MATHEMATICS

ON THE STRUCTURE OF WELL DISTRIBUTED SEQUENCES (IV) *)

BY

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1. Let (s_n) be a sequence of real numbers satisfying $0 \leqslant s_n \leqslant 1$, $(n=1, 2, \ldots)$. Let $I_{(a,b)}(x)$ denote the characteristic function of the interval (a, b), $0 \leqslant a < b \leqslant 1$. The sequence (s_n) is said to be well distributed if

$$\lim_{p \to \infty} \frac{1}{p} \sum_{k=n+1}^{n+p} I_{(a,b)}(s_k) = b - a$$

holds uniformly in n for every interval (a, b).

We denote the *fractional part* of θ by $\{\theta\}$, i.e. $\{\theta\} = \theta - [\theta]$, where $[\theta]$ is the largest integer contained in θ.

Let E be a subset of (0, 1) and let the density $\Delta(a, b)$ of E in the interval (a, b), $0 \leqslant a < b \leqslant 1$, be defined by the following relation

$$\Delta(a, b) = \frac{\text{outer measure } (E \cap (a, b))}{|b-a|}.$$

If E is of measure one, it is clear that $\Delta(a, b) = 1$ for every interval (a, b); likewise if E is of measure 0, it is clear that $\Delta(a, b) = 0$ for every interval (a, b). Sets having the same density for every interval in (0, 1) are called homogeneous. A necessary and sufficient condition for a measurable set to be homogeneous is that its measure be either zero or one. Moreover, if E is measurable, $\Delta(a, b) \geqslant \delta > 0$ for all intervals (a, b), then E is homogeneous and of measure one; (see KNOPP [2], p. 413, Satz 4).

We now prove:

Theorem 1. *Let $(n(k))$ be a sequence of real numbers,*

$$\frac{n(k)}{n(k-1)} = r(k) > M > 4, \qquad (k = 2, 3, \ldots)$$

then, for almost all α, $0 < \alpha \leqslant 1$, the sequence $\{n(k)\alpha\}$ is not well distributed.

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Proof: Let $F(p)$ be the set of α, $0<\alpha\leqslant 1$, such that:

$$\{n(k)\alpha\}\leqslant \tfrac{1}{2},\ k=q+1,\ \dots,\ q+p$$

for some $q=q(\alpha)$, $(p=1, 2, \dots)$. Then if $\alpha \in \bigcap_{p=1}^{\infty} F(p)=E$,

$$\lim_{p\to\infty}\frac{1}{p}\sum_{k=q+1}^{q+p} I_{(0,\frac{1}{2})}(\{n(k)\alpha\})=1$$

and the sequence is not well distributed. We shall show that $\mu(F(p))=1$, $(p=1, 2, \dots)$ and hence $\mu(E)=1$, and this implies our result.

Let E_k be the set of α for which $\{n(k)\alpha\}\leqslant\frac{1}{2}$. This set contains the first half of each of the intervals

$$\left(0,\frac{1}{n(k)}\right),\ \left(\frac{1}{n(k)},\ \frac{2}{n(k)}\right),\ \dots,\ \left(\frac{[n(k)]-1}{n(k)},\ \frac{[n(k)]}{n(k)}\right).$$

Contained in the interval

$$J'(r, n(k))=\left(\frac{r}{n(k)},\ \frac{2r+1}{2n(k)}\right),\qquad (r\leqslant[n(k)]-1)$$

there will be at least $[\frac{1}{2}r(k+1)]-1$ intervals of the form

$$J(r, n(k+1))=\left(\frac{r}{n(k+1)},\ \frac{r+1}{n(k+1)}\right);\ (r\leqslant[n(k+1)]-1),$$

for there can be at most two intervals of the form $J(r, n(k+1))$ which intersect $J'(r, n(k))$ but do not lie completely in $J'(r, n(k))$. Hence, the number of intervals of the form $J'(r, n(k+1))$ completely contained in E_k is at least

$$([\tfrac{1}{2}r(k+1)]-1)[n(k)]\geqslant\tfrac{1}{2}(\tfrac{1}{2}r(k+1)-2)\ n(k),$$

Each of the intervals $J'(r, n(k+1))$ in turn contains at least $[\frac{1}{2}r(k+2)]-1$ intervals of the form $J'(r, n(k+2))$. It follows that

$$\bigcap_{n=k+1}^{k+p} E_n$$

contains at least

$$\tfrac{1}{2}n(k+1)\ (\tfrac{1}{2}r(k+2)-2)\ \dots\ (\tfrac{1}{2}r(k+p)-2)$$

intervals of the form $J'(r, n(k+p))$. This implies that

$$\mu\left(\bigcap_{n=k+1}^{k+p} E_n\right)\geqslant\tfrac{1}{4}\frac{n(k+1)}{n(k+p)}\ (\tfrac{1}{2}r(k+2)-2)\ \dots\ (\tfrac{1}{2}r(k+p)-2)$$

$$=\tfrac{1}{4}\left(\tfrac{1}{2}-\frac{2}{r(k+2)}\right)\cdots\left(\tfrac{1}{2}-\frac{2}{r(k+p)}\right).$$

We have

$$\frac{2}{r(k+s)} < \frac{2}{M}, \text{ hence } \tfrac{1}{2} - \frac{2}{r(k+s)} > \frac{M-4}{2M} \, (s=1, 2, \ldots, p)$$

and

$$\mu\left(\bigcap_{n=k+1}^{k+p} E_n\right) > \tfrac{1}{4}\left(\frac{M-4}{2M}\right)^{p-1}.$$

A similar calculation shows

$$\mu\left((J(r, n(k)) \cap \left(\bigcap_{n=k+1}^{k+p} E_n\right)\right) > \frac{1}{4n(k)}\left(\frac{M-4}{2M}\right)^{p-1}. \tag{1}$$

The criterion for homogeneous sets of measure one may be simplified slightly. If $\Delta(a, b) \geqslant \delta > 0$ for all intervals (a, b), $0 \leqslant a < b < 1$ then it is clear that $\Delta(a, b) \geqslant \delta/2 > 0$ for all intervals (a, b), $0 \leqslant a < b \leqslant 1$. Let (a, b) be an interval such that $0 \leqslant a < b < 1$; then if k_0 is sufficiently large, (a, b) will contain intervals of the form $J(r, n(k_0))$, $(r \leqslant [n(k_0)] - 1)$. In fact the number of such intervals wholly contained in (a, b) will exceed:

$$[|b-a| n(k_0)] - 1.$$

This implies, using (1)

$$\begin{aligned} \mu(F(p) \cap (a, b)) &\geqslant \mu\left(\left(\bigcap_{n=k_0+1}^{k_0+p} E_n\right) \cap (a, b)\right) \\ &\geqslant ([|b-a| n(k_0)] - 1) \cdot \frac{1}{4n(k_0)}\left(\frac{M-4}{2M}\right)^{p-1} \\ &\geqslant \tfrac{1}{4}|b-a| \cdot \left(\frac{M-4}{2M}\right)^{p-1} - \frac{2}{4n(k_0)}\left(\frac{M-4}{2M}\right)^{p-1}. \end{aligned}$$

By choosing $n(k_0)$ sufficiently large

$$\mu(F(p) \cap (a, b)) > \tfrac{1}{8}|b-a| \left(\frac{M-4}{2M}\right)^{p-1}$$

and the density of $F(p)$ in any interval is greater than $\frac{1}{8}(M-4/2M)^{p-1}$. It is evident that $F(p)$ is homogeneous and of measure one. The proof of our theorem is now complete.

We can consider $\{n(k)\alpha\} \leqslant 1 - \varepsilon$ and by a suitable choice of ε prove in a manner similar to the above:

Theorem 2. *Let $(n(k))$ be a sequence of the real numbers*

$$\frac{n(k)}{n(k-1)} = r(k) > M > 2 \qquad (k = 2, 3, \ldots)$$

then, for almost all α, $0 < \alpha \leqslant 1$, the sequence $\{n(k)\alpha\}$ is not well distributed.

We have not yet entirely resolved the conjecture in [5]:

If $(n(k))$ *is a sequence of real numbers such that*

$$\frac{n(k)}{n(k-1)} = r(k) > 1 \qquad (k=2,\ 3,\ \ldots)$$

then $\{n(k)\alpha\}$ *is not well distributed for almost all* α.

Theorem A of [4] and a paper by MURDOCH, [3] would seem to indicate that such an improvement is possible. The proof of Theorem 1 is in part taken from [1], but it was not possible to indicate the alterations without writing out the whole.

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