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Nonstandard Conjunctions and Implications in Fuzzy Logic

János C. Fodor and Tibor Keresztfalvi
Eötvös Loránd University, Budapest, Hungary

ABSTRACT

We point out possible disadvantages of considering exclusively t-norms and t-conorms as proper models for conjunction and disjunction in fuzzy logic. We draw up a general framework for particular investigations, expressed by the so-called closure property. We suggest a constructive approach to the axiomatics of generalized modus ponens (GMP). As a consequence, a system of functional equations is obtained. Idempotent as well as nonidempotent conjunctions fulfilling this system are studied. Three classes of nonstandard conjunctions and implications are formulated so that all of them satisfy the proposed axioms.

KEYWORDS: ***conjunctions; R- and S-implications; generalized modus ponens.***

1. INTRODUCTION

The proper definition of connectives (conjunction, disjunction, negation, implication, etc.) is one of the most important problems in fuzzy logic. Nowadays it is needless to emphasize the dominance of t-norms, t-conorms, strong negations, and related implications. Their sound theoretical foundation as well as their wide variety have given them almost an exclusive role in different theoretical investigations and practical applications. However, people are inclined to use them also as a matter of routine. The following examples support this statement and suggest the study of enlarged classes of operations for fuzzy sets and reasoning.

Address correspondence to János C. Fodor, Department of Computer Science, Eötvös Loránd University, Múzeum krt. 6-8, H-1088 Budapest, Hungary.

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1. When one works with binary conjunctions and there is no need to extend them for three or more arguments, as happens e.g. in the inference pattern called generalized modus ponens (GMP for short), associativity of the conjunction is an unnecessarily restrictive condition. The same is valid for the commutativity property if the two arguments have different semantical backgrounds and it makes no sense to interchange one with the other.
2. In GMP, a number of intuitively desirable properties are not obtained using t-norms and implications defined by t-norms. For more details see Magrez and Smets [1].
3. Obviously, the properties of conjunctions, disjunctions, and negations have to be connected and to be in accordance with those of fuzzy implications. However, if one compares usual axioms for fuzzy implications with properties of R- and S-implications defined by t-norms, t-conorms, and strong negations, then it can easily be observed that these two families have "much nicer" properties than would be axiomatically expected. For more details see Weber [2], Dubois and Prade [3], Fodor [4].
4. There is no way to define strict negations via t-norm-based residuation: the resulted negation is either degenerate or strong; see Remark 4.2 and Theorem 4.3 in [2]. However, the so-called weak t-norms are appropriate conjunctions from this point of view: strict negations appear on using weak-t-norm-based residuation; see Fodor [5].
5. t-norm-based R- and S-implications are, in general, different. For continuous t-norms, these can coincide if and only if the underlying t-norm is isomorphic to the Łukasiewicz t-norm; see for instance Smets and Magrez [6]. Note that a new family of *left-continuous* t-norms has been found by Fodor [7] such that the corresponding R- and S-implications are the same.

These observations, which are very often left out of consideration, have prompted us to revise definitions and properties of operations in fuzzy logic. A new unifying approach is suggested for the investigation of these connectives. It is supported by an important relationship between implications and conjunctions expressed by Equation (4) below.

The paper is organized as follows. After some necessary preliminaries we draw up the theoretical framework for further investigations. Starting from a binary conjunction, a sequence of conjunctions is introduced in a natural way. We want to exclude chaotic behavior of this sequence by requiring the existence of a member of this sequence which agrees with the starting conjunction. In other words, this sequence should be *closed*. This principle is expressed by a functional equation. Its solution is briefly recalled in Section 3. Generalized modus ponens is revisited in Section 4, by choosing a constructive way to investigate its properties. This leads us to a system of functional equations for conjunctions and implications in GMP. Idempotent solutions are studied first, which are useful also in dealing with redundancies in knowledge bases; see [3]. Then a particular class of noncommutative and nonassociative conjunctions and the corresponding class of implications is determined, providing appropriate models for connectives in GMP. Finally, some concluding remarks are presented.

2. BACKGROUND AND THEORETICAL FRAMEWORK

In this section we recall some definitions and results that are more or less known in the literature. Then a theoretical framework is outlined which seems to be appropriate for our further investigations.

A function $n:[0,1] \to [0,1]$ is called a *negation* if it is nonincreasing and $n(0) = 1$, $n(1) = 0$. A negation n is called *strict* if n is continuous and decreasing. A strict negation n is called *strong* if $n(n(x)) = x$ for every $x \in [0,1]$.

A binary operation $*$ on $[0,1]$ is called a *fuzzy conjunction* if it is an extension of the classical Boolean conjunction, i.e.,

$$x * y \in [0,1] \qquad \text{for every} \quad x, y \in [0,1]$$

and

$$0*0 = 0*1 = 1*0 = 0, \qquad 1*1 = 1.$$

A canonical model of fuzzy conjunctions is the family of t-norms, i.e., functions $T:[0,1] \times [0,1] \to [0,1]$ which are commutative, associative, nondecreasing, and such that $T(x,1) = x$ for every $x \in [0,1]$. For more details see e.g. Weber [2] and Schweizer and Sklar [8].

A binary operator $\to$ on $[0,1]$ is a *fuzzy implication* if it is an extension of the Boolean implication, i.e.,

$$x \to y \in [0,1] \qquad \text{for every} \quad x, y \in [0,1]$$

and

$$0 \to 0 = 0 \to 1 = 1 \to 1 = 1, \qquad 1 \to 0 = 0.$$

Let $\odot$ be any binary operation on $[0,1]$. The following transformations of $\odot$ play a central role in this paper:

$$x \mathscr{S}_n(\odot)\, y = n(x \odot n(y)), \tag{1}$$

$$x \mathscr{R}(\odot)\, y = \sup\{z \in [0,1] \mid x \odot z \leq y\}, \tag{2}$$

where n is a strong negation.

Obviously, $\mathscr{S}_n \circ \mathscr{S}_n(\odot) = \odot$ for any binary operation $\odot$ on $[0,1]$ (here $\circ$ denotes composition). Moreover, $\mathscr{S}_n(\to)$ is a fuzzy conjunction if $\to$ is a fuzzy implication. On the other hand, if $*$ is a fuzzy conjunction, then $\mathcal{S}_n(*)$ and $\mathcal{R}(*)$ are fuzzy implications. It is clear that

$$I_S(x, y) = x \mathcal{S}_n(*) y \qquad (S\text{-implication})$$

is based on the classical view of implications, while

$$I_R(x, y) = x \mathcal{R}(*) y \qquad (R\text{-implication})$$

is based on a residuation concept; see e.g. Dubois and Prade [9] when $*$ is a t-norm, and Fodor [4] when $*$ is an arbitrary fuzzy conjunction in the above wide sense.

Suppose $*$ is a fuzzy conjunction. Then one can define a sequence of conjunctions $\{*_j\}$ in the following way:

$$*_0 = *,$$
$$*_j = \mathcal{S}_n \circ \mathcal{R}(*_{j-1}), \qquad j = 1, 2, 3, \ldots. \tag{3}$$

In the sequel we will consider only those conjunctions $*$ for which the above sequence $\{*_j\}$ is *closed* in the sense that there exists a member $*_m \in \{*_j\}$ such that

$$*_m = *. \tag{4}$$

This property excludes undesirable (chaotic) behavior of $\{*_j\}$, and it is the starting point in our further investigations.

3. CLOSURE THEOREMS

All results of this section (with more details and proofs) can be found in Fodor [4, 5]. Fortunately, it is sufficient to investigate the above problem for $m = 1$ and $m = 2$, due to the following theorem.

THEOREM 1 *Let* $\{*_j\}$ *be a sequence of conjunctions defined by* (4). *Then there exists* $*_m \in \{*_j\}$ *such that* $*_m = *$ *if and only if either* $*_1 = *$ *or* $*_2 = *$.

It is clear from the definition of $\{*_j\}$ that $*_1 = *$ is equivalent to

$$\mathcal{R}(*) = \mathcal{S}_n(*), \tag{5}$$

while $*_2 = *$ means that

$$\mathcal{R} \circ \mathcal{S}_n \circ \mathcal{R}(*) = \mathcal{S}_n(*). \tag{6}$$

Moreover, (5) implies (6). The situation described by Equation (6), which was investigated by Dubois and Prade [9] in the case when $*$ is a t-norm and by Fodor [4] in the general case, is illustrated in Figure 1. Complete characterizations of binary operations satisfying either Equation (5) or Equation (6) are given in the following theorem (for more details and proofs see [5]).

THEOREM 2 *A binary operation* $*$ *on* $[0,1]$ *satisfies the equation*

(a) $\mathscr{R}(*) = \mathscr{S}_n(*)$ *if and only if*

$$x * z \le y \quad \Leftrightarrow \quad x * n(y) \le n(z) \quad \forall x, y, z \in [0,1];$$

(b) $\mathscr{R} \circ \mathscr{S}_n \circ \mathscr{R}(*) = \mathscr{S}_n(*)$ *if and only if*

$$x * z \le y \quad \Leftrightarrow \quad z \le x\mathscr{R}(*)y \quad \forall x, y, z \in [0,1].$$

It is worthwhile drawing up the corresponding results when $* = T$ is a t-norm.

COROLLARY 1 *Let* T *be a t-norm (as a binary conjunction on* $[0,1]$*).*

(a) *T is continuous and satisfies Equation* (5) *if and only if there exists an automorphism* φ *of the closed unit interval such that*

$$T(x,y) = \varphi^{-1}(\max\{\varphi(x) + \varphi(y) - 1, 0\})$$

and

$$n(x) = \varphi^{-1}(1 - \varphi(x));$$

(b) *T satisfies Equation* (6) *if and only if T is left-continuous in both places on* $(0,1]$.

In other words, for continuous t-norms, t-norm-based R- and S-implications coincide if and only if T is a φ-transform of the Łukasiewicz t-norm.

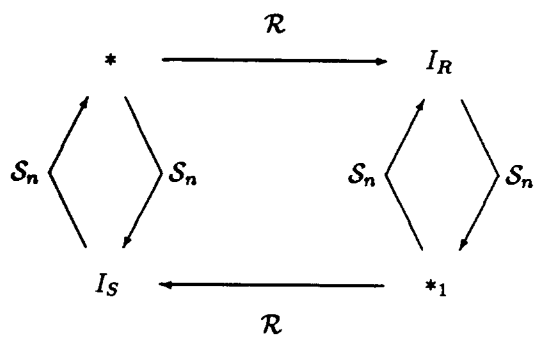

Figure 1. The second case.

No similar characterization is known when left-continuous t-norms are considered in (5). However, each of the following left-continuous t-norms (the nilpotent minimum family) satisfies (5). For more details on this family see Fodor [7].

Suppose φ is an automorphism of the unit interval, and define a binary operation $\min_{\varphi,0}$ as follows.

$$\min_{\varphi,0}(x,y) = \begin{cases} \min(x,y) & \text{if} \quad \varphi(x)+\varphi(y) > 1, \\ 0 & \text{if} \quad \varphi(x)+\varphi(y) \le 1. \end{cases} \tag{7}$$

Let n be the strong negation generated by φ:

$$n(x) = \varphi^{-1}(1-\varphi(x)) \qquad \text{for all} \quad x \in [0,1].$$

Then one can easily obtain the following formulas:

$$x\mathscr{R}\Big(\min_{\varphi,0}\Big)y = \begin{cases} 1 & \text{if} \quad x \le y, \\ \max(n(x),y) & \text{if} \quad x > y \end{cases}$$

and

$$x\,\mathscr{S}_n\Big(\min_{\varphi,0}\Big)y = \begin{cases} 1 & \text{if} \quad x \le y, \\ \max(n(x),y) & \text{if} \quad x > y. \end{cases}$$

That is, Equation (5) is satisfied by $* = \min_{\varphi,0}$.

Another class of conjunctions, for which (5) also holds, will be characterized in Section 5. This class of conjunctions satisfies some properties which makes it suitable for using in approximate reasoning, especially in the generalized modus ponens.

4. GENERALIZED MODUS PONENS

The *generalized modus ponens* (GMP), an inference pattern with fuzzy predicates, is given as follows:

$$\begin{array}{llllll} \text{Rule} & \text{if} & S_1 \text{ has property } A & \text{then} & S_2 \text{ has property } B \\ \text{Fact} & & S_1 \text{ has property } A' & & \\ \hline & & & & S_2 \text{ has property } B' \end{array} \tag{8}$$

where A, A' and B, B' are fuzzy sets of the universes X and Y respectively, i.e., $A, A' \in \mathscr{F}(X)$ and $B, B' \in \mathscr{F}(Y)$. We emphasize that these fuzzy sets are not necessarily normalized.

B' is calculated as

$$B'(y) = \sup_x M(A'(x), I_{A\to B}(x,y)), \tag{9}$$

where M is a fuzzy conjunction and $I_{A \to B}$ is a fuzzy binary relation (usually an implication) on $X \times Y$.

In general, GMP is expected to meet a number of intuitively desirable requirements. Most papers on GMP investigate this problem by first choosing particular classes of conjunctions (e.g. t-norms) and implications (e.g. S- or R-implications based on t-norms) and then testing whether the different requirements are fulfilled. There are lots of possible choices, but still no "best" one; see [1].

Opposed to these approaches, we choose a constructive way to investigate properties of GMP. First we fix only a few basic requirements to be fulfilled, in our opinion, by GMP. Then we state, in the form of axioms, some reasonable properties of conjunction and implication operators. This leads to a system of functional equations for M and $I_{A \to B}$. Then further properties of GMP are verified as consequences, though they usually appear as requirements in the rich literature on GMP (see e.g. the references in [1]). Finally, we show several classes of both idempotent and nonidempotent particular solutions for M and $I_{A \to B}$.

Notice that a different approach, a new model of fuzzy modus ponens, was established also in [1] in order to satisfy all the intuitively required properties. Instead, we keep GMP unchanged while conjunctions and implications are used in a broad sense.

In the literature it is generally required that

R1. if $A' = A$ then $B' = B$ $(A, B \not\equiv 0)$;

R2. if $\operatorname{Supp} A' \cap \operatorname{Supp} A = \emptyset$ then $B' \equiv 1$ $(A, B \not\equiv 0)$;

R3. $B'(y)$ is nondecreasing with respect to $A'(x)$ and $B(y)$ and nonincreasing with respect to $A(x)$ (monotonicity);

R4. if $A' \equiv 0$ then $B' \equiv 0$.

R1 reflects the coincidence of (9) with classical modus ponens. R2 forces the GMP to infer *unknown* when the fact A' has nothing to do with the antecedent A. R3 is clear, and R4 is also obvious: if nothing is observed, then nothing is inferred.

We want to find at least one pair (M, I) such that R1–R4 are satisfied by using (9).

4.1 Axioms

First we assume that $I_{A \to B}$ is defined pointwise, that is,

A1. $I_{A \to B}(x, y)$ depends only on $A(x)$ and $B(y)$, i.e. $I_{A \to B}(x, y) = J(A(x), B(y))$, and so (9) turns into

$$B'(y) = \sup_x M(A'(x), J(A(x), B(y))). \tag{10}$$

A2. J is nonincreasing with respect to its first argument and nondecreasing with respect to its second argument [briefly, $J(\searrow, \nearrow)$];
A3. $J(0, v) = 1 \ \forall v \in [0, 1]$;
A4. $J(1, v) \le v \ \forall v \in [0, 1]$.
A5. M is nondecreasing with respect to both arguments [briefly, $M(\nearrow, \nearrow)$];
A6. $M(0, v) = 0 \ \forall v \in [0, 1)$;
A7. $M(u, v) \le v \ \forall u, v \in [0, 1]$.

Obviously, these axioms are fulfilled when $M = T$ is a t-norm and J is either an R-implication or an S-implication based on T.

4.2 Conditions on M and J Implied by the Crisp Case

Obviously, the GMP should satisfy properties R1–R4 also when A, A', B, B' are crisp sets, so we obtain from (10) on the basis of R1, R2, and R3 that

$$\max\{M(0, J(0, 1)), M(1, J(1, 1))\} = 1, \tag{11}$$

$$\max\{M(0, J(0, 0)), M(1, J(1, 0))\} = 0, \tag{12}$$

$$\max\{M(0, J(1, v)), M(1, J(0, v))\} = 1 \qquad (v \in \{0, 1\}). \tag{13}$$

After simple calculations we finally get from the above equations and from R3 and R4 the following system of equations for any $u, v \in]0, 1]$:

$$\begin{aligned} M(0, J(u, v)) &= 0, \\ M(1, J(0, v)) &= 1, \\ M(u, J(1, 0)) &= 0, \\ M(1, J(u, 1)) &= 1. \end{aligned} \tag{14}$$

Replacing A, A', and B by fuzzy singletons (fuzzy points) of height u and v respectively, we have from R1 for any $u, v \in [0, 1]$ the following equation:

$$M(u, J(u, v)) = v. \tag{15}$$

Note that this last equation cannot be satisfied by using a t-norm T and R- or S-implication based on T. Indeed, if $x, y \in [0, 1]$ and $x < y$, then

$$T(x, I_T(x, y)) = x < y,$$

where $I_T(x, y) = \sup\{z | T(x, z) \le y\}$ is the R-implication defined by T. On the other hand, if $x < y = 1$ and $J(x, y) = n(T(x, n(y)))$ is the S-implication defined by T, then we have

$$T(x, J(x, y)) = T(x, J(x, 1)) = T(x, 1) = x < y = 1.$$

Therefore, we have to find solutions of (15) outside the class of t-norms and corresponding R- or S-implications.

By using our axioms A1–A7, it is easy to see that we have

$$\begin{aligned} M(1,v) &= v, \\ J(1,v) &= v, \\ J(u,1) &= 1. \end{aligned} \tag{16}$$

Then (15) and (16) together imply that

$$M(u,1) = 1 \qquad \forall u > 0. \tag{17}$$

Compare Equations (15), (17) and properties A5, A6, A7 with those of a modus ponens generating function in Trillas and Valverde [10].

Under some continuity conditions, any solution (M, J) of (14)–(16) possesses further nice properties, as we prove in the following theorem.

PROPOSITION 1 *Suppose that* (M, J) *is any solution of* (14)–(16) *satisfying axioms* A1–A8 *and* J *is right-continuous in its first argument. Then the following properties are also satisfied by using* (10):

P1. *if* $A' \subset A$ *then* $B' \subseteq B$;
P2. *if* $A' \equiv 1$ *and* $\inf_x A(x) = 0$ *then* $B' \equiv 1$;
P3. *if* $A \equiv 0$ *and* $A' \not\equiv 0$ *then* $B' \equiv 1$.

Proof To prove P1, we can write

$$\begin{aligned} B'(y) &= \sup_x M(A'(x), J(A(x), B(y))) \\ &\le \sup_x M(A(x), J(A(x), B(y))) \\ &= B(y), \end{aligned}$$

by A6 and (15).

Concerning P2, the following chain of equalities can be written:

$$\begin{aligned} B'(y) &= \sup_x M(A'(x), J(A(x), B(y))) \\ &= \sup_x M(1, J(A(x), B(y))) \\ &= \sup_x J(A(x), B(y)) = J\Big(\inf_x A(x), B(y)\Big) \\ &= J(0, B(y)) = 1, \end{aligned}$$

where we have used (16) and the right continuity of J in its first place.

P3 is obvious because we have

$$B'(y) = \sup_x M(A'(x), J(A(x), B(y))) = \sup M(A'(x), 1) = 1,$$

by (16) and (17). ■

4.3 Idempotent Solutions

In this section we look for solutions (M, J) of the system (14)–(16) such that both M and J are *idempotent*, i.e.,

$$M(x, x) = x \quad \text{for all} \quad x \in [0, 1],$$

$$J(x, x) = x \quad \text{for all} \quad x \in (0, 1].$$

Note that idempotency of conjunctions is useful in dealing with redundancies in knowledge bases; see [3]. On the other hand, idempotency of implications is not a very common property. The equality $J(x, x) = x$ can hold only on $(0, 1]$, since $J(0, 0) = 1$.

First consider M. Monotonicity and idempotency of M together imply that M should be a *mean*, i.e., the following inequality is satisfied for all $u,v \in [0, 1]$:

$$\min(u, v) \leq M(u, v) \leq \max(u, v).$$

The following simple result is easily obtained.

LEMMA 1 *For any idempotent M which satisfies* (14)–(16), *we have*

$$M(u, v) = v \quad \textit{for} \quad u \geq v, \quad u, v \in [0, 1].$$

Proof Any solution M is nondecreasing (see A5) and satisfies $M(1, v) = v$ [by (16)]. Therefore, we have for $u \geq v$ that

$$v = M(v, v) \leq M(u, v) \leq M(1, v) = v,$$

which proves the lemma. ■

In addition to properties P1–P3 in Proposition 1, the following one also holds for idempotent M.

PROPOSITION 2 *Suppose that hypotheses of Proposition* 1 *hold, M is assumed to be idempotent, and M is left-continuous in the first place. Then we have*

P4. *if* $A \equiv 1$ *and* hgt $A' \geq$ hgt B, *then* $B' = B$.

Proof P4 follows from the following equalities:

$$\begin{aligned} B'(y) &= \sup_x M(A'(x), J(A(x), B(y))) \\ &= \sup_x M(A'(x), B(y)) = M\Big(\sup_x A'(x), B(y)\Big) \\ &= B(y), \end{aligned}$$

where the left continuity of M in its first argument and Lemma 1 are used. ∎

Recall that in this section we want to find some particular idempotent solutions (M, J) of (14)–(16). By the property $M(u, 1) = 1$ for $u > 0$ [see (17)], natural candidates for M on the set

$$\{(u, v) | u \le v, u, v \in (0, 1]\}$$

are members of the family given by

$$m_\varphi(u, v) = \varphi^{-1}\big(1 - [1 - \varphi(u)]^\alpha [1 - \varphi(v)]^{1-\alpha}\big), \tag{18}$$

where φ is an automorphism of the unit interval and $\alpha \in [0, 1)$. Note that $\alpha = 1$ is impossible, since in that case we would have for $0 < u < 1$

$$m_\varphi(u, 1) = \varphi^{-1}(1 - [1 - \varphi(u)]) = u < v = 1,$$

a contradiction with (17).

Fortunately, the family defined by (18) is useful for determining a class of solutions for (14)–(16), as we prove now.

THEOREM 3 *For any automorphism φ of the unit interval, the functions M_φ and J_φ defined by*

$$M_\varphi(u, v) = \begin{cases} \varphi^{-1}\big(1 - [1 - \varphi(u)]^\alpha [1 - \varphi(v)]^{1-\alpha}\big) & \text{if } 0 < u \le v, \\ v & \text{if } u > v, \\ 0 & \text{if } u = 0, \end{cases}.$$

$$J_\varphi(u, v) = \begin{cases} \varphi^{-1}\left(1 - \left(\dfrac{1 - \varphi(v)}{[1 - \varphi(u)]^\alpha}\right)^{1/(1-\alpha)}\right) & \text{if } 0 < u \le v, \\ v & \text{if } u > v, \\ 1 & \text{if } u = 0 \end{cases}$$

with $0 \le \alpha < 1$ *are such that Equations* (14)–(16) *are satisfied by* (M_φ, J_φ). *Moreover, both* M_φ *and* J_φ *are idempotent.*

Proof Validity of (14):

$$M_\varphi\big(0, J_\varphi(u,v)\big) = 0, \qquad \text{since} \quad M_\varphi(0,x) = 0 \quad \text{for all} \quad x \in [0,1];$$

$$M_\varphi\big(1, J_\varphi(0,v)\big) = J_\varphi(0,v) = 1;$$

$$M_\varphi\big(u, J_\varphi(1,0)\big) = M_\varphi(u,0) = 0;$$

$$M_\varphi\big(1, J_\varphi(u,1)\big) = M_\varphi(1,1) = 1.$$

To prove (15), consider two cases.

Case 1: $u > v$. Then $J_\varphi(u,v) = v$, and thus we have

$$M_\varphi\big(u, J_\varphi(u,v)\big) = M_\varphi(u,v) = v,$$

by definition of M_φ.

Case 2: $0 < u \le v$. It is easy to prove that in this case $J_\varphi(u,v) \ge u$. Thus, by definition of M_φ and J_φ, we have

$$\begin{aligned} M_\varphi\big(u, J_\varphi(u,v)\big) &= \varphi^{-1}\Big(1 - [1-\varphi(u)]^\alpha \big[1 - \varphi\big(J_\varphi(u,v)\big)\big]^{1-\alpha}\Big) \\ &= \varphi^{-1}\left(1 - [1-\varphi(u)]^\alpha \left[\left(\frac{1-\varphi(v)}{[1-\varphi(u)]^\alpha}\right)^{1/(1-\alpha)}\right]^{1-\alpha}\right) \\ &= \varphi^{-1}\left(1 - [1-\varphi(u)]^\alpha \frac{1-\varphi(v)}{[1-\varphi(u)]^\alpha}\right) \\ &= v. \end{aligned}$$

Equation (16) follows by definition of M_φ and J_φ. Idempotency is obvious. ■

In [11] we suggested another type of idempotent solution satisfying all the assumptions and the system (14)–(16) as follows:

$$M(u,v) = \begin{cases} \dfrac{u}{u+1-v} & \text{if} \quad 0 < u \le v, \\ v & \text{if} \quad u > v, \\ 0 & \text{if} \quad u = 0, \end{cases} \tag{19}$$

$$J(u,v) = \begin{cases} 1 + u - u/v & \text{if} \quad u \le v, \quad v \ne 0, \\ v & \text{if} \quad u > v, \\ 1 & \text{if} \quad u = v = 0. \end{cases}$$

It is worth observing that for $u \leq v$ we have

$$\frac{u}{u+1-v} = \frac{\min(u,v)}{\min(1-u,1-v)+\min(u,v)}.$$

That is, the solution (19) is constructed on the basis of *symmetric sums* studied by Silvert [12]. It is easy to extend the formulas (19), (20) for M, J by using an automorphism φ of the unit interval.

THEOREM 4 *For any automorphism φ of the unit interval, the functions M_φ and J_φ defined by*

$$M_\varphi(u,v) = \begin{cases} \varphi^{-1}\left(\dfrac{\varphi(u)}{\varphi(u)+1-\varphi(v)}\right) & \textit{if } 0 < u \leq v, \\ v & \textit{if } u > v, \\ 0 & \textit{if } u = 0. \end{cases}$$

$$J_\varphi(u,v) = \begin{cases} \varphi^{-1}\left(1+\varphi(u)-\dfrac{\varphi(u)}{\varphi(v)}\right) & \textit{if } u \leq v, \quad v \neq 0, \\ v & \textit{if } u > v, \\ 1 & \textit{if } u = v = 0 \end{cases}$$

are such that equations (14)–(16) *are satisfied by* (M_φ, J_φ). *Moreover, both M_φ and J_φ are idempotent.*

Proof The proof can be carried out simply by checking the required properties. ■

Note that is any particular (M_φ, J_φ) defined either in Theorem 3 or in Theorem 4 is used in (10), then $A' \subset A$ implies $B' \subset B$, which is a stronger property than P1 in Proposition 1.

5. A CLASS OF NONIDEMPOTENT SOLUTIONS

In this section we look for appropriate new operations (both for conjunctions and implications) in the following form:

$$\frac{T(x,y)}{x},$$

where $x \in (0,1]$ and $y \in [0,1]$ and T is a t-norm. The choice of this form was motivated by a formula in [9]:

$$\max\left(\frac{x+y-1}{x}, 0\right).$$

Thus, assume that T is a t-norm. Define a new binary operation on $(0,1] \times [0,1]$ by

$$H(x,y) = \frac{T(x,y)}{x}. \tag{21}$$

The operation H has the following basic properties, for any t-norm T:

- $H(x,y) \in [0,1]$ for any $(x,y) \in (0,1] \times [0,1]$;
- H is nondecreasing with respect to its second argument, but in general, nothing can be said about the first one;
- $H(x,1) = 1$, $H(x,0) = 0$ for any $x \in (0,1]$;
- $H(1,y) = y$ for any $y \in [0,1]$.

We introduce an operation M by

$$M(x,y) := \begin{cases} H(x,y) & \text{if} \quad x > 0, \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

if H is nondecreasing with respect to its both arguments, and an operation J by

$$J(x,y) := \begin{cases} H(x,y) & \text{if} \quad x > 0, \\ 1 & \text{otherwise} \end{cases} \tag{23}$$

if H is nonincreasing with respect to its first argument and nondecreasing with respect to the second one. Then M is a fuzzy conjunction and J is a fuzzy implication in the broad sense of Section 2.

We can define S- and R-implications based on M in the usual way, using the standard strong negation $n(x) = 1 - x$:

$$J_S(x,y) = 1 - M(x, 1-y), \tag{24}$$

$$J_R(x,y) = \sup\{z \mid M(x,z) \le y\}, \tag{25}$$

and similarly S- and R-conjunctions based on J by

$$M_S(x,y) = 1 - J(x, 1-y), \tag{26}$$

$$M_R(x,y) = \inf\{z \mid J(x,z) \ge y\}. \tag{27}$$

Those continuous t-norms T for which (24) and (25) or respectively (26) and (27) coincide (see Figure 2) are characterized by Fodor and Kereszt-

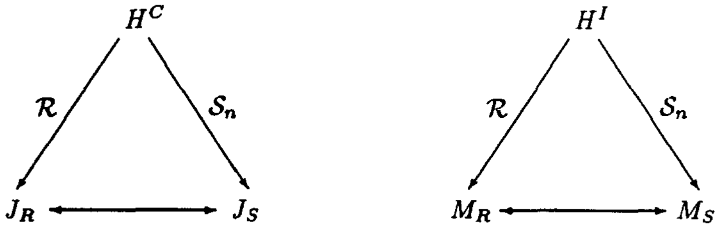

Figure 2. Coincidence of R- and S-transforms.

falvi [14] under an additional condition. By those results, the Hamacher family $\{T_\gamma\}_{\gamma \geq 1}$ (see [13] for details on this family) of t-norms defined by

$$T_\gamma(x, y) := \frac{xy}{\gamma + (1 - \gamma)(x + y - xy)} \tag{28}$$

is such that the functions

$$M_\gamma(u, v) = \begin{cases} \dfrac{v}{\gamma + (1 - \gamma)(u + v - uv)}, & u > 0, \\ 0, & u = 0, \end{cases} \qquad \gamma \geq 1,$$

are fuzzy conjunctions. In addition, for a given $\gamma \geq 1$, the R- and S-implications based on M_γ are the same, and their common expression is given as follows:

$$J_\gamma(u, v) = \begin{cases} \dfrac{\gamma v + (1 - \gamma)uv}{\gamma + (1 - \gamma)(1 - v + uv)} & \text{if } u > 0, \\ 1 & \text{otherwise}. \end{cases}$$

The proof of the following proposition is left to the reader.

PROPOSITION 3 *Each pair* $(M_\gamma, J_\gamma)_{\gamma \geq 1}$ *of fuzzy conjunctions and implications are solution of our system* (14)–(16), *satisfying also axioms* A1–A8.

6. CONCLUSION

In this paper we have investigated fuzzy conjunctions and implications from different points of view. By the results it became clear that one must be rather flexible in choosing connectives for particular reasons. In particular, noncommutative and nonassociative conjunctions and the corresponding implications given in Theorems 3 and 4, or by (19) and (20), can fulfil the expected properties better than t-norms and related implications. Therefore, we would like to encourage readers to use more advanced operators not only in theoretical problems but also in practice.

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