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NOTE

ON ANTIPODAL GRAPHS

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The antipodal graph of a graph G, denoted by $A(G)$, is the graph on the same vertices as of G, two vertices being adjacent if the distance between them is equal to the diameter of G. A graph is said to be antipodal if it is the antipodal graph $A(H)$ of some graph H. We give a necessary and sufficient condition for a graph to be an antipodal graph.

The next three statements were easily proved in [1].

Proposition 1. *$A(G) = G$ if and only if G is complete.*

Proposition 2. *$A(G) = \bar{G}$ if and only if G is of diameter 2 or G is disconnected and the components of G are complete graphs.*

Proposition 3. *$A(G) \subseteq \bar{G}$ for any graph G other than the complete graph.*

The next two assertions are also obvious.

Proposition 4. *If both G and $\bar{G}$ have diameter 3, then $A(G)$ and $A(\bar{G})$ are disconnected.*

Proposition 5. *If G is disconnected, then $A(G)$ is of diameter $\leqslant 2$ and the components of $\overline{A(G)}$ are complete graphs.*

We need the following lemmas for our main theorem.

Lemma 1. *If $\bar{G}$ is a disconnected graph with at least one noncomplete component, then G is not an antipodal graph.*

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Proof. Let us assume that there exists a graph H such that $A(H)=G$ and let d be the diameter of H. By Proposition 1, $d\neq 1$. Therefore $A(H)\subseteq \bar{H}$ by Proposition 3. That is $H\subseteq \bar{G}$. Now H cannot be connected as $\bar{G}$ is disconnected. If H is disconnected then by Proposition 5, the components of $\overline{A(H)}$ are complete graphs, a contradiction. □

Lemma 2. *Let G and $\bar{G}$ be of diameter* 3. *Then G is not an antipodal graph.*

Proof. Let G be the antipodal graph of a graph H. Since both G and $\bar{G}$ are of diameter 3, there exists at least a vertex say u with eccentricity 2 in G. Then the eccentricity of u, $e(u)$, in $\bar{G}$ is 2 or 3.

Let us suppose that $e(u)$ is 3 in $\bar{G}$. Let $\{N_i\}=\{v_i\in H \mid d_H(u,v_i)=i\}$, where $i=1,2,\ldots,d$. There exists at least a vertex say y which belongs to $\{N_d\}$ and adjacent to all the vertices of $\{N_i\}$, $i=1,2,\ldots,d-1$ as $e(u)$ is 3 in $\bar{G}$. Hence for every $v\notin\{N_d\}$, $d_H(y,v)=d$, a contradiction, as there exists at least a $v\notin\{N_d\}$ which lies on a d-path joining y and u.

If $e(u)$ is 2 in $\bar{G}$, then $\bar{G}$ is of diameter 2, again a contradiction. □

Lemma 3. *$A(G)$ is of diameter* 2 *if and only if both G and $\bar{G}$ are of diameter* 2 *or G is disconnected but not totally disconnected.*

Proof. If G is disconnected which is not totally disconnected or if both G and $\bar{G}$ are of diameter 2 then the diameter of $A(G)$ is obviously 2.

Conversely let $A(G)$ be of diameter 2. Let d be the diameter of G. By Proposition 1, $d\neq 1$. Let $d=2$. Then $A(G)=\bar{G}$ by Proposition 2. If $\bar{G}$ is of diameter not equal to 2 then we get a contradiction. Let G be connected and let $d\geqslant 3$. Consider a vertex $u\in G$. Since $d_G(v_i,v_j)\leqslant 2$, $d_{A(G)}(v_i,v_j)=2$ where $v_i\in\{N(u)\}_G$, the set of all vertices adjacent to u in G such that $i\neq j$ and $i,j=1,2,\ldots,\deg_G(u)$. This implies there exists a vertex $x\notin\{N(u)\}_G$ such that u and its adjacent vertices $\{N(u)\}_G$ belong to $\{N(x)\}_{A(G)}$. That is $d_G(u,x)=d=d_G(x,v_i)$ which is a contradiction as all v_i's are adjacent to u where $i=1,2,\ldots,\deg_G(u)$. If G is totally disconnected then $A(G)$ is complete, a contradiction. Hence either G is disconnected but not totally disconnected or both G and $\bar{G}$ are of diameter 2. □

Lemma 4. *If $\bar{G}$ is a connected graph of diameter* >2, *then G is not an antipodal graph.*

Proof. Let there exist a graph H such that $A(H)=G$ and let d be the diameter of H.

Case (i). Let $\bar{G}$ be a connected graph of diameter >3. Then G is of diameter 2. By Lemma 3, either both H and $\bar{H}$ are of diameter 2 or H is disconnected but not totally disconnected. If $d = 2$, then $A(H) = \bar{H}$ which implies $\bar{G} = H$, a contradiction. If H is disconnected, then the components of $\overline{A(H)}$ are complete graphs, again a contradiction.

Case (ii). Let $\bar{G}$ be of diameter 3. If G is of diameter 2 then the proof follows as in Case (i). If G is of diameter 3 then by Lemma 2, G is not antipodal.

The proof of our main theorem now follows from Lemmas 1, 2 and 4.

Theorem. *A graph G is an antipodal graph if and only if it is the antipodal graph of its complement.*

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Reference

[1] R. Aravamudhan and B. Rajendran, Graph equations of antipodal graphs, Presented at the Seminar on Combinatorics and Applications held at ISI., Calcutta, India on 14–17 December, 1982.