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Convergence to a propagating front in a degenerate Fisher-KPP equation with advection

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ABSTRACT

We consider a Fisher-KPP equation with density-dependent diffusion and advection, arising from a chemotaxis–growth model. We study its behavior as a small parameter, related to the thickness of a diffuse interface, tends to zero. We analyze, for small times, the emergence of transition layers induced by a balance between reaction and drift effects. Then we investigate the propagation of the layers. Convergence to a free boundary limit problem is proved and a sharp estimate of the thickness of the layers is provided.

1. Introduction

In this paper we consider a Fisher-KPP equation with density-dependent diffusion and advection, namely

$$(P^\varepsilon)\quad \begin{cases} u_t = \varepsilon\Delta\big(u^m\big) - \nabla\cdot\big(u\nabla v^\varepsilon\big) + \dfrac{1}{\varepsilon}u(1-u) & \text{in } (0,\infty)\times\Omega,\\ \dfrac{\partial(u^m)}{\partial\nu} = 0 & \text{on } (0,\infty)\times\partial\Omega,\\ u(0,x) = u_0(x) & \text{in } \Omega,\end{cases}$$

with $\varepsilon > 0$ a small parameter and $v^\varepsilon(t,x)$ a smooth given function. Here Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geqslant 2$), ν is the Euclidian unit normal vector exterior to $\partial\Omega$ and $m \geqslant 2$. We are concerned with the behavior of the solutions $u^\varepsilon(t,x)$ as $\varepsilon \to 0$.

Assumption 1.1 *(Initial data).* Throughout this paper, we make the following assumptions on the initial data.

(i) Let Ω_0 be a nonempty open bounded set with a smooth boundary and such that $\overline{\Omega_0} \subset \Omega$. Let $\widetilde{u_0} : \overline{\Omega_0} \to \mathbb{R}$ be C^0 in $\overline{\Omega_0}$ and C^2 in Ω_0, strictly positive on Ω_0 and such that $\widetilde{u_0}(x) = 0$ for all $x \in \partial\Omega_0$. Define the map $u_0 : \Omega \to \mathbb{R}$ by

$$u_0(x) := \begin{cases} \widetilde{u_0}(x) & \text{if } x \in \overline{\Omega_0},\\ 0 & \text{if } x \in \Omega\setminus\overline{\Omega_0}.\end{cases}$$

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(ii) Ω_0 is convex.
(iii) There exists $\delta > 0$ such that, if n denotes the Euclidian unit normal vector exterior to the "initial interface" $\Gamma_0 := \partial\Omega_0$, then

$$\left|\frac{\partial \widetilde{u_0}}{\partial n}(y)\right| \geqslant \delta \quad \text{for all } y \in \Gamma_0. \tag{1.1}$$

Remark 1.2. Note that the comparison principle allows to relax the regularity assumption on $\widetilde{u_0}$. See [2] for details.

Assumption 1.3 *(Structure of v^ε).* We assume that

$$v^\varepsilon(t,x) = v(t,x) + \varepsilon v_1^\varepsilon(t,x), \tag{1.2}$$

with v and v_1^ε smooth functions on $[0,\infty) \times \overline{\Omega}$. We assume that, for all $T > 0$, there exists $C > 0$ such that, for all $\varepsilon > 0$ small enough, it holds that $\|v_1^\varepsilon\|_{C^{1,2}([0,T]\times\overline{\Omega})} \leqslant C$. Finally we assume

$$\frac{\partial v^\varepsilon}{\partial \nu} = 0 \quad \text{on } (0,\infty) \times \partial\Omega. \tag{1.3}$$

Remark 1.4. In the sequel we smoothly extend $v(t,x)$ in time and space on the whole of $\mathbb{R} \times \mathbb{R}^N$, as well as $v_1^\varepsilon(t,x)$ in space on $[0,\infty) \times \mathbb{R}^N$. Moreover since we are investigating local in time phenomena we will assume in the sequel, without loss of generality, that the extensions $v(t,x)$ and $v_1^\varepsilon(t,x)$ vanish outside of a large time–space ball.

Problem (P^ε) is a simpler version of a chemotaxis–growth system with logistic nonlinearity, where $v^\varepsilon(t,x)$ is not a given function but is coupled to u either through the parabolic equation $\varepsilon v_t = \Delta v + u - \gamma v$ or through the elliptic equation $0 = \Delta v + u - \gamma v$, supplemented with the Neumann boundary condition (1.3) (see e.g. [21]). Note that, in the case of linear diffusion (corresponding to $m = 1$) and a bistable nonlinearity, the asymptotic behavior of the corresponding system as $\varepsilon \to 0$ has been studied using Green's function associated to the homogeneous Neumann boundary value problem on Ω for the operator $-\Delta + \gamma$ (see [8] and [1]).

Motivation and biological background

Before describing our results, let us briefly comment about the relevance of (P^ε) in population dynamics models. The evolution equation in Problem (P^ε) combines logistic growth, chemotaxis and degenerate diffusion. We recall below how these terms appear in mathematical models that attempt to capture remarkable biological features.

Reaction–diffusion equations with a logistic nonlinearity were introduced in the pioneering works [12,18]. The simplest equation reads

$$u_t = \Delta u + u(1-u),$$

and has been widely used to model phenomena arising in population genetics [12] or in biological invasions [22]. Its main mathematical property is to sustain travelling wave solutions with a semi-infinite interval of admissible wave speeds, with the minimal one having a crucial biological interpretation.

Chemotaxis, i.e. the tendency of biological individuals to direct their movements according to certain chemicals in their environment, is induced in (P^ε) by the advection term $-\nabla \cdot (u\nabla v^\varepsilon)$: the population, whose density is $u(t,x)$, has an oriented motion in the direction of a positive gradient of the chemotactic substance, whose concentration is $v^\varepsilon(t,x)$. The first PDE model to describe such movements was proposed in [17] and involves linear diffusion for u and a parabolic equation coupling v to u. The Keller–Segel model has received considerable attention in mathematical literature, particularly focusing on the finite-time blow-up of solutions (see [16] for a recent review). This provides a mathematical tool to analyze aggregation phenomena as observed in bacteria colonies. Chemotaxis systems involving linear diffusion and a growth term, either logistic or bistable, have later been considered in, e.g., [21,8,1,24].

Variants of the Fisher-KPP equation involving a degenerate diffusion have been proposed in order to take into account population density pressure. Actually one can introduce density-dependent birth or death rates as an attempt to control the size of a population. Nevertheless as shown in [13], the introduction of a nonlinearity into the dispersal behavior of a species, which behaves in an otherwise linear way, may lead, in an inhomogeneous environment, to a similar regulatory effect. Moreover this assumption is consistent with ecological observations as reported for instance in [9], where it is shown that arctic ground squirrels migrate from densely populated areas into sparsely populated areas, even when the latter is less favorable (due to reduced availability of burrow sites or exposure to intensive predation). For such species, migration to avoid crowding, rather than random motion, is the primary cause of dispersal. To describe such movements, the authors in [22] and [13] use the directed motion model where individuals can only stay put or move down the population gradient; this model yields the degenerate equation

$$u_t = \Delta\big(u^2\big) + G(x)u, \tag{1.4}$$

in which the population regulates its size below the carrying capacity set by the supply of nutrients. Later in [14] a larger class of equations with degenerate diffusion and nonlinear reaction was considered, namely

$$u_t = \Delta\big(u^m\big) + f(u), \quad m \geqslant 2. \tag{1.5}$$

Note that in the absence of $f(u)$, Eq. (1.5) reduces to the so-called porous medium equation

$$u_t = \Delta\big(u^m\big), \tag{1.6}$$

which has been extensively investigated in the literature. We refer to the book [23] and the references therein. The main feature of this equation is that it is degenerate at the points where $u = 0$. As a consequence, a loss of regularity of solutions occurs and disturbances propagate with finite speed, a property which has a relevant interpretation in a biological context (see for instance [6]).

Formal asymptotic analysis

Problem (P^ε) possesses a unique solution $u^\varepsilon(t,x)$ in a sense that is explained in Section 3. As $\varepsilon \to 0$, the qualitative behavior of this solution is the following. In the very early stage, the nonlinear diffusion term $\varepsilon\Delta(u^m)$ is negligible compared with the drift term $-\nabla u \cdot \nabla v^\varepsilon$ and the reaction term $\varepsilon^{-1}u(1-u)$. Hence, in some sense, the equation is well approximated by a coupling between the transport equation $u_t + \nabla u \cdot \nabla v^\varepsilon = 0$ and the ordinary differential equation $u_t = \varepsilon^{-1}u(1-u)$. Therefore, as suggested by the analysis in [2], u^ε quickly approaches the values 0 or 1, and an interface is formed between the regions $\{u^\varepsilon \approx 0\}$ and $\{u^\varepsilon \approx 1\}$ (*emergence of the layers*). Note that, in this very early stage, the balance of the transport equation and the ordinary differential equation will generate an interface not exactly around Γ_0 but in a slightly drifted place. Once such an interface is developed, the diffusion term becomes large near the interface, and comes to balance with the drift and the reaction terms so that the interface starts to propagate, on a much slower time scale (*propagation of the front*).

Our goal in this paper is to provide a rigorous analysis that supports this formal approach and makes it more precise. To study the interfacial behavior, we consider the asymptotic limit of (P^ε) as $\varepsilon \to 0$. Then the limit solution will be a step function $\tilde{u}(t,x)$ taking the value 1 on one side of a moving interface, and 0 on the other side. We show that this sharp interface, which we will denote by Γ_t, obeys the law of motion

$$\big(P^0\big) \quad \begin{cases} V_n = c^* + \dfrac{\partial v}{\partial n} & \text{on } \Gamma_t, \\ \Gamma_t|_{t=0} = \Gamma_0, \end{cases}$$

where V_n is the normal velocity of Γ_t in the exterior direction, c^* the minimal speed of travelling waves solutions of a related degenerate one-dimensional problem (see Section 5 for details) and n the outward normal vector on Γ_t.

Plan

The organization of this paper is as follows. We present our results in Section 2. In Section 3, we briefly recall known results concerning the well-posedness of Problem (P^ε); in particular, it admits a comparison principle so that the sub- and super-solutions method can be used to investigate the behavior of the solutions u^ε. In Section 4, we prove a generation of interface property for Problem (P^ε). In Section 5 we investigate the motion of interface. Finally, we prove our main result in Section 6.

2. Results and comments

The question of the convergence of Problem (P^ε) to (P^0) has been addressed in [11]. However, the author considers only a very restricted class of initial data, namely those having a specific profile with well-developed transition layers. In other words the generation of interface from arbitrary initial data is not studied. In the present paper we study both the emergence and the propagation of interface. Moreover we prove a sharp $\mathcal{O}(\varepsilon)$ estimate of the thickness of the transition layers of the solutions u^ε.

The authors in [15] prove the convergence of the solutions of (P^ε) with arbitrary initial data with convex compact support to solutions of (P^0), when there is no advection (i.e. $v^\varepsilon \equiv 0$). They provide an $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ estimate of the thickness of the transition layers. Therefore, even in the particular case $v^\varepsilon \equiv 0$, our $\mathcal{O}(\varepsilon)$ estimate was not known.

As mentioned in the introduction, the drift term and the reaction term in (P^ε) are of the same magnitude for small times. Therefore the emergence of the layers, initiated by the ODE $u_t = \varepsilon^{-1}u(1-u)$, will occur in the neighborhood of a slightly drifted initial interface $\Gamma_0^{\varepsilon,\mathrm{drift}}$. To analyze such a phenomenon we shall use the Lagrangian coordinates. Recall that we have smoothly extended $v(t,x)$ in time–space on the whole of $\mathbb{R} \times \mathbb{R}^N$, with $v \equiv 0$ outside of a large time–space ball. Then, for $(t_0,x_0) \in \mathbb{R} \times \mathbb{R}^N$, we denote by $\varphi_{(t_0,x_0)}$ the solution, defined on $\mathbb{R}$, of the Cauchy problem

$$\begin{cases} \dfrac{dX}{dt}(t) = \nabla v\big(t, X(t)\big), \\ X(t_0) = x_0. \end{cases} \tag{2.1}$$

We denote by Φ the associated flow defined on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N$, that is

$$\Phi(t_1,t_2,x_3):=\varphi_{(t_2,x_3)}(t_1). \tag{2.2}$$

Recall that $\Gamma_0=\partial\Omega_0=\partial(\operatorname{Supp} u_0)$ is the initial interface. From $t=0$ to

$$t^\varepsilon:=\varepsilon|\ln\varepsilon| \quad \text{(generation time)}, \tag{2.3}$$

we let each point on Γ_0 evolve with the law (2.1) and then define a *drifted initial interface* $\Gamma_0^{\varepsilon,\text{drift}}$ by

$$\Gamma_0^{\varepsilon,\text{drift}}:=\{\Phi(t^\varepsilon,0,x)\colon x\in\Gamma_0\}. \tag{2.4}$$

Next we consider the free boundary problem

$$\big(P^0_{\varepsilon,\text{drift}}\big)\quad \begin{cases} V_n=c^*+\dfrac{\partial v}{\partial n} & \text{on } \Gamma_t^{\varepsilon,\text{drift}},\\ \Gamma_t^{\varepsilon,\text{drift}}\big|_{t=0}=\Gamma_0^{\varepsilon,\text{drift}}. \end{cases}$$

Well-posedness of (P^0) and of $(P^0_{\varepsilon,\text{drift}})$

Using the level set formulation (see, e.g., [5]), the motion law in Problem (P^0) can be rewritten as a first order Hamilton–Jacobi equation with a convex Hamiltonian. This approach, combined with the results in [19], has been used in [11] in order to prove the following.

Theorem 2.1 *(Well-posedness of (P^0)). (See [11].) Let $\Omega_0\Subset\Omega$ be a smooth subdomain of Ω and let $\Gamma_0=\partial\Omega_0$ be the given smooth initial interface. Then there exists $T^{max}(\Gamma_0)>0$ such that Problem (P^0) has a unique smooth solution on $[0,T]$ for any $0<T<T^{max}(\Gamma_0)$. More precisely, there exists a family of smooth subdomains $(\Omega_t)_{t\in(0,T]}$ with $\Omega_t\Subset\Omega$ such that, denoting $\Gamma_t=\partial\Omega_t$, $\Gamma:=\bigcup_{0\leqslant t\leqslant T}(\{t\}\times\Gamma_t)$ is the unique solution to Problem (P^0) on $[0,T]$.*

Moreover, $T^{max}(\Gamma_0)$ depends smoothly on Γ_0. Therefore we can choose $\varepsilon_0>0$ small enough and $T>0$ such that

$$0<T<\inf_{0\leqslant\varepsilon\leqslant\varepsilon_0}T^{max}\big(\Gamma_0^{\varepsilon,\text{drift}}\big), \tag{2.5}$$

which guarantees the existence of a unique smooth solution on $[0,T]$ to both (P^0) and $(P^0_{\varepsilon,\text{drift}})$ for any $0<\varepsilon\leqslant\varepsilon_0$. We denote by $\Gamma^{\varepsilon,\text{drift}}=\bigcup_{0\leqslant t\leqslant T}(\{t\}\times\Gamma_t^{\varepsilon,\text{drift}})$ the smooth solution to $(P^0_{\varepsilon,\text{drift}})$ and by $\Omega_t^{\varepsilon,\text{drift}}$ the region enclosed by $\Gamma_t^{\varepsilon,\text{drift}}$. In the sequel we work on $[0,T]$, with T satisfying (2.5), and define $Q_T:=(0,T)\times\Omega$.

Our main result, Theorem 2.2, contains generation, motion and thickness of the transition layers properties. It asserts that: given an initial data u_0, the solution u^ε quickly (at time $t^\varepsilon=\varepsilon|\ln\varepsilon|$) becomes close to 1 or 0, except in a small neighborhood of the drifted interface $\Gamma_{t^\varepsilon}^{\varepsilon,\text{drift}}$, creating a steep transition layer around $\Gamma_{t^\varepsilon}^{\varepsilon,\text{drift}}$ (*generation of interface*). The theorem then states that the solution u^ε remains close to the step function associated with $(P^0_{\varepsilon,\text{drift}})$ on the time interval $[t^\varepsilon,T]$ (*motion of interface*); in other words, the motion of the transition layer is well approximated by the limit interface equation $(P^0_{\varepsilon,\text{drift}})$. Moreover, the estimate (2.6) in Theorem 2.2 implies that, once a transition layer is formed, its thickness remains within order $\mathcal{O}(\varepsilon)$ for the rest of time.

Theorem 2.2 *(Generation, motion and thickness of the layers). Let $\eta\in(0,1/2)$ be arbitrary. Then, there exists $\mathcal{C}>0$ such that, for all $\varepsilon>0$ small enough and all*

$$t^\varepsilon=\varepsilon|\ln\varepsilon|\leqslant t\leqslant T,$$

we have

$$u^\varepsilon(t,x)\in\begin{cases} [0,1+\eta] & \text{if } x\in\mathcal{N}_{\mathcal{C}\varepsilon}(\Gamma_t^{\varepsilon,\text{drift}}),\\ [1-\eta,1+\eta] & \text{if } x\in\Omega_t^{\varepsilon,\text{drift}}\setminus\mathcal{N}_{\mathcal{C}\varepsilon}(\Gamma_t^{\varepsilon,\text{drift}}),\\ \{0\} & \text{if } x\in(\Omega\setminus\overline{\Omega_t^{\varepsilon,\text{drift}}})\setminus\mathcal{N}_{\mathcal{C}\varepsilon}(\Gamma_t^{\varepsilon,\text{drift}}), \end{cases} \tag{2.6}$$

with $\mathcal{N}_r(\Gamma_t^{\varepsilon,\text{drift}}):=\{x\colon \operatorname{dist}(x,\Gamma_t^{\varepsilon,\text{drift}})<r\}$ the tubular r-neighborhood of $\Gamma_t^{\varepsilon,\text{drift}}$.

Note that (2.6) shows that, for any $0<a<1$, for all $t^\varepsilon\leqslant t\leqslant T$, the a-level set

$$L_t^\varepsilon(a):=\{x\colon u^\varepsilon(t,x)=a\}$$

lives in a tubular $\mathcal{O}(\varepsilon)$ neighborhood of the interface $\Gamma_t^{\varepsilon,\text{drift}}$. In other words, we provide a new $\mathcal{O}(\varepsilon)$ estimate of the thickness of the transition layers of the solutions u^ε. Concerning the localization of the level sets $L_t^\varepsilon(a)$, it is made with respect to a slightly drifted free boundary Problem $(P^0_{\varepsilon,\text{drift}})$. Nevertheless, since the solution of $(P^0_{\varepsilon,\text{drift}})$ on $[0,T]$ is continuous w.r.t. the initial hypersurface $\Gamma_0^{\varepsilon,\text{drift}}$, we recover, as $\varepsilon \to 0$, the original free boundary Problem (P^0) and obtain the expected result. More precisely, let us define the step function $\tilde{u}(t,x)$ by

$$\tilde{u}(t,x) := \begin{cases} 1 & \text{in } \Omega_t, \\ 0 & \text{in } \Omega \setminus \overline{\Omega_t} \end{cases} \quad \text{for } t \in (0,T]. \tag{2.7}$$

As a consequence of Theorem 2.2, we obtain the following convergence result which shows that $\tilde{u}$ is the sharp interface limit of u^ε as $\varepsilon \to 0$.

Corollary 2.3 *(Convergence). As $\varepsilon \to 0$, u^ε converges to $\tilde{u}$, defined in* (2.7)*, everywhere in $\bigcup_{0<t\leqslant T}(\{t\} \times \Omega_t)$ and $\bigcup_{0<t\leqslant T}(\{t\} \times (\Omega \setminus \overline{\Omega_t}))$.*

3. Comparison principle, well-posedness for (P^ε)

Since the diffusion term degenerates when $u = 0$ a loss of regularity of solutions occurs. We define below a notion of weak solution for Problem (P^ε), which is very similar to the one proposed in [3] for the one-dimensional problem with homogeneous Dirichlet boundary conditions. Concerning the initial data, we suppose here that $u_0 \in L^\infty(\Omega)$ and $u_0 \geqslant 0$ a.e. Note that in this section, and only in this section, we assume, for ease of notation, that $\varepsilon = 1$ and that $v^\varepsilon \equiv v$; we then denote the associated Problem (P^ε) by (P). In the sequel $f(u) = u(1-u)$.

Definition 3.1. A function $u : [0,\infty) \to L^1(\Omega)$ is a solution of Problem (P) if, for all $T > 0$,

(i) $u \in C([0,\infty); L^1(\Omega)) \cap L^\infty(Q_T)$;
(ii) for all $\varphi \in C^2(\overline{Q_T})$ such that $\varphi \geqslant 0$ and $\frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega$, it holds that

$$\int_\Omega u(T)\varphi(T) - \iint_{Q_T} \left(u\varphi_t + u^m \Delta\varphi + u\nabla v \cdot \nabla\varphi\right) = \int_\Omega u_0\varphi(0) + \iint_{Q_T} f(u)\varphi. \tag{3.1}$$

A sub-solution (a super-solution) of Problem (P) is a function satisfying (i) and (ii) with equality replaced by $\leqslant$ (respectively $\geqslant$).

Theorem 3.2 *(Existence and comparison principle). Let $T > 0$ be arbitrary. The following properties hold.*

(i) *Let u^- (u^+) be a sub-solution (respectively a super-solution) with initial data u_0^- (respectively u_0^+).*

$$\text{If } u_0^- \leqslant u_0^+ \quad \text{a.e.} \quad \text{then } u^- \leqslant u^+ \text{ in } Q_T;$$

(ii) *Problem (P) has a unique solution u on $[0,\infty)$ and*

$$0 \leqslant u \leqslant \max\left(1, \|u_0\|_{L^\infty(\Omega)}\right) \quad \text{in } Q_T; \tag{3.2}$$

(iii) $u \in C(\overline{Q_T})$.

Since (1.3) holds, the proof of Theorem 3.2 is standard and follows the same steps of that of [3, Theorem 5]. The continuity of u follows from [10].

The following lemma proved in [15], will be very useful when constructing smooth sub- and super-solutions in later sections.

Lemma 3.3 *(Being sub- and super-solutions). Let u be a continuous nonnegative function in $\overline{Q_T}$. Define*

$$\Omega_t^{supp} := \left\{x \in \Omega \colon u(t,x) > 0\right\}, \qquad \Gamma_t^{supp} := \partial\Omega_t^{supp},$$

for all $t \in [0,T]$. Suppose the family $\Gamma^{supp} := \bigcup_{0\leqslant t\leqslant T}(\{t\} \times \Gamma_t^{supp})$ is sufficiently smooth and let ν_t^{supp} be the outward normal vector on Γ_t^{supp}. Suppose moreover that

(i) $\nabla(u^m)$ *is continuous in* $\overline{Q_T}$;
(ii) $\mathcal{L}^\varepsilon[u] := u_t - \Delta(u^m) + \nabla \cdot (u\nabla v) - f(u) = 0$ *in* $\{(t,x) \in \overline{Q_T} \colon u(t,x) > 0\}$;
(iii) $\frac{\partial(u^m)}{\partial \nu_t^{supp}} = 0$ *on* $\partial\Omega_t^{supp}$*, for all* $t \in [0,T]$.

Then u is a solution of Problem (P). *Similarly a function satisfying* (i) *and* (ii)–(iii) *with equality replaced by* $\leqslant$ $(\geqslant)$ *is a sub-solution* (*respectively a super-solution*) *of Problem* (P).

4. Emergence of the transition layers

In this section, we investigate the generation of interface which occurs very quickly around $\Gamma^{\varepsilon,\mathrm{drift}}_{t^\varepsilon}$. We prove that, given a virtually arbitrary initial datum u_0, the solution u^ε of (P^ε) quickly becomes close to 1 or 0 in most part of Ω. More precisely – recalling that $\Phi(t_1, t_2, x_3)$, defined in (2.2), denotes the flow associated with the Cauchy problem (2.1) – the following holds.

Theorem 4.1 *(Emergence of the layers). Let* $\eta \in (0, 1/2)$ *be arbitrary. Then there exists* $M_0 > 0$ *such that, for all* $\varepsilon > 0$ *small enough, the following hold with* $t^\varepsilon = \varepsilon|\ln\varepsilon|$:

(i) *for all* $x \in \Omega$, *we have that*

$$0 \leqslant u^\varepsilon\big(t^\varepsilon, x\big) \leqslant 1 + \eta; \tag{4.1}$$

(ii) *for all* $x \in \Omega$, *we have that*

$$\text{if } u_0\big(\Phi\big(0, t^\varepsilon, x\big)\big) \geqslant M_0\varepsilon \quad \text{then } u^\varepsilon\big(t^\varepsilon, x\big) \geqslant 1 - \eta; \tag{4.2}$$

(iii) *for all* $x \in \Omega$, *we have*

$$\text{if } \operatorname{dist}\big(\Phi\big(0, t^\varepsilon, x\big), \Omega_0\big) \geqslant M_0\varepsilon \quad \text{then } u^\varepsilon\big(t^\varepsilon, x\big) = 0, \tag{4.3}$$

where we recall that $\Omega_0 = \{x\colon\ u_0(x) > 0\}$ (*see Assumption* 1.1).

In order to prove the above theorem, we shall construct sub- and super-solutions. As mentioned before, in this very early stage, we have to take into account both the reaction and the drift terms. We start with some preparations.

4.1. A related ODE and the flow Φ

An ODE

The solution of the problem without diffusion nor advection, namely $\bar u_t = \varepsilon^{-1} f(\bar u)$ supplemented with the condition $\bar u(0, x) = u_0(x)$, is written in the form $\bar u(t, x) = Y(\frac{t}{\varepsilon}, u_0(x))$, where $Y(\tau, \xi)$ denotes the solution of the ordinary differential equation $Y_\tau(\tau, \xi) = f(Y(\tau, \xi))$ supplemented with the initial condition $Y(0, \xi) = \xi$. Nevertheless, in order to take care of the term $-u\Delta v^\varepsilon$, we need a slight modification of f.

Let $\tilde f$ be the smooth odd function that coincides with $f(u) = u(1-u)$ on $[0, \infty)$: $\tilde f$ has exactly three zeros $-1 < 0 < 1$ and

$$\tilde f'(-1) = -1 < 0, \qquad \tilde f'(0) = 1 > 0, \qquad \tilde f'(1) = -1 < 0, \tag{4.4}$$

i.e. $\tilde f$ is of the bistable type. Next, we define

$$\tilde f_\delta(u) := \tilde f(u) + \delta.$$

For $|\delta|$ small enough, this function is still of the bistable type: if $\delta_0 > 0$ is small enough, then for any $\delta \in (-\delta_0, \delta_0)$, $\tilde f_\delta$ has exactly three zeros $\alpha_-(\delta) < a(\delta) < \alpha_+(\delta)$ and there exists a positive constant C such that

$$\big|\alpha_-(\delta) + 1\big| + \big|a(\delta)\big| + \big|\alpha_+(\delta) - 1\big| \leqslant C|\delta|, \tag{4.5}$$

$$\big|\mu(\delta) - 1\big| \leqslant C|\delta|, \tag{4.6}$$

where $\mu(\delta)$ is the slope of $\tilde f_\delta$ at the unstable zero, namely

$$\mu(\delta) := \tilde f_\delta'\big(a(\delta)\big) = \tilde f'\big(a(\delta)\big). \tag{4.7}$$

Now for each $\delta \in (-\delta_0, \delta_0)$, we define $Y(\tau, \xi; \delta)$ as the solution of the ordinary differential equation

$$\begin{cases} Y_\tau(\tau, \xi; \delta) = \tilde f_\delta\big(Y(\tau, \xi; \delta)\big) & \text{for } \tau > 0, \\ Y(0, \xi; \delta) = \xi, \end{cases} \tag{4.8}$$

where ξ varies in $(-C_0, C_0)$, with

$$C_0 := \|u_0\|_{L^\infty(\Omega)} + 1. \tag{4.9}$$

We claim that $Y(\tau, \xi; \delta)$ has the following properties.

Lemma 4.2 *(Behavior of Y). There exist positive constants δ_0 and C such that the following hold for all $(\tau,\xi;\delta) \in (0,\infty)\times(-C_0,C_0)\times(-\delta_0,\delta_0)$.*

(i) *If $\xi > a(\delta)$ then $Y(\tau,\xi;\delta) > a(\delta)$,*
If $\xi < a(\delta)$ then $Y(\tau,\xi;\delta) < a(\delta)$;
(ii) $|Y(\tau,\xi;\delta)| \leqslant C_0$;
(iii) $0 < Y_\xi(\tau,\xi;\delta) \leqslant Ce^{\mu(\delta)\tau}$;
(iv) $|\frac{Y_{\xi\xi}}{Y_\xi}(\tau,\xi;\delta)| \leqslant C(e^{\mu(\delta)\tau}-1)$.

Properties (i) and (ii) are direct consequences of the bistable profile of $\tilde{f}_\delta$. For proofs of (iii) and (iv) we refer to [1].

The flow Φ

Let us briefly recall known facts concerning the flow $\Phi(t_1,t_2,x_3)$. By definition we have

$$\frac{\partial\Phi}{\partial t_1}(t,t_0,x_0) = \nabla v\big(t,\Phi(t,t_0,x_0)\big). \tag{4.10}$$

Next, note that, by uniqueness,

$$\Phi\big(t,t_0,\Phi(t_0,t,x_0)\big) = x_0,$$

for all $(t,t_0,x_0) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^N$. Differentiating this identity with respect to t_0, we get

$$\frac{\partial\Phi}{\partial t_2}(t,t_0,x) + D_3\Phi(t,t_0,x)\frac{\partial\Phi}{\partial t_1}(t_0,t,x_0) = 0_{\mathbb{R}^N},$$

where $x := \Phi(t_0,t,x_0)$ and where $D_3\Phi(t_1,t_2,x_3)$ denotes the Jacobian matrix of Φ w.r.t. the third variable. Hence, using (4.10) we infer that

$$\frac{\partial\Phi}{\partial t_2}(t,t_0,x) + D_3\Phi(t,t_0,x)\nabla v(t_0,x) = 0_{\mathbb{R}^N}, \tag{4.11}$$

which is of crucial importance for our analysis.

4.2. Proof of (4.1) and (4.2)

We use the notation $z^+ = \max(z,0)$. Our sub- and super-solutions are given by

$$w_\varepsilon^\pm(t,x) := \left[Y\left(\frac{t}{\varepsilon}, u_0\big(\Phi(0,t,x)\big) \pm \varepsilon^2 C^\star\big(e^{\mu(\pm\varepsilon M)t/\varepsilon}-1\big); \pm\varepsilon M\right)\right]^+, \tag{4.12}$$

or equivalently by

$$w_\varepsilon^\pm\big(t,\Phi(t,0,x)\big) := \left[Y\left(\frac{t}{\varepsilon}, u_0(x) \pm \varepsilon^2 C^\star\big(e^{\mu(\pm\varepsilon M)t/\varepsilon}-1\big); \pm\varepsilon M\right)\right]^+. \tag{4.13}$$

Here $Y(\tau,\xi;\delta)$ is the solution of (4.8), $\mu(\delta)$ the slope defined in (4.7), $\Phi(t_1,t_2,x_3)$ the flow defined in (2.2) and M is chosen such that, for all $\varepsilon > 0$ small enough, $M \geqslant C_0\|\Delta v^\varepsilon\|_{L^\infty(Q_T)}$, with C_0 defined by (4.9).

Lemma 4.3 *(Sub- and super-solutions for small times). There exists $C^\star > 0$ such that, for all $\varepsilon > 0$ small enough, $(w_\varepsilon^-, w_\varepsilon^+)$ is a pair of sub- and super-solutions for Problem (P^ε), in the domain $[0,t^\varepsilon]\times\overline{\Omega}$.*

Before proving the lemma, we remark that $w_\varepsilon^-(0,x) = w_\varepsilon^+(0,x) = u_0(x)$. Consequently, by the comparison principle, we have

$$w_\varepsilon^-(t,x) \leqslant u^\varepsilon(t,x) \leqslant w_\varepsilon^+(t,x) \quad \text{for all } (t,x) \in \big[0,t^\varepsilon\big]\times\overline{\Omega}. \tag{4.14}$$

Proof of Lemma 4.3. In order to prove that $(w_\varepsilon^-, w_\varepsilon^+)$ is a pair of sub- and super-solutions for Problem (P^ε) – if $C^\star$ is appropriately chosen – we check the sufficient conditions stated in Lemma 3.3.

On the one hand, concerning the sub-solution w_ε^-, for (t,x) such that $x \in \Omega_t^{supp}[w_\varepsilon^-] := \{x\colon\ w_\varepsilon^-(t,x) > 0\}$ we have, at point (t,x),

$$\nabla\big(w_\varepsilon^-\big)^m = \big(mY^{m-1}Y_\xi\big)\left(\frac{t}{\varepsilon}, u_0\big(\Phi(0,t,x)\big) - \varepsilon^2 C^\star\big(e^{\mu(-\varepsilon M)t/\varepsilon}-1\big); -\varepsilon M\right)\nabla_x\big(u_0\big(\Phi(0,t,x)\big)\big).$$

If $(t,x)\to(t_0,x_0)$ such that $x_0\in\Gamma_{t_0}^{supp}[w_\varepsilon^-]:=\partial\Omega_{t_0}^{supp}[w_\varepsilon^-]$ then the equality above implies

$$\lim_{(t,x)\to(t_0,x_0)}\nabla\big(w_\varepsilon^-\big)^m(t,x)=0_{\mathbb{R}^N},$$

and conditions (i) and (iii) of Lemma 3.3 are checked for the sub-solution.

On the other hand, concerning the super-solution w_ε^+, note that

$$\xi:=u_0\big(\Phi(0,t,x)\big)+\varepsilon^2C^\star\big(e^{\mu(\varepsilon M)t/\varepsilon}-1\big)$$

is positive. Therefore the cubic profile of $\tilde f_\delta$ shows that, for $t>0$,

$$\Omega_t^{supp}\big[w_\varepsilon^+\big]=\Omega,$$
$$\Gamma_t^{supp}\big[w_\varepsilon^+\big]:=\partial\Omega_t^{supp}\big[w_\varepsilon^+\big]=\partial\Omega.$$

Recall that $u_0=0$ in a neighborhood $\mathcal{V}$ of $\partial\Omega$; if x is sufficiently close to $\partial\Omega$, $\Phi(0,t,x)$ lives in $\mathcal{V}$ for all $t\in[0,t^\varepsilon]$ (with $\varepsilon>0$ sufficiently small). Therefore (4.12) shows that w_ε^+ is independent on x near $\partial\Omega$ and condition (iii) of Lemma 3.3 for the super-solution is checked (and condition (i) is obviously checked).

Then it remains to prove that

$$\mathcal{L}^\varepsilon\big[w_\varepsilon^-\big]:=\big(w_\varepsilon^-\big)_t-\varepsilon\Delta\big(w_\varepsilon^-\big)^m+\nabla\cdot\big(w_\varepsilon^-\nabla v^\varepsilon\big)-\frac{1}{\varepsilon}f\big(w_\varepsilon^-\big)\leqslant 0,$$

in $\{(t,x)\in[0,t^\varepsilon]\times\overline{\Omega}\colon\ w_\varepsilon^-(t,x)>0\}$ and that $\mathcal{L}^\varepsilon[w_\varepsilon^+]\geqslant 0$ in $\{(t,x)\in[0,t^\varepsilon]\times\overline{\Omega}\}$. We will only prove the latter inequality since the proof of the former is similar.

We compute

$$\partial_t w_\varepsilon^+=\frac{1}{\varepsilon}Y_\tau+\frac{\partial}{\partial t}\big[u_0\big(\Phi(0,t,x)\big)\big]Y_\xi+\varepsilon\mu(\varepsilon M)C^\star e^{\mu(\varepsilon M)t/\varepsilon}Y_\xi,$$
$$\nabla w_\varepsilon^+=\nabla_x\big[u_0\big(\Phi(0,t,x)\big)\big]Y_\xi,$$
$$\nabla\big[\big(w_\varepsilon^+\big)^m\big]=\nabla_x\big[u_0\big(\Phi(0,t,x)\big)\big]\big(Y^m\big)_\xi,$$
$$\Delta\big[\big(w_\varepsilon^+\big)^m\big]=\big|\nabla_x\big[u_0\big(\Phi(0,t,x)\big)\big]\big|^2\big(Y^m\big)_{\xi\xi}+\Delta_x\big[u_0\big(\Phi(0,t,x)\big)\big]\big(Y^m\big)_\xi,$$

where the function Y and its derivatives are taken at the point

$$(\tau,\xi;\delta):=\big(t/\varepsilon,u_0\big(\Phi(0,t,x)\big)+\varepsilon^2C^\star\big(e^{\mu(\varepsilon M)t/\varepsilon}-1\big);\varepsilon M\big).$$

Note that

$$\frac{\partial}{\partial t}\big[u_0\big(\Phi(0,t,x)\big)\big]=\nabla u_0\big(\Phi(0,t,x)\big)\cdot\frac{\partial\Phi}{\partial t_2}(0,t,x),$$

and that

$$\nabla_x\big[u_0\big(\Phi(0,t,x)\big)\big]=\big(D_3\Phi(0,t,x)\big)^T\nabla u_0\big(\Phi(0,t,x)\big),$$

with $(D_3\Phi(t_1,t_2,x_3))^T$ the transpose of the Jacobian matrix of Φ w.r.t. the third variable.

Therefore, using $f(w_\varepsilon^+)=\tilde f(w_\varepsilon^+)=\tilde f_{\varepsilon M}(Y)-\varepsilon M$ and the equation $Y_\tau=\tilde f_{\varepsilon M}(Y)$, we infer that

$$\mathcal{L}^\varepsilon\big[w_\varepsilon^+\big]=E_1+E_2+\varepsilon Y_\xi E_3,$$

where

$$E_1=M+Y\Delta v^\varepsilon,$$
$$E_2=\nabla u_0\big(\Phi(0,t,x)\big)\cdot\bigg(\frac{\partial\Phi}{\partial t_2}(0,t,x)+D_3\Phi(0,t,x)\nabla v(t,x)\bigg)Y_\xi,$$
$$E_3=C^\star\mu(\varepsilon M)e^{\mu(\varepsilon M)t/\varepsilon}+\big(D_3\Phi(0,t,x)\big)^T\nabla u_0\big(\Phi(0,t,x)\big)\cdot\nabla v_1^\varepsilon(t,x)$$
$$\quad-\Delta_x\big[u_0\big(\Phi(0,t,x)\big)\big]\frac{(Y^m)_\xi}{Y_\xi}-\big|\nabla_x\big[u_0\big(\Phi(0,t,x)\big)\big]\big|^2\frac{(Y^m)_{\xi\xi}}{Y_\xi}.$$

We note that, for $\varepsilon>0$ sufficiently small, $\delta=\varepsilon M\in(-\delta_0,\delta_0)$ and that, in the range $0\leqslant t\leqslant t^\varepsilon=\varepsilon|\ln\varepsilon|$,

$$\xi=u_0\big(\Phi(0,t,x)\big)+\varepsilon^2C^\star\big(e^{\mu(\varepsilon M)t/\varepsilon}-1\big)\in(-C_0,C_0),$$

so that estimates of Lemma 4.2 on $Y(\tau,\xi;\delta)$ will apply.

Since we have chosen $M \geqslant C_0\|\Delta v^\varepsilon\|_{L^\infty(Q_T)}$, $E_1 \geqslant 0$ holds. Moreover, (4.11) implies $E_2 = 0$. In the sequel we denote by C various positive constants which may change from place to place but do not depend on ε. From Lemma 4.2 (ii)–(iv) we see that $|\frac{(Y^m)_\xi}{Y_\xi}| = |mY^{m-1}| \leqslant C$ and that

$$\left|\frac{(Y^m)_{\xi\xi}}{Y_\xi}\right| \leqslant m(m-1)\left|Y^{m-2}Y_\xi\right| + mY^{m-1}\left|\frac{Y_{\xi\xi}}{Y_\xi}\right| \leqslant C + C\left(e^{\mu(\varepsilon M)t/\varepsilon} - 1\right),$$

since $m \geqslant 2$. Hence

$$E_3 \geqslant \left(C^\star\mu(\varepsilon M) - C\right)e^{\mu(\varepsilon M)t/\varepsilon} - C.$$

Since $\mu(\varepsilon M) \to 1$ as $\varepsilon \to 0$, by choosing $C^\star \gg C$ we see that $E_3 \geqslant 0$ for all $\varepsilon > 0$ small enough.

Recalling that $Y_\xi > 0$, we get $\mathcal{L}^\varepsilon[w_\varepsilon^+] \geqslant 0$ and the lemma is proved. □

We are now in the position to prove (4.1) and (4.2).

Proof of (4.1) and (4.2). Let $\eta \in (0, 1/2)$ be arbitrary. Then [1, Lemma 3.11] provides a constant $C_Y > 0$ such that, for all $\varepsilon > 0$ small enough, for all $\xi \in (-C_0, C_0)$,

$$Y\left(|\ln\varepsilon|, \xi; \pm\varepsilon M\right) \leqslant 1 + \eta; \tag{4.15}$$

$$\text{if } \xi \geqslant C_Y\varepsilon \quad \text{then } Y\left(|\ln\varepsilon|, \xi; \pm\varepsilon M\right) \geqslant 1 - \eta. \tag{4.16}$$

By setting $t = t^\varepsilon = \varepsilon|\ln\varepsilon|$ in (4.14), we obtain

$$\begin{aligned} &Y\left(|\ln\varepsilon|, u_0\left(\Phi\left(0, t^\varepsilon, x\right)\right) - C^\star\varepsilon^2 e^{\mu(-\varepsilon M)|\ln\varepsilon|} + C^\star\varepsilon^2; -\varepsilon M\right)^+ \\ &\quad \leqslant u^\varepsilon\left(t^\varepsilon, x\right) \leqslant Y\left(|\ln\varepsilon|, u_0\left(\Phi\left(0, t^\varepsilon, x\right)\right) + C^\star\varepsilon^2 e^{\mu(\varepsilon M)|\ln\varepsilon|} - C^\star\varepsilon^2; \varepsilon M\right)^+. \end{aligned} \tag{4.17}$$

Therefore, the assertion (4.1) of Theorem 4.1 is a direct consequence of (4.17) and (4.15). Next we prove (4.2). Note that in view of (4.6), we have $\varepsilon e^{\mu(-\varepsilon M)|\ln\varepsilon|} \to 1$ as $\varepsilon \to 0$. Therefore, for $\varepsilon > 0$ small enough (since $Y_\xi > 0$),

$$u^\varepsilon\left(t^\varepsilon, x\right) \geqslant Y\left(|\ln\varepsilon|, u_0\left(\Phi\left(0, t^\varepsilon, x\right)\right) - \frac{3}{2}C^\star\varepsilon + C^\star\varepsilon^2; -\varepsilon M\right)^+. \tag{4.18}$$

Choose $M_0 \gg 0$ so that $M_0\varepsilon - \frac{3}{2}C^\star\varepsilon + C^\star\varepsilon^2 \geqslant \max(C_Y\varepsilon, a(-\varepsilon M))$, with C_Y as in (4.16). Then, for any $x \in \Omega$ such that $u_0(\Phi(0, t^\varepsilon, x)) \geqslant M_0\varepsilon$, we have

$$u_0\left(\Phi\left(0, t^\varepsilon, x\right)\right) - \frac{3}{2}C^\star\varepsilon + C^\star\varepsilon^2 \geqslant C_Y\varepsilon.$$

Combining this, (4.18) and (4.16), we see that

$$u^\varepsilon\left(t^\varepsilon, x\right) \geqslant 1 - \eta.$$

This completes the proof of (4.2). □

4.3. Proof of (4.3)

Let us recall that a finite speed of propagation property, as is (4.3), is proved in [15]: the authors construct a super-solution using a related travelling wave U of minimal speed, and they obtain an $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ estimate of the thickness of the transition layers. We borrow some ideas from this paper but, in order to obtain the improved $\mathcal{O}(\varepsilon)$ estimate, we again use the solution Y of the ordinary differential equation (4.8).

Let z^ε be the solution of the Cauchy problem (recall that $v^\varepsilon(t, x)$ has been extended on $[0, \infty) \times \mathbb{R}^N$ in Remark 1.4)

$$(Q^\varepsilon) \quad \begin{cases} z_t = \varepsilon\Delta\left(z^m\right) - \nabla\cdot\left(z\nabla v^\varepsilon\right) + \dfrac{1}{\varepsilon}f(z) & \text{in } (0, \infty)\times\mathbb{R}^N, \\ z(0, x) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

Lemma 4.4 *(Super-solutions for (Q^ε) for small times). Choose $K \geqslant 1$ and $C_\star > 0$ appropriately. For all $x_0 \in \partial\Omega_0 = \partial\,\mathrm{Supp}\,u_0$, denote by n_0 the unit outward normal vector to $\partial\Omega_0$ at x_0. For $t \geqslant 0$, $x \in \mathbb{R}^n$, define the function*

$$z_\varepsilon^+(t, x) := K\left[Y\left(\frac{t}{\varepsilon}, -\left(\Phi(0, t, x) - x_0\right)\cdot n_0 + \varepsilon^2 C_\star\left(e^{\mu(\varepsilon M)t/\varepsilon} - 1\right); \varepsilon M\right)\right]^+.$$

Here $Y(\tau,\xi;\delta)$ is the solution of (4.8), $\mu(\delta)$ the slope defined in (4.7), $\Phi(t_1,t_2,x_3)$ the flow defined in (2.2) and M is chosen such that, for $\varepsilon>0$ small enough, $M \geqslant C_0\|\Delta v^\varepsilon\|_{L^\infty(Q_T)}$. Then, for all $\varepsilon>0$ small enough,

$$u_0(x) \leqslant z_\varepsilon^+(0,x) \quad \text{for all } x \in \mathbb{R}^N, \tag{4.19}$$

and

$$\mathcal{L}^\varepsilon\big[z_\varepsilon^+\big] := \big(z_\varepsilon^+\big)_t - \varepsilon\Delta\big(z_\varepsilon^+\big)^m + \nabla\cdot\big(z_\varepsilon^+\nabla v^\varepsilon\big) - \frac{1}{\varepsilon}f\big(z_\varepsilon^+\big) \geqslant 0, \tag{4.20}$$

in the domain $[0,t^\varepsilon]\times\mathbb{R}^N$.

Proof. Recall that Ω_0 is convex. Therefore, in view of (1.1), we can choose $K\geqslant 1$ sufficiently large so that, for all $x_0\in\partial\Omega_0$ and all $x\in\Omega_0$,

$$u_0(x) \leqslant -K(x-x_0)\cdot n_0. \tag{4.21}$$

We prove (4.19). If $\Phi(0,0,x)=x\notin\Omega_0$ this is obvious since $u_0(x)=0$. Let us now assume that $\Phi(0,0,x)=x\in\Omega_0$. Since Ω_0 is convex, it lies on one side of the tangent hyperplane at x_0 so that $(x-x_0)\cdot n_0<0$. Recall that $Y(0,\xi;\delta)=\xi$ so that $z_\varepsilon^+(0,x)=-K(x-x_0)\cdot n_0$ and (4.19) follows from (4.21).

We now prove (4.20). As in the proof of Lemma 4.3, straightforward computations combined with (4.8) and (4.11) yield

$$\begin{aligned}
\varepsilon\mathcal{L}^\varepsilon\big[z_\varepsilon^+\big] &= Kf(Y) - f(KY) + \varepsilon K\big(M + Y\Delta v^\varepsilon\big)\\
&\quad + \varepsilon^2 KY_\xi\bigg\{C_\star\mu(\varepsilon M)e^{\mu(\varepsilon M)t/\varepsilon} - D_3\Phi(0,t,x)n_0\cdot\nabla v_1^\varepsilon(t,x)\\
&\quad + \Delta_x\big[\Phi(0,t,x)\cdot n_0\big]K^{m-1}\frac{(Y^m)_\xi}{Y_\xi} - \big|\nabla_x\big[\Phi(0,t,x)\cdot n_0\big]\big|^2K^{m-1}\frac{(Y^m)_{\xi\xi}}{Y_\xi}\bigg\}.
\end{aligned}$$

Note that $Kf(Y)-f(KY)=K(K-1)Y^2\geqslant 0$. Then, by using similar arguments to those in the proof of Lemma 4.3, we see that $\mathcal{L}^\varepsilon[z_\varepsilon^+]\geqslant 0$, if $C_\star>0$ is sufficiently large. □

We now prove (4.3).

Proof of (4.3). We shall first prove that property (4.3) holds for z^ε the solution of the Cauchy problem (Q^ε). Recall that $a(\delta)$ is the unstable zero of $\tilde f_\delta = \tilde f + \delta$ so that $a(\varepsilon M)<0$. Moreover, in view of (4.5) and (4.6), we can choose $M_0>0$ large enough so that, for $\varepsilon>0$ small enough,

$$-M_0\varepsilon + C_\star\varepsilon e^{(\mu(\varepsilon M)-1)|\ln\varepsilon|} - C_\star\varepsilon^2 < a(\varepsilon M).$$

For $x\in\Omega$ such that $\operatorname{dist}(\Phi(0,t^\varepsilon,x),\Omega_0)\geqslant M_0\varepsilon$, we choose $x_0\in\partial\Omega_0$ such that $\operatorname{dist}(\Phi(0,t^\varepsilon,x),\Omega_0)=\|\Phi(0,t^\varepsilon,x)-x_0\|$ and define z_ε^+ as in Lemma 4.4. It follows from Lemma 4.4 and the comparison principle that, for all $\varepsilon>0$ small enough, all $(t,x)\in[0,t^\varepsilon]\times\mathbb{R}^N$,

$$0\leqslant z^\varepsilon(t,x)\leqslant z_\varepsilon^+(t,x). \tag{4.22}$$

Since, for $t=t^\varepsilon=\varepsilon|\ln\varepsilon|$,

$$\begin{aligned}
-\big(\Phi\big(0,t^\varepsilon,x\big)-x_0\big)\cdot n_0 + \varepsilon^2C_\star\big(e^{\mu(\varepsilon M)t^\varepsilon/\varepsilon}-1\big) &= -\big\|\Phi\big(0,t^\varepsilon,x\big)-x_0\big\| + C_\star\varepsilon e^{(\mu(\varepsilon M)-1)|\ln\varepsilon|} - C_\star\varepsilon^2\\
&\leqslant -M_0\varepsilon + C_\star\varepsilon e^{(\mu(\varepsilon M)-1)|\ln\varepsilon|} - C_\star\varepsilon^2\\
&< a(\varepsilon M),
\end{aligned}$$

it follows from Lemma 4.2 (i) that

$$Y\bigg(\frac{t^\varepsilon}{\varepsilon}, -\big(\Phi\big(0,t^\varepsilon,x\big)-x_0\big)\cdot n_0 + \varepsilon^2C_\star\big(e^{\mu(\varepsilon M)t^\varepsilon/\varepsilon}-1\big); \varepsilon M\bigg) < a(\varepsilon M) < 0,$$

and therefore $z_\varepsilon^+(t^\varepsilon,x)=0$, which in turn implies $z^\varepsilon(t^\varepsilon,x)=0$. Hence (4.3) holds for z^ε the solution of (Q^ε).

Now, a straightforward modification of [15, Corollary 4.1] shows that there exists $\tilde T>0$ such that, for all $\varepsilon>0$ small enough,

$$u^\varepsilon(t,x) = z^\varepsilon(t,x),$$

for all $(t,x)\in(0,\tilde T)\times\Omega$. This proves (4.3) for u^ε the solution of (P^ε). □

5. The propagating front

The goal of this section is to construct efficient sub- and super-solutions that control u^ε during the latter time range, when the motion of interface occurs. We begin with some preparations.

5.1. Materials

In the linear diffusion case ($m = 1$), it is well known that the equation $u_t = \Delta u + u(1-u)$ admits travelling wave solutions with some semi-infinite interval of admissible wave speed. The same property holds for the nonlinear diffusion case, namely equation $u_t = \Delta(u^m) + u(1-u)$, $m > 1$. Nevertheless, it turns out that the travelling wave with minimal speed $c^* > 0$ is both compactly supported from one side and sharp. In the following, U denotes the unique solution of

$$\begin{cases} \big(U^m\big)''(z) + c^* U'(z) + U(z)\big(1 - U(z)\big) = 0 & \text{for all } z \in \mathbb{R}, \\ U(-\infty) = 1, & \\ U(z) > 0 & \text{for all } z < 0, \\ U(z) = 0 & \text{for all } z \geqslant 0. \end{cases} \tag{5.1}$$

Lemma 5.1 *(Behavior of U). For all $z \in (-\infty, 0)$ we have $U'(z) < 0$. The travelling wave U is smooth outside 0 and*

$$U'(0) \quad \begin{cases} = 0 & \text{if } 1 < m < 2, \\ \in (-\infty, 0) & \text{if } m = 2, \\ = -\infty & \text{if } m > 2. \end{cases}$$

Moreover, there exist $C > 0$ and $\beta > 0$ such that the following properties hold.

$$\big|\big(U^m\big)'(z)\big| \leqslant C U(z) \quad \text{for all } z \in \mathbb{R}, \tag{5.2}$$

$$0 < 1 - U(z) \leqslant C e^{-\beta|z|} \quad \text{for all } z \leqslant 0, \tag{5.3}$$

$$\big|z U'(z)\big| \leqslant C U(z) \quad \text{for all } z \leqslant -1. \tag{5.4}$$

For more details and proofs we refer the reader to [4,7,15], as well as to [20] for related results.

Another ingredient is a "cut-off signed distance function" $d^\varepsilon(t,x)$ which is defined as follows. Let $\widetilde{d}^\varepsilon = \widetilde{d}^{\varepsilon,\mathrm{drift}}$ be the signed distance function to $\Gamma^{\varepsilon,\mathrm{drift}}$, the smooth solution of the free boundary Problem $(P^0_{\varepsilon,\mathrm{drift}})$, namely

$$\widetilde{d}^\varepsilon(t,x) := \begin{cases} -\operatorname{dist}(x, \Gamma_t^{\varepsilon,\mathrm{drift}}) & \text{for } x \in \Omega_t^{\varepsilon,\mathrm{drift}}, \\ \operatorname{dist}(x, \Gamma_t^{\varepsilon,\mathrm{drift}}) & \text{for } x \in \Omega \setminus \overline{\Omega_t^{\varepsilon,\mathrm{drift}}}, \end{cases} \tag{5.5}$$

where $\operatorname{dist}(x, \Gamma_t^{\varepsilon,\mathrm{drift}})$ is the distance from x to the hypersurface $\Gamma_t^{\varepsilon,\mathrm{drift}}$. We remark that $\widetilde{d}^\varepsilon = 0$ on $\Gamma^{\varepsilon,\mathrm{drift}}$ and that $|\nabla \widetilde{d}^\varepsilon| = 1$ in a neighborhood of $\Gamma^{\varepsilon,\mathrm{drift}}$: there exists $d_0 > 0$ such that, for all $\varepsilon > 0$ small enough, $|\nabla \widetilde{d}^\varepsilon(t,x)| = 1$ if $|\widetilde{d}^\varepsilon(t,x)| < 2d_0$. By reducing d_0 if necessary we can assume that $\widetilde{d}^\varepsilon$ is smooth in $\{(t,x) \in [0,T] \times \overline{\Omega}\colon |\widetilde{d}^\varepsilon(t,x)| < 3d_0\}$.

Next, let $\zeta(s)$ be a smooth increasing function on $\mathbb{R}$ such that

$$\zeta(s) = \begin{cases} s & \text{if } |s| \leqslant d_0, \\ -2d_0 & \text{if } s \leqslant -2d_0, \\ 2d_0 & \text{if } s \geqslant 2d_0. \end{cases}$$

We then define the cut-off signed distance function $d^\varepsilon = d^{\varepsilon,\mathrm{drift}}$ by

$$d^\varepsilon(t,x) := \zeta\big(\widetilde{d}^\varepsilon(t,x)\big). \tag{5.6}$$

Note that

$$\text{if } \big|d^\varepsilon(t,x)\big| < d_0 \quad \text{then } \big|\nabla d^\varepsilon(t,x)\big| = 1, \tag{5.7}$$

and that the equation of motion $(P^0_{\varepsilon,\mathrm{drift}})$ is recast as

$$\big(d_t^\varepsilon + c^* + \nabla d^\varepsilon \cdot \nabla v\big)(t,x) = 0 \quad \text{on } \Gamma_t^{\varepsilon,\mathrm{drift}} = \big\{x \in \Omega\colon d^\varepsilon(t,x) = 0\big\}. \tag{5.8}$$

Moreover, there exists a constant $D > 0$ such that, for all $\varepsilon > 0$ small enough,

$$\big|\nabla d^\varepsilon(t,x)\big| + \big|\Delta d^\varepsilon(t,x)\big| \leqslant D \quad \text{for all } (t,x) \in \overline{Q_T}. \tag{5.9}$$

Finally, in view of (5.7) and (5.8), the mean value theorem provides a constant $N > 0$ such that, for all $\varepsilon > 0$ small enough,

$$\big|d_t^\varepsilon + c^* \big|\nabla d^\varepsilon\big|^2 + \nabla d^\varepsilon \cdot \nabla v\big|(t,x) \leqslant N\big|d^\varepsilon(t,x)\big| \quad \text{for all } (t,x) \in \overline{Q_T}. \tag{5.10}$$

5.2. Sub- and super-solutions

We define

$$u_\varepsilon^\pm(t,x) := \big(1 \pm q(t)\big) U\left(\frac{d^\varepsilon(t,x) \mp \varepsilon p(t)}{\varepsilon}\right), \tag{5.11}$$

where

$$p(t) := -e^{-t/\varepsilon} + e^{Lt} + K,$$
$$q(t) := \sigma\big(e^{-t/\varepsilon} + \varepsilon L e^{Lt}\big),$$

and where U and d^ε were defined in Section 5.1. In the following lemma, $\Omega_t^{supp}[u_\varepsilon^\pm]$, $\Gamma_t^{supp}[u_\varepsilon^\pm]$, $\Gamma^{supp}[u_\varepsilon^\pm]$ and $\nu_t^{supp}[u_\varepsilon^\pm]$ are defined as in Lemma 3.3.

Lemma 5.2 *(Sub- and super-solutions for the propagating front). Choose $\sigma > 0$ small enough so that*

$$c^*(m-1)D^2(1+2\sigma)^{m-2}\sigma \leqslant \frac{1}{2}, \tag{5.12}$$

where D is the constant that appears in (5.9). *Choose $K \geqslant 1$. Then, if $L > 0$ is large enough, we have, for $\varepsilon > 0$ small enough,*

$$\mathcal{L}^\varepsilon\big[u_\varepsilon^-\big] \leqslant 0 \leqslant \mathcal{L}^\varepsilon\big[u_\varepsilon^+\big] \quad \text{in } (0,T)\times\Omega, \tag{5.13}$$

$$\frac{\partial(u_\varepsilon^-)^m}{\partial\nu_t^{supp}[u_\varepsilon^-]} = 0 \quad \text{on } \partial\Omega_t^{supp}\big[u_\varepsilon^-\big], \text{ for all } t\in[0,T], \tag{5.14}$$

$$\frac{\partial(u_\varepsilon^+)^m}{\partial\nu_t^{supp}[u_\varepsilon^+]} = 0 \quad \text{on } \partial\Omega_t^{supp}\big[u_\varepsilon^+\big], \text{ for all } t\in[0,T]. \tag{5.15}$$

Proof. Properties (5.14) and (5.15) follow from $(U^m)'(0)=0$ (see (5.2)). We prove below that $\mathcal{L}^\varepsilon[u_\varepsilon^+] \geqslant 0$, the proof of $\mathcal{L}^\varepsilon[u_\varepsilon^-] \leqslant 0$ follows the same lines. Note that we only need to consider the case $d^\varepsilon(t,x) \leqslant \varepsilon p(t)$ since, if $d^\varepsilon(t,x) > \varepsilon p(t)$ then $u_\varepsilon^+(t,x) = 0$. Straightforward computations and Eq. (5.1) yield

$$\varepsilon\big(u_\varepsilon^+\big)_t = \varepsilon q' U + (1+q)\big(d_t^\varepsilon - \varepsilon p'\big)U',$$
$$\varepsilon\nabla u_\varepsilon^+ = (1+q)U'\nabla d^\varepsilon,$$
$$\varepsilon^2\Delta\big(u_\varepsilon^+\big)^m = (1+q)^m\big|\nabla d^\varepsilon\big|^2\big(-c^*U' - U(1-U)\big) + \varepsilon(1+q)^m\Delta d^\varepsilon\big(U^m\big)',$$

where arguments are omitted. Thus we get

$$\varepsilon\mathcal{L}^\varepsilon\big[u_\varepsilon^+\big] = E_1 + E_2 + E_3,$$

where

$$E_1 = U'(1+q)\big[d_t^\varepsilon - \varepsilon p' + c^*(1+q)^{m-1}\big|\nabla d^\varepsilon\big|^2 + \nabla d^\varepsilon\cdot\nabla v^\varepsilon\big] =: U'(1+q)E_1^\star,$$
$$E_2 = U\big\{-(1+q) + (1+q)^m\big|\nabla d^\varepsilon\big|^2 + U\big[(1+q)^2 - (1+q)^m\big|\nabla d^\varepsilon\big|^2\big] + \varepsilon q'\big\},$$
$$E_3 = -\varepsilon(1+q)^m\Delta d^\varepsilon\big(U^m\big)' + \varepsilon(1+q)\Delta v^\varepsilon U.$$

In the sequel we define $a(t) := 1 + q(t)$ and denote by C_i various positive constants which do not depend on ε.

Since $\|\Delta v^\varepsilon\|_{L^\infty(Q_T)}$ is uniformly bounded w.r.t. $\varepsilon > 0$ (see Assumption 1.3), we deduce from (5.9) and (5.2) that $|E_3| \leqslant \varepsilon C_3(a^m + a)U$ so that

$$E_2 + E_3 \geqslant U\big\{-a + a^m + U\big(a^2 - a^m\big) - \varepsilon C_3 a^m - \varepsilon C_3 a + \big(\big|\nabla d^\varepsilon\big|^2 - 1\big)a^m(1-U) + \varepsilon q'\big\}. \tag{5.16}$$

We claim that, for $\varepsilon > 0$ small enough,

$$\big|\big(\big|\nabla d^\varepsilon\big|^2 - 1\big)(1-U)\big| \leqslant \varepsilon C_2. \tag{5.17}$$

Indeed, if $-d_0 < d^\varepsilon(t,x) \leqslant \varepsilon p(t)$, it follows from (5.7) that, for $\varepsilon > 0$ small enough, $|\nabla d^\varepsilon(t,x)| = 1$. Next, if $d^\varepsilon(t,x) \leqslant -d_0$, (5.3) implies that

$$0 < (1-U)\left(\frac{d^\varepsilon(t,x) - \varepsilon p(t)}{\varepsilon}\right) \leqslant (1-U)\left(-\frac{d_0}{\varepsilon}\right) \leqslant Ce^{-\beta\frac{d_0}{\varepsilon}},$$

and (5.17) holds for $\varepsilon > 0$ small enough. Therefore (5.16) and (5.17) imply

$$E_2 + E_3 \geqslant U\{-a + a^m + U(a^2 - a^m) - \varepsilon(C_2 + C_3)a^m - \varepsilon C_3 a + \varepsilon q'\}. \tag{5.18}$$

Next, since

$$E_1^\star = d_t^\varepsilon + c^*|\nabla d^\varepsilon|^2 + \nabla d^\varepsilon \cdot \nabla v - \varepsilon p' + c^*(a^{m-1} - 1)|\nabla d^\varepsilon|^2 + \varepsilon \nabla d^\varepsilon \cdot \nabla v_1^\varepsilon,$$

using (5.10), (5.9) and Assumption 1.3, we see that

$$\begin{aligned} E_1^\star &\leqslant N|d^\varepsilon(t,x)| - \varepsilon p'(t) + c^*(a^{m-1} - 1)|\nabla d^\varepsilon|^2 + \varepsilon \nabla d^\varepsilon \cdot \nabla v_1^\varepsilon \\ &\leqslant N|d^\varepsilon(t,x) - \varepsilon p(t)| + \varepsilon(Np(t) - p'(t)) + c^*(a^{m-1} - 1)D^2 + \varepsilon C D \\ &\leqslant N|d^\varepsilon(t,x) - \varepsilon p(t)| + \varepsilon(Np(t) - p'(t)) + c^*(m-1)D^2(1+2\sigma)^{m-2}q(t) + \varepsilon C D. \end{aligned}$$

The last inequality above comes from the fact that, for $\varepsilon > 0$ small enough, we have $0 \leqslant q(t) \leqslant \sigma(1 + \varepsilon L e^{LT}) \leqslant 2\sigma$, which in turn implies that

$$0 \leqslant a^{m-1} - 1 \leqslant (m-1)(1+2\sigma)^{m-2}q(t). \tag{5.19}$$

In the following, we distinguish two cases.

First, assume that $0 \leqslant d^\varepsilon(t,x) \leqslant \varepsilon p(t)$ so that, for $\varepsilon > 0$ small enough,

$$\begin{aligned} E_1^\star &\leqslant \varepsilon(2Np(t) - p'(t)) + c^*(m-1)D^2(1+2\sigma)^{m-2}q(t) + \varepsilon C D \\ &\leqslant e^{-t/\varepsilon}(-\varepsilon 2N - 1 + c^*(m-1)D^2(1+2\sigma)^{m-2}\sigma) \\ &\quad + e^{Lt}(\varepsilon 2N - \varepsilon L + \varepsilon c^*(m-1)D^2(1+2\sigma)^{m-2}\sigma L) + \varepsilon 2NK + \varepsilon C D. \end{aligned}$$

In view of (5.12) we get

$$E_1^\star \leqslant \varepsilon\left(e^{Lt}\left(2N - \frac{1}{2}L\right) + 2NK + CD\right) \leqslant 0,$$

if $L > 0$ is sufficiently large. This implies that $E_1 = aU'E_1^\star \geqslant 0$.

Now, assume that $d^\varepsilon(t,x) \leqslant 0$ so that

$$\frac{d^\varepsilon(t,x) - \varepsilon p(t)}{\varepsilon} \leqslant -K \leqslant -1. \tag{5.20}$$

If $E_1^\star \leqslant 0$ the conclusion $E_1 \geqslant 0$ follows. Let us now assume $E_1^\star > 0$. The above study for the case $0 \leqslant d^\varepsilon(t,x) \leqslant \varepsilon p(t)$ implies a fortiori that

$$\varepsilon(Np(t) - p'(t)) + c^*(m-1)D^2(1+2\sigma)^{m-2}q(t) + \varepsilon C D \leqslant 0.$$

Therefore

$$|E_1^\star| \leqslant N|d^\varepsilon(t,x) - \varepsilon p(t)|,$$

and we deduce from (5.20) and (5.4) that

$$|E_1| \leqslant \varepsilon C_1 a U.$$

Summarizing, we obtain that, in any cases,

$$\varepsilon \mathcal{L}^\varepsilon[u_\varepsilon^+] \geqslant U\{-a + a^m + U(a^2 - a^m) - \varepsilon C_4 a^m + \varepsilon q'\},$$

since $a = 1 + q > 1$ and with $C_4 := C_1 + C_2 + 2C_3$. Since $U < 1$, $a > 1$ and $m \geqslant 2$, the inequality $-a + a^m + U(a^2 - a^m) \geqslant -a + a^2 = q + q^2 \geqslant q$ holds. Therefore, using $|a| \leqslant 1 + 2\sigma$ and substituting the expressions for $q(t)$ and $q'(t)$, we see that

$$\begin{aligned} \varepsilon \mathcal{L}^\varepsilon[u_\varepsilon^+] &\geqslant U\{\varepsilon\sigma L e^{Lt} - \varepsilon C_4(1+2\sigma)^m + \sigma\varepsilon^2 L^2 e^{Lt}\} \\ &\geqslant U\varepsilon\{\sigma L - C_4(1+2\sigma)^m\} \\ &\geqslant 0, \end{aligned}$$

if $L > 0$ is sufficiently large.

This completes the proof of Lemma 5.2. □

6. Proof of Theorem 2.2

By fitting the sub- and super-solutions for the generation into the ones for the motion, we are now in the position to prove our main result.

Let $\eta \in (0, 1/2)$ be arbitrary. Choose σ that satisfies (5.12) and

$$\sigma \leqslant \frac{\eta}{2}. \tag{6.1}$$

By Theorem 4.1, there exists $M_0 > 0$ such that (4.1), (4.2) and (4.3) hold with the constant η replaced by $\sigma/2$. Recall that $u_\varepsilon^\pm$ are the sub- and super-solutions constructed in (5.11).

Lemma 6.1 *(Ordering initial data). There exists $\tilde{K} > 0$ such that for all $K \geqslant \tilde{K}$, all $L > 0$, all $\varepsilon > 0$ small enough, we have*

$$u_\varepsilon^-(0,x) \leqslant u^\varepsilon\big(t^\varepsilon, x\big) \leqslant u_\varepsilon^+(0,x), \tag{6.2}$$

for all $x \in \Omega$.

Proof. We first prove

$$u_\varepsilon^-(0,x) = (1-\sigma-\varepsilon\sigma L)U\left(\frac{d^\varepsilon(0,x)+K\varepsilon}{\varepsilon}\right) \leqslant u^\varepsilon\big(t^\varepsilon, x\big). \tag{6.3}$$

If x is such that $d^\varepsilon(0,x) \geqslant -K\varepsilon$, this is obvious since the definition of U then implies $u_\varepsilon^-(0,x) = 0$. Next assume that x is such that $d^\varepsilon(0,x) < -K\varepsilon$. Let us denote by $d(t,x)$ the signed distance function to Γ_t. Note that, in view of hypothesis (1.1), the mean value theorem provides the existence of a constant $\tilde{K}_0 > 0$ such that

$$\text{if } d(0,y) \leqslant -\tilde{K}_0\varepsilon \quad \text{then } u_0(y) \geqslant M_0\varepsilon. \tag{6.4}$$

Moreover in view of the definition of $\Gamma_0^{\varepsilon,\mathrm{drift}}$ in (2.4) and the compactness of Γ_0, there exists $K_0 > 0$ such that, for $\varepsilon > 0$ small enough,

$$\text{if } d^\varepsilon(0,x) \leqslant -K_0\varepsilon \quad \text{then } d\big(0, \Phi\big(0, t^\varepsilon, x\big)\big) \leqslant -\tilde{K}_0\varepsilon. \tag{6.5}$$

Hence, if we choose $K \geqslant K_0$, we deduce from (6.5), (6.4) and (4.2) (with η replaced by $\sigma/2$) that $u^\varepsilon(t^\varepsilon, x) \geqslant 1 - \frac{\sigma}{2}$. Since $U \leqslant 1$, this proves (6.3).

Next we prove

$$u^\varepsilon\big(t^\varepsilon, x\big) \leqslant (1+\sigma+\varepsilon\sigma L)U\left(\frac{d^\varepsilon(0,x)-K\varepsilon}{\varepsilon}\right) = u_\varepsilon^+(0,x). \tag{6.6}$$

In view of the definition of $\Gamma_0^{\varepsilon,\mathrm{drift}}$ in (2.4) and the compactness of Γ_0, there exists $K_1 > 0$ such that

$$\text{if } d^\varepsilon(0,x) \geqslant K_1\varepsilon \quad \text{then } \operatorname{dist}\big(\Phi\big(0, t^\varepsilon, x\big), \Omega_0\big) \geqslant M_0\varepsilon. \tag{6.7}$$

If x is such that $d^\varepsilon(0,x) \geqslant K_1\varepsilon$ then it follows from (6.7) and Theorem 4.1 (iii) that $u^\varepsilon(t^\varepsilon, x) = 0$, which proves (6.6). Next assume that x is such that $d^\varepsilon(0,x) < K_1\varepsilon$. Since U is nonincreasing we have

$$(1+\sigma+\varepsilon\sigma L)U\left(\frac{d^\varepsilon(0,x)-K\varepsilon}{\varepsilon}\right) \geqslant (1+\sigma)U(K_1-K) \geqslant 1+\frac{\sigma}{2},$$

if $K \gg K_1$ (recall that $U(-\infty) = 1$). Then (6.6) follows from (4.1) (with η replaced by $\sigma/2$). $\square$

We now prove Theorem 2.2.

Proof of Theorem 2.2. We fix $K \geqslant 1$ large enough so that Lemma 6.1 holds, and $L > 0$ large enough so that Lemma 5.2 holds. Therefore, from the comparison principle, we deduce that

$$u_\varepsilon^-(t,x) \leqslant u^\varepsilon\big(t+t^\varepsilon, x\big) \leqslant u_\varepsilon^+(t,x) \quad \text{for } 0 \leqslant t \leqslant T - t^\varepsilon. \tag{6.8}$$

Note that, since

$$\lim_{\varepsilon\to 0} u_\varepsilon^\pm(t,x) = \begin{cases} 1 & \text{if } d^\varepsilon(t,x) < 0, \\ 0 & \text{if } d^\varepsilon(t,x) > 0, \end{cases} \tag{6.9}$$

for $t > 0$, (6.8) is enough to prove the convergence result, namely Corollary 2.3. We now choose $\mathcal{C}$ large enough so that

$$\left(1-\frac{3}{4}\eta\right)U\left(-\mathcal{C}+e^{LT}+K\right) \geqslant 1-\eta. \tag{6.10}$$

Note that this choice forces

$$\mathcal{C} \geqslant e^{LT}+K. \tag{6.11}$$

In the sequel we prove (2.6).

Obviously, if $\varepsilon > 0$ is small enough, the constant map $z^+ \equiv 1+\eta$ is a super-solution. Therefore we deduce from Theorem 4.1 (i) that $u^\varepsilon(t+t^\varepsilon, x) \in [0, 1+\eta]$, for all $0 \leqslant t \leqslant T - t^\varepsilon$.

Next we take $x \in \Omega_t^{\varepsilon,\mathrm{drift}} \setminus \mathcal{N}_{\mathcal{C}\varepsilon}(\Gamma_t^{\varepsilon,\mathrm{drift}})$, i.e.

$$d^\varepsilon(t,x) \leqslant -\mathcal{C}\varepsilon. \tag{6.12}$$

For $\varepsilon > 0$ small enough, we have

$$\begin{aligned} u_\varepsilon^-(t,x) &\geqslant \left(1-\sigma-\varepsilon\sigma L e^{LT}\right)U\left(-\mathcal{C}+e^{LT}+K\right) \\ &\geqslant \left(1-\frac{3}{2}\sigma\right)U\left(-\mathcal{C}+e^{LT}+K\right) \\ &\geqslant \left(1-\frac{3}{4}\eta\right)U\left(-\mathcal{C}+e^{LT}+K\right) \\ &\geqslant 1-\eta, \end{aligned}$$

where we have successively used (6.1) and (6.10). In view of (6.8) this implies that $u^\varepsilon(t+t^\varepsilon, x) \geqslant 1-\eta$, for all $0 \leqslant t \leqslant T - t^\varepsilon$.

Finally we take $x \in (\Omega \setminus \overline{\Omega_t^{\varepsilon,\mathrm{drift}}}) \setminus \mathcal{N}_{\mathcal{C}\varepsilon}(\Gamma_t^{\varepsilon,\mathrm{drift}})$, i.e.

$$d^\varepsilon(t,x) \geqslant \mathcal{C}\varepsilon. \tag{6.13}$$

Using (6.11) we see that, for $\varepsilon > 0$ small enough, $d^\varepsilon(t,x) - \varepsilon p(t) \geqslant 0$ so that $u_\varepsilon^+(t,x) = 0$, which, in view of (6.8) implies that $u^\varepsilon(t+t^\varepsilon, x) = 0$, for $0 \leqslant t \leqslant T - t^\varepsilon$.

This completes the proof of Theorem 2.2. □

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