=\max_{\bar\Omega}u(x,t)$, $M_2(t)=\max_{\bar\Omega}v(x,t)$. *Then*

$$M_1^{n+h}(t)+M_2^{q+k}(t)\geqslant c_0(T-t)^{\frac{-(q+k)(n+h)}{nq-hk}}, \tag{2.3}$$

where c_0 *is a positive constant which will be given by* (2.5).

Proof. It is easy to see that $M_1(t)$ and $M_2(t)$ are Lipschitz continuous and satisfy

$$\lim_{t\to T}M_1(t)=\infty, \quad \text{or} \quad \lim_{t\to T}M_2(t)=\infty,$$
$$M_1'(t)\leqslant M_1^{\alpha+p}(t)M_2^{q}(t), \qquad M_2'(t)\leqslant M_2^{\beta+m}(t)M_1^{n}(t) \quad \text{a.e. } [0,T).$$

By Young's inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[M_1^{n+h}(t)+M_2^{q+k}(t)\right]\leqslant(n+h+q+k)M_1^n(t)M_2^q(t)\leqslant K\left[M_1^{n+h}(t)+M_2^{q+k}(t)\right]^{\frac{n(q+k)+q(n+h)}{(n+h)(q+k)}}, \tag{2.4}$$

where

$$K=(n+h+q+k)K_0^{\frac{n(q+k)+q(n+h)}{(n+h)(q+k)}}, \qquad K_0=\frac{\max\{n(q+k),q(n+h)\}}{n(q+k)+q(n+h)}.$$

Integrating (2.4) from t to T, we obtain that

$$M_1^{n+h}(t)+M_2^{q+k}(t)\geqslant c_0(T-t)^{\frac{-(q+k)(n+h)}{nq-hk}},$$

where

$$c_0=\left(\frac{(nq-hk)K}{(q+k)(n+h)}\right)^{\frac{-(q+k)(n+h)}{nq-hk}}. \tag{2.5}$$

The proof is complete. □

Lemma 2.2. *Assume that* (H1)–(H3) *hold. Then we have*

$$u_t-\eta u^{1/\rho+1}\geqslant 0, \qquad v_t-\eta v^{1/\theta+1}\geqslant 0, \quad (x,t)\in\bar Q_T.$$

Proof. Denote $J_1 = u_t - \eta u^{1/\rho+1}$, $J_2 = v_t - \eta v^{1/\theta+1}$. Using Theorem 1.1, we have $u_t, v_t \geqslant 0$, $(x,t) \in \bar{Q}_T$. A direct calculation yields

$$\begin{aligned}
&J_{1t} - u^{\alpha}\Delta J_1 - 2\eta\alpha u^{1/\rho} J_1 - q u^{\alpha+p} v^{q-1}(x_0,t) J_2(x_0,t)\\
&\quad = \alpha u^{-1} J_1^2 + \eta \frac{(\rho+1)}{\rho^2} u^{1/\rho+\alpha-1} |\nabla u|^2 + \alpha\eta^2 u^{2/\rho+1} + q\eta u^{\alpha+p} v^{q+1/\theta}(x_0,t)\\
&\qquad - \eta(1+1/\rho) u^{1/\rho+\alpha+p} v^q(x_0,t) + p u_t u^{p+\alpha-1} v^q(x_0,t)\\
&\quad \geqslant \alpha\eta^2 u^{2/\rho+1} + q\eta u^{\alpha+p} v^{q+1/\theta}(x_0,t) - \eta(1+1/\rho) u^{1/\rho+\alpha+p} v^q(x_0,t).
\end{aligned}$$

Notice that $q\theta/(1+q\theta) + 1/(2+\rho h) = 1$, by Young's inequality we have

$$u^{1/\rho} v^q(x_0,t) \leqslant \frac{\varepsilon^{-q\theta}}{2+\rho h}\big(u^{1/\rho}\big)^{2+\rho h} + \frac{\varepsilon q\theta}{q\theta+1}\big(v^q(x_0,t)\big)^{1+1/(q\theta)}.$$

Choose $\varepsilon = \rho(q\theta+1)/[\theta(\rho+1)]$, then we get

$$\begin{aligned}
&J_{1t} - u^{\alpha}\Delta J_1 - 2\eta\alpha u^{1/\rho} J_1 - q u^{\alpha+p} v^{q-1}(x_0,t) J_2(x_0,t)\\
&\quad \geqslant \alpha\eta^2 u^{2/\rho+1} + q\eta u^{\alpha+p} v^{q+1/\theta}(x_0,t) - \eta(1+1/\rho) u^{1/\rho+\alpha+p} v^q(x_0,t)\\
&\quad \geqslant \alpha\eta(\eta-\eta_1) u^{2/\rho+1} \geqslant 0.
\end{aligned}$$

Similarly

$$J_{2t} - v^{\beta}\Delta J_2 - 2\eta\beta v^{1/\theta} J_2 - n v^{\beta+m} u^{n-1}(x_0,t) J_1(x_0,t) \geqslant 0.$$

In view of $J_1 = J_2 = 0$ for $(x,t) \in \Gamma_T$ and $J_1(x,0), J_2(x,0) \geqslant 0$ for $x \in \bar{\Omega}$. By the comparison principle we have

$$u_t - \eta u^{1/\rho+1} \geqslant 0, \qquad v_t - \eta v^{1/\theta+1} \geqslant 0, \quad (x,t) \in \bar{Q}_T. \tag{2.6}$$

So we arrive at the conclusion. □

Proof of Theorem 2.1. By (2.6), we have

$$M_1'(t) \geqslant \eta M_1^{1/\rho+1}(t), \qquad M_2'(t) \geqslant \eta M_2^{1/\theta+1}(t) \quad \text{a.e. } [0,T). \tag{2.7}$$

Since (u,v) blows up in finite time T, without loss of generality, we may assume that $\lim_{t\to T} M_1(t) = \infty$. Integrating the first inequality of (2.7) from t to T, it yields

$$M_1(t) \leqslant \eta^{-\rho}\rho^{\rho}(T-t)^{-\rho}. \tag{2.8}$$

By (2.3) and the definition of η, we can prove that $\lim_{t\to T} M_2(t) = \infty$. Integrating the second inequality of (2.6) from t to T, we have

$$M_2(t) \leqslant \eta^{-\theta}\theta^{\theta}(T-t)^{-\theta}.$$

On the other hand, note that the definition of η, it follows from (2.3) and (2.8) that

$$M_2(t) \geqslant \left(\frac{c_0}{2}\right)^{1/(q+k)} (T-t)^{-\theta}, \quad \forall t \in (0,T).$$

Similarly,

$$M_1(t) \geqslant \left(\frac{c_0}{2}\right)^{1/(n+h)} (T-t)^{-\rho}, \quad \forall t \in (0,T).$$

The proof is completed. □

3. The uniform blow-up profile

In this section we study the uniform blow-up profile of (u, v) for the case: $p \leqslant 1-\alpha$, $m \leqslant 1-\beta$. Note that (u, v) blows up in finite time, there holds

$$nq \geqslant (1-p)(1-m) > (1-p-\alpha)(1-m-\beta).$$

So the parameters h, k, ρ and θ, defined in the previous section, satisfy $0 \leqslant h, k < 1$, $nq > hk$ and $\rho, \theta > 0$. Set

$$S_1 = \gamma^{-\rho}(q+k)^{k/\gamma}(n+h)^{q/\gamma}, \qquad S_2 = \gamma^{-\theta}(n+h)^{h/\gamma}(q+k)^{n/\gamma},$$

where $\gamma = nq - hk > 0$.

Theorem 3.1. *Assume that* (H1)–(H3) *hold. If $\alpha\rho < 1$, $\beta\theta < 1$, and $\Delta u_0 \leqslant 0$, $\Delta v_0 \leqslant 0$ on $\bar{\Omega}$, then the following statements hold uniformly on any compact subset of Ω.*

(i) *When $p < 1-\alpha$ and $m < 1-\beta$, then*

$$\lim_{t\to T}\frac{u(x,t)}{(T-t)^{-\rho}} = S_1, \qquad \lim_{t\to T}\frac{v(x,t)}{(T-t)^{-\theta}} = S_2.$$

(ii) *When $p = 1-\alpha$ and $m < 1-\beta$, then*

$$\lim_{t\to T}\frac{\ln u(x,t)}{|\ln(T-t)|} = \frac{q+k}{qn}, \qquad \lim_{t\to T}\frac{\ln v(x,t)}{|\ln(T-t)|} = \frac{1}{q}.$$

(iii) *When $p < 1-\alpha$ and $m = 1-\beta$, then*

$$\lim_{t\to T}\frac{\ln u(x,t)}{|\ln(T-t)|} = \frac{1}{n}, \qquad \lim_{t\to T}\frac{\ln v(x,t)}{|\ln(T-t)|} = \frac{n+h}{qn}.$$

(iv) *When $p = 1-\alpha$ and $m = 1-\beta$, then*

$$\lim_{t\to T}\frac{\ln u(x,t)}{|\ln(T-t)|} = \frac{1}{n}, \qquad \lim_{t\to T}\frac{\ln v(x,t)}{|\ln(T-t)|} = \frac{1}{q}.$$

In order to prove Theorem 3.1, we first prove some lemmas.

Lemma 3.1. *Assume that* (H1)–(H3) *hold, and $\Delta u_0 \leqslant 0$, $\Delta v_0 \leqslant 0$ on $\bar{\Omega}$. Then $\Delta u \leqslant 0$ and $\Delta v \leqslant 0$ on any compact subset of Ω.*

Proof. The proof is similar to that of Lemma 5.1 in [20]. □

Denote

$$f(t) = v^q(x_0,t), \qquad F(t) = \int_0^t f(s)\,ds, \qquad g(t) = u^n(x_0,t), \qquad G(t) = \int_0^t g(s)\,ds.$$

In the following, $f(t) \sim g(t)$ means that $\lim_{t\to T}\frac{f(t)}{g(t)} = 1$.

Lemma 3.2. *Assume that* (H1)–(H3) *hold. Then*

$$\lim_{t\to T} f(t) = \lim_{t\to T} F(t) = \infty, \qquad \lim_{t\to T} g(t) = \lim_{t\to T} G(t) = \infty.$$

Proof. Let

$$M_1(t) = \max_{\bar{\Omega}} u(x,t), \qquad M_2(t) = \max_{\bar{\Omega}} v(x,t),$$

then $M_1(t)$ and $M_2(t)$ are Lipschitz continuous and satisfy

$$M_1'(t) \leqslant M_1^{\alpha+p}(t) f(t), \qquad M_2'(t) \leqslant M_2^{\beta+m}(t) g(t) \quad \text{a.e. } [0,T). \tag{3.1}$$

By Theorem 2.1, we may assume that $M_1(0) > 1$, $M_2(0) > 1$. In view of $h \geqslant 0$, integrating the first inequality of (3.1) from 0 to t, we get

$$\frac{M_1^h(t)}{h} \leqslant \int_0^t f(s)\,\mathrm{d}s + \frac{M_1^h(0)}{h} =: \int_0^t f(s)\,\mathrm{d}s + M \quad \text{if } h > 0, \tag{3.2}$$

$$\ln M_1(t) \leqslant \int_0^t f(s)\,\mathrm{d}s + \ln M_1(0) =: \int_0^t f(s)\,\mathrm{d}s + \tilde{M} \quad \text{if } h = 0. \tag{3.3}$$

Since $\lim_{t\to T} M_1(t) = \infty$, it follows that $\lim_{t\to T} F(t) = \infty$. Note that $v_t \geqslant 0$, we see that $f(t)$ is monotone non-decreasing. It follows that $\lim_{t\to T} f(t) = \infty$ since $\lim_{t\to T} F(t) = \infty$. Similarly we have $\lim_{t\to T} g(t) = \lim_{t\to T} G(t) = \infty$.

Lemma 3.3. *Assume that* (H1)–(H3) *hold. If $\alpha\rho < 1$, $\beta\theta < 1$, and $\Delta u_0 \leqslant 0$, $\Delta v_0 \leqslant 0$ on $\bar{\Omega}$, then the following statements hold uniformly on any compact subset of Ω.*

(i) *When $p < 1-\alpha$ and $m < 1-\beta$, then*

$$\lim_{t\to T} \frac{u^h(x,t)}{hF(t)} = \lim_{t\to T} \frac{\|u(\cdot,t)\|_\infty^h}{hF(t)} = 1, \qquad \lim_{t\to T} \frac{v^k(x,t)}{kG(t)} = \lim_{t\to T} \frac{\|v(\cdot,t)\|_\infty^k}{kG(t)} = 1.$$

(ii) *When $p = 1-\alpha$ and $m < 1-\beta$, then*

$$\lim_{t\to T} \frac{\ln u(x,t)}{F(t)} = \lim_{t\to T} \frac{\|\ln u(\cdot,t)\|_\infty}{F(t)} = 1, \qquad \lim_{t\to T} \frac{v^k(x,t)}{kG(t)} = \lim_{t\to T} \frac{\|v(\cdot,t)\|_\infty^k}{kG(t)} = 1.$$

(iii) *When $p < 1-\alpha$ and $m = 1-\beta$, then*

$$\lim_{t\to T} \frac{u^h(x,t)}{hF(t)} = \lim_{t\to T} \frac{\|u(\cdot,t)\|_\infty^h}{hF(t)} = 1, \qquad \lim_{t\to T} \frac{\ln v(x,t)}{G(t)} = \lim_{t\to T} \frac{\|\ln v(\cdot,t)\|_\infty}{G(t)} = 1.$$

(iv) *When $p = 1-\alpha$ and $m = 1-\beta$, then*

$$\lim_{t\to T} \frac{\ln u(x,t)}{F(t)} = \lim_{t\to T} \frac{\|\ln u(\cdot,t)\|_\infty}{F(t)} = 1, \qquad \lim_{t\to T} \frac{\ln v(x,t)}{G(t)} = \lim_{t\to T} \frac{\|\ln v(\cdot,t)\|_\infty}{G(t)} = 1.$$

Proof. For the case (i), we have $h > 0$. Denote

$$w(x,t) = F(t) - \frac{u^h(x,t)}{h}, \qquad \phi(t) = \int_\Omega w(y,t)\varphi(y)\,\mathrm{d}y,$$

where $\varphi(x)$ is the principal eigenfunction of $-\Delta$ in Ω with the null Dirichlet boundary condition, and satisfies $\varphi > 0$ in Ω, $\int_\Omega \varphi(x)\,\mathrm{d}x = 1$. Let $\lambda_1 > 0$ be the corresponding eigenvalue. A directly computation shows that

$$\begin{aligned}
\phi'(t) &= \int_\Omega \big(f(t) - u^{h-1}(y,t)u_t(y,t)\big)\varphi(y)\,\mathrm{d}y = -\int_\Omega \big(u^{-p}(y,t)\Delta u(y,t)\varphi(y)\big)\,\mathrm{d}y \\
&= \int_\Omega \bigg(-\frac{1}{1-p}\Delta\big(u^{-p+1}(y,t)\big) - pu^{-(p+1)}(y,t)|\nabla u|^2\bigg)\varphi(y)\,\mathrm{d}y \leqslant \frac{-1}{1-p}\int_\Omega \Delta\big(u^{-p+1}(y,t)\big)\varphi(y)\,\mathrm{d}y \\
&= \frac{\lambda_1}{1-p}\int_\Omega u^{1-p}(y,t)\varphi(y)\,\mathrm{d}y = C\int_\Omega \big(F(t) - w(y,t)\big)^{\frac{1-p}{h}}\varphi(y)\,\mathrm{d}y.
\end{aligned}$$

Using $(a+b)^p \leqslant 2^{p-1}(a^p+b^p)$ for $a,b \geqslant 0$ and $p \geqslant 1$, and $\int_\Omega \varphi(y)\,\mathrm{d}y = 1$, we get

$$\phi'(t) \leqslant C\left(F^{\frac{1-p}{h}}(t) + \int_\Omega \left(w^-(y,t)\right)^{\frac{1-p}{h}} \varphi(y)\,\mathrm{d}y\right),$$

where $w^-(x,t) = \max\{-w(x,t),0\}$. By (3.2), we have

$$w(x,t) \geqslant -M, \quad (x,t) \in \Omega \times [0,T). \tag{3.4}$$

This implies $w^-(x,t) \leqslant M$. Hence $\phi'(t) \leqslant C(F^{\frac{1-p}{h}}(t)+1)$. Integrating this inequality from 0 to t yields

$$\phi(t) \leqslant C\left(1 + \int_0^t F^{\frac{1-p}{h}}(s)\,\mathrm{d}s\right).$$

Therefore,

$$\begin{aligned}\int_\Omega \left|w(y,t)\right|\varphi(y)\,\mathrm{d}y &= \int_{\{w\geqslant 0\}} w(y,t)\varphi(y)\,\mathrm{d}y - \int_{\{w<0\}} w(y,t)\varphi(y)\,\mathrm{d}y \\ &\leqslant \int_\Omega w(y,t)\varphi(y)\,\mathrm{d}y - 2\int_{\{w<0\}} w(y,t)\varphi(y)\,\mathrm{d}y \leqslant \phi(t) + C \\ &\leqslant C\left(1 + \int_0^t F^{\frac{1-p}{h}}(s)\,\mathrm{d}s\right).\end{aligned} \tag{3.5}$$

For any given $\zeta > 0$, define $\Omega_\zeta = \{y \in \Omega\colon \operatorname{dist}(y,\partial\Omega) \geqslant \zeta\}$. Note that $0<h<1$, by Lemma 3.1, we have $-\Delta w \leqslant 0$. Note that (3.5), we can use Lemma 4.5 in [15] and get that

$$\max_{\bar{\Omega}_\zeta} w(x,t) \leqslant \frac{C}{\zeta^{N+1}}\left(1 + \int_0^t F^{\frac{1-p}{h}}(s)\,\mathrm{d}s\right). \tag{3.6}$$

It follows from (3.4) and (3.6) that, for $x \in \bar{\Omega}_\zeta$ and $t \in (0,T)$,

$$-\frac{M}{F(t)} \leqslant \frac{w(x,t)}{F(t)} = 1 - \frac{u^h}{hF(t)} \leqslant \frac{C}{\zeta^{N+1}F(t)}\left(1 + \int_0^t F^{\frac{1-p}{h}}(s)\,\mathrm{d}s\right). \tag{3.7}$$

By (3.2) and Theorem 2.1, we get that, as t close to T,

$$F(t) \geqslant CM_1^h(x,t) \geqslant C(T-t)^{-h\rho}, \tag{3.8}$$

$$\begin{aligned}F(t) &= \int_0^t f(s)\,\mathrm{d}s = \int_0^t v^q(x_0,s)\,\mathrm{d}s \leqslant \int_0^t M_2^q(s)\,\mathrm{d}s \\ &\leqslant \eta^{-q\theta}\theta^{q\theta}\int_0^t (T-s)^{-q\theta}\,\mathrm{d}s \leqslant \frac{-\eta^{-q\theta}\theta^{q\theta}}{1-q\theta}(T-t)^{-q\theta+1} \\ &= \frac{\eta^{-q\theta}\theta^{q\theta}}{h\rho}(T-t)^{-h\rho}.\end{aligned} \tag{3.9}$$

Note that $\alpha\rho < 1$, it follows from (3.8) and (3.9) that

$$\lim_{t\to T} \frac{1}{F(t)}\int_0^t F^{\frac{1-p}{h}}(s)\,\mathrm{d}s = 0.$$

This combined with (3.7) yields that the following holds uniformly on $\bar{\Omega}_\zeta$:

$$\lim_{t\to T}\frac{u^h(x,t)}{hF(t)}=1. \tag{3.10}$$

We claim that

$$\liminf_{t\to T}\frac{\|u(\cdot,t)\|_\infty^h}{hF(t)}\geqslant 1. \tag{3.11}$$

If this is not true, then there exists $0<\varepsilon<1$, $t_i\to T$ and $x_i\in\Omega$ such that

$$u(x_i,t_i)=\max_{\bar{\Omega}}u(x,t_i),\qquad \frac{u^h(x_i,t_i)}{hF(t_i)}\leqslant 1-\varepsilon.$$

We may assume that $x_i\to x^*\in\bar{\Omega}$. Using (3.10), it is easy to derive that $x^*\in\partial\Omega$. For the small constant $\zeta>0$, we see that $x_i\notin\bar{\Omega}_\zeta=\{y\in\Omega\colon\ \mathrm{dist}(y,\partial\Omega)\geqslant\zeta\}$ for all $i\gg 1$. Since $\max_{\bar{\Omega}_\zeta}u(x,t_i)<u(x_i,t_i)$, it follows that

$$\frac{u^h(x,t_i)}{hF(t_i)}<\frac{u^h(x_i,t_i)}{hF(t_i)}\leqslant 1-\varepsilon,\quad \forall x\in\bar{\Omega}_\zeta.$$

This contradicts (3.10).

On the other hand, it follows from (3.2) that

$$\limsup_{t\to T}\frac{\|u(\cdot,t)\|_\infty^h}{hF(t)}\leqslant 1.$$

This combined with (3.11) yields

$$\lim_{t\to T}\frac{\|u(\cdot,t)\|_\infty^h}{hF(t)}=1.$$

Similarly, we can prove that the following holds uniformly on $\bar{\Omega}_\zeta$:

$$\lim_{t\to T}\frac{v^k(x,t)}{kG(t)}=\lim_{t\to T}\frac{\|v(\cdot,t)\|_\infty^k}{kG(t)}=1.$$

For the case (ii), we have $h=0$. Define

$$z(x,t)=F(t)-\ln u(x,t),\qquad \lambda(t)=\int_\Omega z(y,t)\varphi(y)\,\mathrm{d}y.$$

A direct computation shows that

$$\begin{aligned}
\lambda'(t)&=\int_\Omega\big(f(t)-u^{-1}(y,t)u_t(y,t)\big)\varphi(y)\,\mathrm{d}y\\
&=-\int_\Omega\big(u^{\alpha-1}(y,t)\Delta u(y,t)\varphi(y)\big)\,\mathrm{d}y\\
&=\int_\Omega-\left[\frac{1}{\alpha}\Delta u^\alpha(y,t)-(\alpha-1)u^{-2+\alpha}(y,t)|\nabla u|^2\right]\varphi(y)\,\mathrm{d}y\\
&\leqslant-\frac{1}{\alpha}\int_\Omega\varphi(y)\Delta u^\alpha(y,t)\,\mathrm{d}y\\
&=\frac{\lambda_1}{\alpha}\int_\Omega u^\alpha(y,t)\varphi(y)\,\mathrm{d}y\\
&=C\int_\Omega\exp\big\{\alpha\big[F(t)-z(y,t)\big]\big\}\varphi(y)\,\mathrm{d}y.
\end{aligned}$$

Using (3.3), we have

$$z(x,t) \geqslant -\tilde{M}, \quad (x,t) \in \Omega \times [0,T). \tag{3.12}$$

Thus

$$\lambda'(t) \leqslant C \int_\Omega \exp\{\alpha F(t)\}\varphi(y)\,\mathrm{d}y = C\exp\{\alpha F(t)\}.$$

Integrating from 0 to t, it yields

$$\lambda(t) \leqslant \lambda(0) + C\int_0^t \exp\{\alpha F(s)\}\,\mathrm{d}s \leqslant C\left(1 + \int_0^t \exp\{\alpha F(s)\}\,\mathrm{d}s\right).$$

Similar to the proof of (3.5) we have

$$\int_\Omega |z(y,t)|\varphi(y)\,\mathrm{d}y \leqslant C\left(1 + \int_0^t \exp\{\alpha F(s)\}\,\mathrm{d}s\right). \tag{3.13}$$

For any given $\zeta > 0$, similar to the above we define $\Omega_\zeta = \{y \in \Omega\colon \operatorname{dist}(y, \partial\Omega) \geqslant \zeta\}$. By Lemma 3.1, $-\Delta z \leqslant 0$. Note that (3.13), we can use Lemma 4.5 in [15] and get

$$\max_{\bar{\Omega}_\zeta} z(x,t) \leqslant \frac{C}{\zeta^{N+1}}\left(1 + \int_0^t \exp\{\alpha F(s)\}\,\mathrm{d}s\right). \tag{3.14}$$

It follows from (3.12) and (3.14) that

$$-\frac{\tilde{M}}{F(t)} \leqslant \frac{z(x,t)}{F(t)} = 1 - \frac{\ln u(x,t)}{F(t)} \leqslant \frac{C}{\zeta^{N+1}F(t)}\left(1 + \int_0^t \exp\{\alpha F(s)\}\,\mathrm{d}s\right), \quad x \in \bar{\Omega}_\zeta,\ t \in (0,T).$$

Without loss of generality, we assume that $T > 1$. By Theorem 2.1 we have

$$\begin{aligned} F(t) &= \int_0^t f(s)\,\mathrm{d}s \leqslant \int_0^t M_2^q(s)\,\mathrm{d}s \leqslant \eta^{-1}\theta\int_0^t (T-s)^{-1}\,\mathrm{d}s \\ &= \eta^{-1}\theta\ln(T-t)^{-1} + \eta^{-1}\theta\ln T \leqslant \ln(T-t)^{-1} + \ln T. \end{aligned}$$

Using (3.3), we get

$$F(t) \geqslant C\ln M_1(t) \geqslant C\ln(T-t)^{-\rho} \quad \text{as } t \to T.$$

Thus, for $x \in \bar{\Omega}_\zeta$ and $t \in (0,T)$,

$$-\frac{\tilde{M}}{F(t)} \leqslant 1 - \frac{\ln u(x,t)}{F(t)} \leqslant \frac{C}{\zeta^{N+1}\ln(T-t)^{-\rho}}\left(1 + \int_0^t \exp\{\alpha\ln(T-s)^{-1} + \alpha\ln T\}\,\mathrm{d}s\right). \tag{3.15}$$

Using $1-\alpha > 0$, it is easy to derive

$$\lim_{t\to T}\frac{1}{\ln(T-t)^{-\rho}}\int_0^t \exp\{\alpha\ln(T-s)^{-1} + \alpha\ln T\}\,\mathrm{d}s = 0. \tag{3.16}$$

Note that $F(t) \to \infty$ as $t \to T$, it follows from (3.15) and (3.16) that the following holds uniformly on $\bar{\Omega}_\zeta$:

$$\lim_{t\to T}\frac{\ln u(x,t)}{F(t)} = 1.$$

Similar to the proof of (3.11), we have

$$\liminf_{t\to T}\frac{\|\ln u(\cdot,t)\|_\infty}{F(t)}\geqslant 1.$$

It follows from (3.3) that

$$\limsup_{t\to T}\frac{\|\ln u(\cdot,t)\|_\infty}{F(t)}\leqslant 1.$$

Thus

$$\lim_{t\to T}\frac{\|\ln u(\cdot,t)\|_\infty}{F(t)}=1.$$

Similarly, we can prove that the following holds uniformly on $\bar{\Omega}_\zeta$:

$$\lim_{t\to T}\frac{v^k(x,t)}{kG(t)}=\lim_{t\to T}\frac{\|v(\cdot,t)\|_\infty^k}{kG(t)}=1.$$

The proofs of (iii) and (iv) are similarly. □

Lemma 3.4. *Let* (H1)–(H3) *hold. Assume that $\alpha\rho<1$, $\beta\theta<1$, and $\Delta u_0\leqslant 0$, $\Delta v_0\leqslant 0$ on Ω. Then for any given positive constants δ, ε, and τ satisfying $0<\delta$, $\varepsilon<1$ and $\tau>1$, there exists $\tilde{T}<T$ such that, for all $t\in[\tilde{T},T)$, the following statements hold*:

(i) *If $p=1-\alpha$ and $m<1-\beta$, then*

$$\begin{cases} n\delta F(t)\leqslant \ln\{\delta\varepsilon^{-1}\tau^{\frac{q}{k}}\}+\ln\dfrac{n}{q+k}+\dfrac{q+k}{k}\ln[kG(t)],\\ \ln\{\tau\varepsilon\delta^{\frac{q}{k}}\}+\ln\dfrac{n}{q+k}+\dfrac{q+k}{k}\ln[kG(t)]\leqslant n\tau F(t).\end{cases}$$

(ii) *If $p<1-\alpha$ and $m=1-\beta$, then*

$$\begin{cases} q\delta G(t)\leqslant \ln\{\delta\varepsilon^{-1}\tau^{\frac{n}{h}}\}+\ln\dfrac{q}{n+h}+\dfrac{n+h}{h}\ln[hF(t)],\\ \ln\{\tau\varepsilon\delta^{\frac{n}{h}}\}+\ln\dfrac{q}{n+h}+\dfrac{n+h}{h}\ln[hF(t)]\leqslant q\tau G(t).\end{cases}$$

(iii) *If $p=1-\alpha$ and $m=1-\beta$, then*

$$n\delta F(t)\leqslant\ln\frac{n\delta}{\varepsilon q\tau}+\tau qG(t),\qquad q\delta G(t)+\ln\frac{n\varepsilon\tau}{\delta q}\leqslant\tau nF(t).$$

Proof. (i) $p=1-\alpha, m<1-\beta$. By (ii) of Lemma 3.3, we know that for any given compact subset $\Omega_0\Subset\Omega$, which contains x_0, there exists $0<t_0<T$ such that the following hold on $\bar{\Omega}_0$:

$$\delta F(t)\leqslant\ln u(x,t)\leqslant\tau F(t),\qquad \delta kG(t)\leqslant v^k(x,t)\leqslant\tau kG(t),\quad t\in[t_0,T).$$

Therefore,

$$\exp\{n\delta F(t)\}\leqslant G'(t)\leqslant\exp\{n\tau F(t)\},\qquad [\delta kG(t)]^{\frac{q}{k}}\leqslant F'(t)\leqslant[\tau kG(t)]^{\frac{q}{k}},\quad t\in[t_0,T).$$

It follows that

$$\frac{[\delta kG(t)]^{\frac{q}{k}}}{\exp\{n\tau F(t)\}}\leqslant\frac{\mathrm{d}F(t)}{\mathrm{d}G(t)}\leqslant\frac{[\tau kG(t)]^{\frac{q}{k}}}{\exp\{n\delta F(t)\}},\quad t\in[t_0,T).\tag{3.17}$$

In view of the right-hand side of (3.17), we have

$$\exp\{n\delta F(t)\}\,\mathrm{d}F(t)\leqslant[\tau kG(t)]^{\frac{q}{k}}\,\mathrm{d}G(t),\quad t\in[t_0,T).$$

Integrating the above inequality from t_0 to t, we get

$$\frac{1}{n\delta}\exp\{n\delta F(t)\}\big|_{t_0}^{t}\leqslant(\tau k)^{\frac{q}{k}}\frac{k}{k+q}G^{\frac{k+q}{k}}(t)\big|_{t_0}^{t}\leqslant(\tau k)^{\frac{q}{k}}\frac{k}{k+q}G^{\frac{k+q}{k}}(t).$$

Due to $\lim_{t\to T}F(t)=\infty$, there exists $\tilde{t}_0$: $t_0\leqslant\tilde{t}_0<T$ such that

$$\frac{1}{n\delta}\exp\{n\delta F(t_0)\}\leqslant(1-\varepsilon)\frac{1}{n\delta}\exp\{n\delta F(t)\},\quad t\in[\tilde{t}_0,T).$$

Hence,

$$\frac{\varepsilon}{n\delta}\exp\{n\delta F(t)\}\leqslant\tau^{\frac{q}{k}}\frac{1}{k+q}\big[kG(t)\big]^{\frac{q+k}{k}},\quad t\in[\tilde{t}_0,T).$$

Thus we have

$$n\delta F(t)\leqslant\ln\{\delta\varepsilon^{-1}\tau^{\frac{q}{k}}\}+\ln\frac{n}{q+k}+\frac{q+k}{k}\ln\big[kG(t)\big],\quad t\in[\tilde{t}_0,T).\tag{3.18}$$

Applying the similar analysis as the above to the left-hand side of (3.17), there exists t_0^*: $t_0\leqslant t_0^*<T$ such that, for $t\in[t_0^*,T)$,

$$\ln\{\tau\varepsilon\delta^{\frac{q}{k}}\}+\ln\frac{n}{q+k}+\frac{q+k}{k}\ln\big[kG(t)\big]\leqslant n\tau F(t).\tag{3.19}$$

Set $\tilde{T}=\max\{\tilde{t}_0,t_0^*\}$, then (3.18) and (3.19) hold for $t\in[\tilde{T},T)$.

Analogous to the case (i), we can draw the cases (ii) and (iii). □

Proof of Theorem 3.1. For the case (i). By (i) of Lemma 3.3 we have that, as $t\to T$,

$$F'(t)=v^q(x_0,t)\sim\big[kG(t)\big]^{\frac{q}{k}},\qquad G'(t)=u^n(x_0,t)\sim\big[hF(t)\big]^{\frac{n}{h}}.$$

It follows that

$$\big[kG(t)\big]^{\frac{k+q}{k}}\sim\frac{(k+q)}{(h+n)}\big[hF(t)\big]^{\frac{h+n}{h}}.$$

Consequently,

$$F(t)\sim h^{-1}S_1^h(T-t)^{-h\rho},\qquad G(t)\sim k^{-1}S_2^k(T-t)^{-k\theta}.$$

This fact combined with the conclusion (i) of Lemma 3.3 asserts that the following hold uniformly on any compact subset of Ω:

$$\lim_{t\to T}\frac{u(x,t)}{(T-t)^{-\rho}}=S_1,\qquad\lim_{t\to T}\frac{v(x,t)}{(T-t)^{-\theta}}=S_2.$$

For the case (ii). Choose sequences $\{\delta_i\}_{i=1}^\infty$, $\{\varepsilon_i\}_{i=1}^\infty$ and $\{\tau_i\}_{i=1}^\infty$ satisfying $0<\delta_i,\varepsilon_i<1$, $\tau_i>1$ and $\delta_i\to1$, $\varepsilon_i\to1$, $\tau_i\to1$. Putting $(\delta,\varepsilon,\tau)=(\delta_i,\varepsilon_i,\tau_i)$ in Lemma 3.4, we get a sequence $\{T_i\}_{i=1}^\infty$ satisfying $T_i<T$ and $T_i\to T$, such that the corresponding conclusion (i) of Lemma 3.4 holds for $T_i\leqslant t<T$.

In view of $p=1-\alpha$ and $m<1-\beta$, by the second conclusion of (ii) of Lemma 3.3, there exists $\{\tilde{T}_i\}_{i=1}^\infty$ with $\tilde{T}_i<T$, $\tilde{T}_i\to T$, such that

$$\big[\delta_i kG(t)\big]^{\frac{q}{k}}\leqslant v^q(x_0,t)=f(t)=F'(t)\leqslant\big[\tau_i kG(t)\big]^{\frac{q}{k}},\quad\forall\tilde{T}_i\leqslant t<T.\tag{3.20}$$

Set $T_i^*=\max\{T_i,\tilde{T}_i\}$. Then for any $T_i^*\leqslant t<T$, (3.20) and the conclusion (i) of Lemma 3.4 hold. Thus we have

$$\begin{aligned}F'(t)&\geqslant\big[\delta_i kG(t)\big]^{\frac{q}{k}}\geqslant\delta_i^{\frac{q}{k}}\exp\left\{\frac{qn\delta_i}{q+k}F(t)\right\}\delta_i^{-\frac{q}{q+k}}\left(\frac{(q+k)\varepsilon_i}{n}\right)^{\frac{q}{q+k}}\tau_i^{-\frac{q^2}{k(k+q)}}\\&=\left(\frac{\delta_i}{\tau_i}\right)^{\frac{q^2}{k(k+q)}}\left(\frac{(q+k)\varepsilon_i}{n}\right)^{\frac{q}{q+k}}\exp\left\{\frac{qn\delta_i}{q+k}F(t)\right\},\\F'(t)&\leqslant\left(\frac{\tau_i}{\delta_i}\right)^{\frac{q^2}{k(k+q)}}\left(\frac{k+q}{n\varepsilon_i}\right)^{\frac{q}{k+q}}\exp\left\{\frac{qn\tau_i}{q+k}F(t)\right\}.\end{aligned}$$

Hence, for $T_i^* \leqslant t < T$,

$$\begin{cases} \exp\left\{-\dfrac{qn\delta_i}{q+k}F(t)\right\}F'(t) \geqslant \left(\dfrac{\delta_i}{\tau_i}\right)^{\frac{q^2}{k(k+q)}}\left(\dfrac{(q+k)\varepsilon_i}{n}\right)^{\frac{q}{q+k}}, \\ \exp\left\{-\dfrac{qn\tau_i}{q+k}F(t)\right\}F'(t) \leqslant \left(\dfrac{\tau_i}{\delta_i}\right)^{\frac{q^2}{k(k+q)}}\left(\dfrac{k+q}{n\varepsilon_i}\right)^{\frac{q}{k+q}}. \end{cases} \tag{3.21}$$

Let $A = -\ln qn + \frac{k}{q+k}\ln(q+k) + \frac{q}{q+k}\ln n$ and using $\lim_{t\to T} F(t) = \infty$, integrating (3.21) from t to T,

$$\frac{1}{\tau_i}\big(c_i + \big|\ln(T-t)\big|\big) \leqslant \frac{qn}{q+k}F(t) \leqslant \frac{1}{\delta_i}\big(C_i + \big|\ln(T-t)\big|\big), \tag{3.22}$$

where

$$c_i = A - \ln\tau_i - \frac{q^2}{(q+k)k}\ln\frac{\tau_i}{\delta_i} + \frac{q}{q+k}\ln\varepsilon_i,$$

$$C_i = A - \ln\delta_i - \frac{q^2}{(q+k)k}\ln\frac{\delta_i}{\tau_i} + \frac{q}{q+k}\ln\varepsilon_i^{-1}.$$

By joining (3.22) and (i) of Lemma 3.4, it follows that, for $T_i^* \leqslant t < T$,

$$\frac{\delta_i}{\tau_i}\big\{\hat{c}_i + \big|\ln(T-t)\big|\big\} \leqslant \frac{q}{k}\ln\big\{kG(t)\big\} \leqslant \frac{\tau_i}{\delta_i}\big\{\hat{C}_i + \big|\ln(T-t)\big|\big\}, \tag{3.23}$$

where

$$\hat{c}_i = c_i - \frac{\tau_i q}{\delta_i(q+k)}\ln\big\{\delta_i\varepsilon_i^{-1}\tau_i^{\frac{q}{k}}\big\} - \frac{\tau_i q}{\delta_i(q+k)}\ln\left\{\frac{n}{q+k}\right\},$$

$$\hat{C}_i = C_i - \frac{\delta_i q}{\tau_i(q+k)}\ln\big\{\varepsilon_i\tau_i\delta_i^{\frac{q}{k}}\big\} - \frac{\delta_i q}{\tau_i(q+k)}\ln\left\{\frac{n}{q+k}\right\}.$$

It follows from (3.22) and (3.23) that, when $T_i^* \leqslant t < T$,

$$\begin{cases} \dfrac{c_i + |\ln(T-t)|}{\tau_i|\ln(T-t)|} \leqslant \dfrac{qnF(t)}{(q+k)|\ln(T-t)|} \leqslant \dfrac{C_i + |\ln(T-t)|}{\delta_i|\ln(T-t)|}, \\ \dfrac{\delta_i\{\hat{c}_i + |\ln(T-t)|\}}{\tau_i|\ln(T-t)|} \leqslant \dfrac{q\ln\{kG(t)\}}{k|\ln(T-t)|} \leqslant \dfrac{\tau_i\{\hat{C}_i + |\ln(T-t)|\}}{\delta_i|\ln(T-t)|}. \end{cases} \tag{3.24}$$

Note that $\delta_i, \varepsilon_i, \tau_i \to 1$, and

$$c_i, C_i \to A, \qquad \hat{c}_i, \hat{C}_i \to A - \frac{q}{q+k}\ln\left\{\frac{n}{q+k}\right\}.$$

Letting $i \to \infty$ in (3.24), it yields

$$\lim_{t\to T}\frac{\ln\{kG(t)\}}{|\ln(T-t)|} = \frac{k}{q}, \qquad \lim_{t\to T}\frac{F(t)}{|\ln(T-t)|} = \frac{q+k}{qn}. \tag{3.25}$$

It follows from the second conclusion of (ii) of Lemma 3.3 that

$$k\ln v(x,t) \sim \ln\big[kG(t)\big]$$

uniformly on any compact subset of Ω. Thanks to the first conclusion of (3.25),

$$\lim_{t\to T}\frac{\ln v(x,t)}{|\ln(T-t)|} = \frac{1}{q}$$

holds uniformly on any compact subset of Ω. Similarly,

$$\lim_{t\to T}\frac{\ln u(x,t)}{|\ln(T-t)|} = \frac{q+k}{qn}$$

holds uniformly on any compact subset of Ω.

By the same way, we can prove conclusions (iii) and (iv). □

4. Blow-up set and blow-up rate in space: A special case

In this section, we study the blow-up set and the blow-up rates in space with respect to the radial variable of blow-up solution when the domain Ω is a ball. We need the following additional assumption:

(H4) $\Omega = B_R(0)$, $x_0 = 0$, and $u_0(x)$ and $v_0(x)$ are radially symmetric and non-increasing continuous functions.

Under the above assumptions (H1), (H2) and (H4), we have $u(x,t) = u(r,t)$, $v(x,t) = v(r,t)$ with $r = |x|$, and

$$u_t \geqslant 0, \qquad v_t \geqslant 0, \qquad u_r \leqslant 0, \qquad v_r \leqslant 0, \quad (r,t) \in (0,R) \times (0,T),$$
$$u(0,t) = \max_{\bar\Omega} u(x,t), \qquad v(0,t) = \max_{\bar\Omega} v(x,t).$$

Theorem 4.1. *Assume that* (H1), (H2) *and* (H4) *hold. If $p > 1$ and $m > 1$, then $x = 0$ is the only blow-up point of (u, v).*

Proof. We use the ideas of [7] and [11] to complete the proof. Without loss of generality, we assume that u blows up in finite time T. On the contrary we assume that u blows up in another point $x' \neq 0$, i.e. $\lim_{t\to T} u(x',t) = \infty$. Let $r^* = |x'| > 0$. Because of $u(r,t)$ is non-increasing in r, we have $\lim_{t\to T} u(r,t) = \infty$ for any $r \in [0, r^*]$.

Denote $B_R^\sigma(0) = \{x \in B_R(0)\colon x_1 > \sigma\}$ with $\sigma = r^*/3 > 0$. Let

$$J(x,t) = u_{x_1} + \psi(x_1)u^s(x,t), \quad (x,t) \in \overline{B_R^\sigma(0)} \times [0,T),$$

where $1 < s < p$, $\psi(x_1) = \varepsilon(x_1 - \sigma)^2$, and $\varepsilon > 0$ will be determined later. The carefully calculation gives

$$\begin{aligned}
J_t - u^\alpha \Delta J &= \big(u^{p+\alpha} v^q(0,t)\big)_{x_1} + \alpha u_{x_1} u^{-1}\big(u_t - u^{p+\alpha} v^q(0,t)\big) + s\psi(x_1)u^{s-1}\big(u_t - u^\alpha \Delta u\big) \\
&\quad - 2\varepsilon u^{s+\alpha} - 4s\varepsilon u^{s+\alpha-1}(x_1 - \sigma)u_{x_1} - s(s-1)\psi(x_1)u^{s+\alpha-2}|\nabla u|^2 \\
&\leqslant (p+\alpha)u^{p+\alpha-1} v^q(0,t)u_{x_1} - \alpha u^{p+\alpha-1} v^q(0,t)u_{x_1} + s\psi(x_1)u^{s+p+\alpha-1} v^q(0,t) \\
&\quad - 2\psi(x_1)u^{s+\alpha}(x_1 - \sigma)^{-2} - 4s\varepsilon u^{s+\alpha-1}(x_1 - \sigma)u_{x_1} \\
&= \big[p u^{p+\alpha-1} v^q(0,t) - 4\varepsilon s(x_1 - \sigma)u^{s+\alpha-1}\big]J \\
&\quad - \psi(x_1)u^{s+\alpha}\big[(p-s)u^{p-1} v^q(0,t) - 4\varepsilon s(x_1 - \sigma)u^{s-1} + 2(x_1 - \sigma)^{-2}\big] \\
&\leqslant c(x,t)J - \psi(x_1)u^{s+\alpha}\big[(p-s)u^{p-1} v^q(0,t) - 4\varepsilon s R u^{s-1} + 2R^{-2}\big],
\end{aligned} \tag{4.1}$$

where $c(x,t) = p u^{p+\alpha-1} v^q(0,t) - 4\varepsilon s(x_1 - \sigma)u^{s+\alpha-1}$. Notice that $v(0,t) \geqslant v(0,0) > 0$, $u(r,t) > 0$ for $(r,t) \in [0,R) \times [0,T)$ and $1 < s < p$, there exists $0 < \varepsilon_1 < 1$ such that for $0 < \varepsilon \leqslant \varepsilon_1$,

$$(p-s)u^{p-1} v^q(0,t) - 4\varepsilon s R u^{s-1} + 2R^{-2} \geqslant 0.$$

It follows from (4.1) that

$$J_t - u^\alpha \Delta J - c(x,t)J \leqslant 0, \quad (x,t) \in B_R^\sigma(0) \times (0,T).$$

We claim that u_{0r} is non-positive and non-trivial (otherwise $u_0(r) \equiv 0$, hence $u \equiv 0$ which contradicts the assumption that u blows up in finite time T). By the standard method we can deduce that $u_r(r,t) < 0$ in $\overline{B_R^\sigma(0)} \times (0,T)$. Thus $u_{x_1}(x,t) < 0$ in $\overline{B_R^\sigma(0)} \times (0,T)$. Hence

$$J(x,t) = u_{x_1}(x,t) < 0, \quad (x,t) \in \partial B_R^\sigma \times (0,T).$$

Taking t_0: $0 < t_0 < T$ and considering t_0 as the initial time, we may assume that $u_{x_1}(x,0) < 0$ on $\overline{B_R^\sigma(0)}$. So, there exists a constant $0 < \varepsilon_2 < 1$, such that when $0 < \varepsilon \leqslant \varepsilon_2$,

$$J(x,0) = u_{x_1}(x,0) + \psi(x_1)u_0^s(x) \leqslant u_{x_1}(x,0) + \varepsilon R^2 u_0^s(0) \leqslant 0, \quad x \in \overline{B_R^\sigma(0)}.$$

Choose $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Since for any fixed T_0: $0 < T_0 < T$, the function $c(x,t)$ is bounded on $\overline{B_R^\sigma(0)} \times [0,T_0]$, by the maximum principle we have

$$J(x,t) \leqslant 0, \quad (x,t) \in B_R^\sigma(0) \times [0,T_0],$$

and so

$$J(x,t) \leqslant 0, \quad (x,t) \in B_R^\sigma(0) \times [0,T).$$

That is

$$\psi(x_1) \leqslant -u^{-s}(x,t)u_{x_1}(x,t), \quad (x,t) \in B_R^\sigma(0) \times [0,T). \tag{4.2}$$

Let $y = (2\sigma, 0, \ldots)$ and $z = (r^*, 0, \ldots)$. Then $y, z \in B_R^\sigma(0)$. Integrating (4.2) from y to z

$$0 < \int_y^z \psi(x_1)\,\mathrm{d}x_1 \leqslant \frac{1}{s-1} u^{1-s}(z,t), \quad 0 < t < T.$$

Since $\lim_{t\to T} u^{1-s}(z,t) = 0$, we get a contradiction from the above inequality. □

Under some additional assumptions on the initial data, the blow-up rate in space can be evaluated as follows.

Theorem 4.2. *Assume that $p > 1$, $m > 1$ and* (H1)*,* (H2) *and* (H4) *hold. Suppose that there exist $c > 0$ and $0 \leqslant \xi \leqslant 1$ such that*

$$u_0'(r) \leqslant -cr^\xi, \qquad v_0'(r) \leqslant -cr^\xi, \quad r \in [0,R].$$

Then

$$u(r,t) \leqslant Cr^{-\gamma_1}, \qquad v(r,t) \leqslant Cr^{-\gamma_2}, \quad (r,t) \in (0,R] \times [0,T),$$

hold for some constant $C > 0$ and any $\gamma_1 > 2/(p-1)$, $\gamma_2 > 2/(m-1)$.

Proof. We only give an evaluation of $u(r,t)$. Similar as Theorem 4.1, we still apply the ideas of [7] and [11] to discuss the above statement. Let

$$J(r,t) = u_r(r,t) + c(r)u^\ell(r,t), \quad (r,t) \in [0,R] \times [0,T),$$

where $1 < \ell < p$, $c(r) = \varepsilon r^{1+\delta}$, $\delta > 0$, and $\varepsilon > 0$ to be determined later. A direct computation shows that

$$\begin{aligned}
&J_t - u^\alpha\left(\frac{N-1}{r}J_r + J_{rr}\right)\\
&\quad = \left[u_t - u^\alpha\left(\frac{N-1}{r}u_r + u_{rr}\right)\right]_r + \alpha u^{\alpha-1}u_r\left(\frac{N-1}{r}u_r + u_{rr}\right) - u^\alpha\frac{N-1}{r^2}u_r\\
&\qquad + \ell c(r)u^{\ell-1}\left[u_t - u^\alpha\left(\frac{N-1}{r}u_r + u_{rr}\right)\right] - (N-1)r^{-1}c'(r)u^{\ell+\alpha}\\
&\qquad - c''(r)u^{\ell+\alpha} - 2\ell c'(r)u^{\ell+\alpha-1}u_r - \ell(\ell-1)c(r)u^{\ell+\alpha-2}u_r^2\\
&\quad \leqslant \left[pu^{p+\alpha-1}v^q(0,t) - u^\alpha(N-1)r^{-2} - 2\ell(1+\delta)\varepsilon r^\delta u^{\ell-1+\alpha}\right]u_r\\
&\qquad + \ell c(r)u^{\ell-1+p+\alpha}v^q(0,t) - u^{\alpha+\ell}(N-1)r^{-2}(1+\delta)c(r) - (1+\delta)\delta r^{-2}c(r)u^{\ell+\alpha}\\
&\quad = b(r,t)J - c(r)u^{\ell+\alpha}\left[pu^{p-1}v^q(0,t) - (N-1)r^{-2} - 2\ell(1+\delta)\varepsilon r^\delta u^{\ell-1}\right.\\
&\qquad \left. - \ell u^{-1+p}v^q(0,t) + (N-1)r^{-2}(1+\delta) + (1+\delta)\delta r^{-2}\right]\\
&\quad = b(r,t)J - c(r)u^{\ell+\alpha}\left[(p-\ell)u^{p-1}v^q(0,t) - 2\ell(1+\delta)\varepsilon r^\delta u^{\ell-1} + r^{-2}\delta(N+\delta)\right]\\
&\quad \leqslant b(r,t)J - c(r)u^{\ell+\alpha}\left[(p-\ell)u^{p-1}v^q(0,t) - 2\ell(1+\delta)\varepsilon R^\delta u^{\ell-1} + R^{-2}\delta(N+\delta)\right],
\end{aligned}$$

where $b(r,t) \equiv pu^{p+\alpha-1}v^q(0,t) - (N-1)r^{-2}u^\alpha - 2\ell(1+\delta)\varepsilon r^\delta u^{\ell+\alpha-1}$.

In view of $v(r,t) > 0$, $v(0,t) = \max_{0\leqslant r\leqslant R} v(r,t)$, and $v(0,t) \geqslant v(0,0) > 0$, $1 < \ell < p$, there exists $0 < \varepsilon_1 < 1$ such that, for $0 < \varepsilon \leqslant \varepsilon_1$,

$$(p-\ell)u^{p-1}v^q(0,t) + \delta(N+\delta)R^{-2} - 2\ell(1+\delta)\varepsilon R^\delta u^{\ell-1} \geqslant 0, \quad (r,t) \in (0,R) \times (0,T). \tag{4.3}$$

Thus

$$J_t - u^\alpha\left(\frac{N-1}{r}J_r + J_{rr}\right) - b(r,t)J \leqslant 0, \quad (r,t) \in (0,R)\times(0,T).$$

In addition, as $u(r,t) > 0$ and $u(R,t) = 0$, we see that $u_r(R,t) \leqslant 0$. Therefore

$$J(0,t) = u_r(0,t) = 0, \qquad J(R,t) = u_r(R,t) \leqslant 0, \quad t \in (0,T).$$

For $t = 0$. Note that $0 \leqslant \xi \leqslant 1$, there exists $0 < \varepsilon_2 < 1$ such that, for $0 < \varepsilon \leqslant \varepsilon_2$,

$$J(r,0) = u_0'(r) + \varepsilon r^{1+\delta} u_0^\ell(r) \leqslant r^\xi\left[-c + \varepsilon R^{1+\delta-\xi} u_0^\ell(0)\right] \leqslant 0, \quad r \in [0,R]. \tag{4.4}$$

Choose $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, then (4.3) and (4.4) hold. Similar to the proof of Theorem 4.1, we have $J(r,t) \leqslant 0$, i.e.

$$-u^{-\ell} u_r \geqslant \varepsilon r^{1+\delta}, \quad (r,t) \in [0,R]\times[0,T).$$

Integrating the above inequality from 0 to r we have

$$u(r,t) \leqslant \left(\frac{\varepsilon(\ell-1)}{2+\delta} r^{2+\delta} + u^{1-\ell}(0,t)\right)^{-1/(\ell-1)} \leqslant \left(\frac{\varepsilon(\ell-1)}{2+\delta}\right)^{-1/(\ell-1)} r^{-\frac{2+\delta}{\ell-1}}, \quad (x,t) \in (0,R]\times[0,T).$$

Note that $\delta > 0$ and $1 < \ell < p$ are arbitrary, and $(2+\delta)/(\ell-1) \to 2/(p-1)$ as $\delta \to 0$ and $\ell \to p$, and $(2+\delta)/(\ell-1) \to \infty$ as $\ell \to 1$. For any $\gamma_1 > 2/(p-1)$, there exist $\delta > 0$ and $\ell \in (1,p)$ such that $\gamma_1 = (2+\delta)/(\ell-1)$. Hence $u(r,t) \leqslant Cr^{-\gamma_1}$ for any $\gamma_1 > 2/(p-1)$. $\square$

Acknowledgments

The authors are grateful to the referee and the associate editor for their helpful comments and suggestions.

References

[1] J.M. Chadam, A. Piece, H.M. Yin, The blow-up property of solutions to some diffusion equations with localized nonlinear reactions, J. Math. Anal. Appl. 169 (1992) 313–328.
[2] Y.P. Chen, C.H. Xie, Blow-up for porous medium equation with a localized source, Appl. Math. Comput. 159 (2004) 79–93.
[3] W.B. Deng, Global existence and finite time blow-up for a degenerate reaction–diffusion system, Nonlinear Anal. 60 (5) (2005) 977–991.
[4] L.L. Du, Blow-up for a degenerate reaction–diffusion system with nonlinear localized sources, J. Math. Anal. Appl. 324 (2006) 304–320.
[5] L.L. Du, A further blow-up analysis for a localized porous medium equation, Appl. Math. Comput. 179 (2006) 200–208.
[6] A. Friedman, B. Mcleod, Blow-up of solutions of nonlinear degenerate parabolic equations, Arch. Ration. Mech. Anal. 96 (1986) 55–80.
[7] A. Friedman, B. Mcleod, Blow-up of positive solutions of semilinear heat equations, Indiana Univ. Math. J. 34 (1985) 425–447.
[8] F.C. Li, S.X. Huang, C.H. Xie, Global existence and blow-up of solutions to a nonlocal reaction–diffusion system, Discrete Contin. Dyn. Syst. 9 (1992) 1519–1532.
[9] F.C. Li, C.H. Xie, Existence and blow-up for a degenerate parabolic equation with nonlocal source, Nonlinear Anal. 52 (2003) 523–534.
[10] H.A. Levine, The role of critical exponents in blow-up theorems, SIAM Rev. 32 (1990) 262–288.
[11] H.L. Li, M.X. Wang, Properties of blow-up solutions to a parabolic system with nonlinear localized terms, Discrete Contin. Dyn. Syst. 13 (2005) 683–700.
[12] H.L. Li, M.X. Wang, Uniform blow-up profiles and boundary layer for a parabolic system with localized nonlinear reaction terms, Sci. China Ser. A 48 (2) (2005) 185–197.
[13] C.V. Pao, Blow-up of solution for a nonlocal reaction–diffusion problem in combustion theory, J. Math. Anal. Appl. 166 (1992) 591–600.
[14] P. Souplet, Blow-up in nonlocal reaction–diffusion equations, SIAM J. Math. Anal. 29 (6) (1998) 1301–1334.
[15] P. Souplet, Uniform blow-up profiles and boundary behavior for diffusion equations with nonlocal nonlinear source, J. Differential Equations 153 (1999) 374–406.
[16] X.F. Song, S.N. Zheng, Z.X. Jiang, Blow-up analysis for a nonlinear diffusion systems, Z. Angew. Math. Phys. 56 (2005) 1–10.
[17] M.X. Wang, C.H. Xie, A degenerate and strongly coupled quasilinear parabolic system not in divergence form, Z. Angew. Math. Phys. 55 (2004) 741–755.
[18] M.X. Wang, Y.M. Wang, Properties of positive solutions for non-local reaction–diffusion problem, Math. Methods Appl. Sci. 19 (1996) 1141–1156.
[19] S. Wang, M.X. Wang, C.H. Xie, A nonlinear degenerated diffusion equation not in divergence form, Z. Angew. Math. Phys. 51 (2000) 149–159.
[20] S.N. Zheng, L.D. Wang, Blow-up rate and profile for a degenerate parabolic system coupled via nonlocal sources, Comput. Math. Appl. 52 (2006) 1387–1402.