Module Structures on the $K$-Theory of Graded Rings

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Let $R$ be a commutative ring, $A = A_0 \oplus A_1 \oplus \cdots$ a graded $R$-algebra, and $A_+$ the graded ideal $A_1 \oplus A_2 \oplus \cdots$. Then $K_n(A) = K_n(A_0) \oplus K_n(A, A_+)$. We show that the groups $K_n(A, A_+)$ are naturally modules over the ring $W(R)$ of Witt vectors. They also have a natural filtration whose associated graded groups are $R$-modules. When $R$ contains a field of characteristic zero, $K_n(A, A_+)$ is an $R$-module, and the filtration is by $R$-submodules.

Although algebraic $K$-groups are a priori nothing more than abelian groups, much of our ability to perform calculations rests on module structures which can be imposed on large parts of $K$-theory. This viewpoint originated in 1971 with van der Kallen's observation in [vdK] that when $t$ is in a commutative ring $R$ we have $K_2(R)[E]/(E') = K_2(R) \oplus \Omega_R$, where $\Omega_R$ is the $R$-module of absolute Kahler differentials of $R$. Four years later, Bloch generalized this, showing that the relative groups $NK_1(R) = K_1(R[x], x)$ and $\lim_n K_n(R[E]/(E^n), E)$ were modules over the ring $W(R)$ of Witt vectors over $R$. (See [B1, B2, S2]; a summary is given in [W1].) In this note, we fit these phenomena into a more general context.

THEOREM 0.1. Let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded $R$-algebra, where $R$ is a commutative ring (concentrated in degree 0). If $A_+$ denotes the graded ideal $A_1 \oplus A_2 \oplus \cdots$, then there is a continuous $W(R)$-module structure on each group $K_n(A, A_+)$. This $W(R)$-module structure is natural on the category of graded $R$-algebras, and agrees with the known module structures for $A = A_0[x]$ and $A = A_0[E]/(E^n)$.

If $R$ contains $\mathbb{Q}$ (the rational numbers), then each $K_n(A, A_+)$ has a natural $R$-module structure via the ring map $\lambda: R \to W(R)$.

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The phrase "continuous $W(R)$-module" needs explanation. There is a descending filtration on $W(R)$ by ideals $I_n$, making $W(R)$ into a topological ring. For example, if $R$ contains $\mathbb{Q}$, then as a ring we have

$$W(R) \simeq \prod_{i=1}^{\infty} R, \quad \text{while} \quad I_m \simeq \prod_{i=m}^{\infty} R.$$ 

Note that $I_mI_n$ does not lie inside $I_{m+n}$.

Now let $M$ be a $W(R)$-module. We say that $M$ is a continuous module if the annihilator of every element of $M$ is open in $W(R)$, i.e., if $M$ is the union of the submodules $F^mM = \{m \in M | I_m m = 0 \}$. $M$ is separated if the intersection of the $I_mM$ is zero. I do not know whether or not the continuous $W(R)$-modules $K_t(A, A_+)$ are separated, although it is clear that the symbol part will be separated. However, if $R$ contains $\mathbb{Q}$, it is clear that every continuous $W(R)$-module is separated. (I am grateful to Wilberd van der Kallen for pointing out the need for care here.) From these general comments we deduce

**Corollary 0.2.** Suppose that $R$ contains $\mathbb{Q}$. Then the groups $I_nK_t(A, A_+)$ form a natural decreasing filtration on $K_t(A, A_+)$, whose intersection is zero. All groups involved are naturally $R$-modules, including the associated graded groups.

**Example 0.3.** Suppose that $A_+$ is nilpotent and commutative, so that $K_t(A, A_+)$ is the multiplicative group $1 + A_+$. The action of the element $(1 - rt^m)$ of $W(R)$ on $K_t(A, A_+)$ is given by the formula:

$$(1 - rt^m) \ast (1 - a) = (1 - r^{n/d}a^{m/d})^{d}, \quad a \in A_n, \quad d = \gcd(m, n).$$

The submodule $I_mK_t(A, A_+)$ is contained in $1 + A_{\geq m}$, $A_{\geq m} = A_m \oplus A_{m+1} \oplus \cdots$. If $R$ contains the rational numbers, the $R$-module structure is given by the formula:

$$r \ast (1 + a) = (1 + a^r) = 1 + ra + \frac{r(r - 1)}{2} a^2 + \cdots.$$ 

In fact, the map sending $a$ to $e^a$ is an $R$-module isomorphism between $A_+$ and $K_t(A, A_+)$.

As another example, let $A = \mathbb{C}[x_0, \ldots, x_n]/I$ be the homogeneous coordinate ring of a smooth curve $X$ embedded in complex projective $n$-space. Srinivas proved in [Sr] that when $H^1(X, O(1)) \neq 0$ the group $K_0(A, A_+)$ is an abelian group of uncountable rank. In fact, it is a vector space over $\mathbb{C}$.

A similar remark applies to the 2-dimensional normal domain
$A = C[x, y, z]/(x^2 + y^3 = z^2)$. We know that both $\tilde{K}_0(A) = K_0(A, A_+) \text{ and } K_{-1}(A) = K_{-1}(A, A_+)$ are nonzero. (This was originally due to Bloch and Murthy; see [Sr], [W3] and [Reid].) Hence $K_0(A)$ and $K_{-1}(A)$ are both nonzero vector spaces over $C$. As abelian groups, therefore, they are divisible of uncountable rank.

The outline of this paper is as follows. We define the action of $W(R)$ on $K_*(A, A_+)$ in Section 1 and prove that it is well defined in Section 2. In Section 3 we give another pairing, due to Bloch, and show that it agrees with our module structure. In Section 4 we establish some basic structural results for the module structure. We devote Section 5 to establishing formulas for the action on $K_2(A, A_+)$ when $A_+$ is nilpotent.

Throughout this paper, $R$ will denote a commutative ring, and $A = A_0 \oplus A_1 \oplus \cdots$ will denote a graded $R$-algebra. $R$ is to be concentrated in degree zero, and $A_+$ will denote the ideal $A_0 \oplus \cdots$ of $A$. The letters $a, b$ (resp. $q, r, s$) will always denote elements of $A$ (resp., of $R$), and the letters $t, x$ and $y$ will stand for indeterminates.

I would like to express my gratitude to Jan Stienstra and Wilberd van der Kallen for helpful conversations. In addition, I would like to point out that I presented the calculations in Section 5 in 1981 at the Topology Conference at the University of Western Ontario.

1. The Action of $W(R)$

The ring $W(R)$ of Witt vectors over $R$ has as its underlying additive group the group $1 + tR[[t]]$. This is a topological group, the subgroups $I_n = 1 + t^n R[[t]]$ forming a basic family of open neighborhoods of the identity. Every element of $W(R)$ has a unique convergent expansion $\omega(t) = \Pi(1 - r_m t^m)$. Using $*$ for the ring product, the ring structure on $W(R)$ is completely determined by the formula:

$$(1 - rt^m) * (1 - st^n) = (1 - r^{n/d}s^{m/d}t^{mn/d})^d, \quad d = \gcd(m, n). \quad (1.1)$$

We want to make $K_r(A, A_+)$ into a continuous $W(R)$-module in a natural way. It is enough to define natural maps $(1 - rt^m) * : K_r(A, A_+) \to K_{r-1}(A, A_+)$ for every $r \in R$ and $m \geq 1$, and then to verify the following.

**Axioms 1.2.** For every $v$ in $K_r(A, A_+)$:

(a) There is an $M \geq 0$ such that $(1 - rt^m) * v = 0$ for every $m \geq M$ and every $r$.

(b) Whenever $\Pi(1 - q_t t') \cdot \Pi(1 - r_m t^m) = \Pi(1 - s_n t^n)$ in the group $1 + R[[t]]$, then in $K_r(A, A_+)$:

$$\sum (1 - q_t t') * v + \sum (1 - r_m t^m) * v = \sum (1 - s_n t^n) * v.$$
(c) \((1-t) * v = v\). \((1-t)\) is the unit of the ring \(W(R)\).
(d) \([(1 - rt^m) \ast (1 - st^n)] \ast v = (1 - rt^m) \ast [(1 - st^n) \ast v]\).

To verify the Axioms (1.2) for every \(A\), it is enough to verify that (1.2) holds when \(A\) is the polynomial ring \(A_0[x]\) with \(x\) in degree one. To see this, let \(B\) denote the \(R\)-algebra \(A[x]\). Grade \(B\) by setting \(A\) in degree zero and \(x\) in degree one, so that the ring homomorphism \(\phi: A \to B\) which sends \(a_i\) in \(A_i\) to \(a_i x^i\) is a degree-preserving map. The induced map \(\phi^*: K(A, A_+) \to K(B, B_+)\) is an injection, because it is a summand of the map \(\phi^*: K(A) \to K(B)\), and this map is split by the nongraded map \(B \to A\) sending \(x\) to 1. If the Axioms (1.2) hold for \(K(B, B_+)\), then they must hold for \(K(A, A_+)\) as well.

In the remainder of this section, we define the map \((1 - rt^m) \ast \) on \(K(A, A_+)\). In the next section, we will verify the Axioms (1.2) for the special case \(A = A_0[y]\), proving that the \(K(A, A_+)\) are continuous \(W(R)\)-modules.

We will work with the category \(P(B)\) of finitely generated projective right \(B\)-modules. If \(F: P(B) \to P(C)\) is an additive functor, \(F(B)\) is a left \(B\)-module via the isomorphism \(B \cong \text{Hom}(B, B)\), and therefore a \(B-C\) bimodule, i.e., an object of \(B\text{-mod-}C\). The possibility of going back and forth between \(F\) and \(F(B)\) is made possible by the following elementary result, whose proof we omit (cf. [Bass, p. 57]).

**Lemma 1.3.** If \(B\) and \(C\) are rings, there is an equivalence of categories:

\[
\{\text{additive functors } P(B) \to P(C) \text{ and natural transformations}\} \cong \{ B-C \text{ bimodules in } P(C) \text{ and bimodule maps}\}.
\]

Under this equivalence, \(F\) corresponds to \(F(B)\) and the \(B-C\) bimodule \(P\) corresponds to the functor \(F_p(M) = M \otimes_B P\).

For the rest of this section, we fix \(r\) in \(R\) and an integer \(m \geq 1\). We want to define an additive functor \(F: P(A) \to P(A)\), and we do this by defining an \(A\)-bimodule \(P\). As a right module, \(P\) is free on basis \(\{e_0, \ldots, e_{m-1}\}\). For \(j \geq m\), we make the convention that \(e_j\) means \(e_{j - m} r\), and we define the left \(A\)-module structure by

\[a_i e_j = e_{i+j} a_i\quad \text{for } a_i \text{ in } A_i.\]

**Remark 1.4.** Here is another way to understand the functor \(F\). Set \(S = R[\tilde{s}] / (s^m - r)\), and let \(\sigma: A \otimes S \to A \otimes S\) be the graded \(S\)-algebra map sending \(a_i \otimes 1\) in \(A_i \otimes S\) to \(a_i \otimes s^i\). If \(j: A \to A \otimes S\) denotes the inclusion, then \(F\) is the functor

\[
P(A) \xrightarrow{j^*} P(A \otimes S) \xrightarrow{\sigma^*} P(A \otimes S) \xrightarrow{j_*} P(A).
\]
In fact, $P = F(A) = j \ast \sigma \ast j \ast A$ is just $A \otimes S$, with $e_i$ in $P$ corresponding to $1 \otimes s'$ in $A \otimes S$. For example, when $m = 1$ the ring map $\sigma: A \rightarrow A$ is $\sigma(a_r) = a_r r^i$ and $F$ is the base-change map $\sigma^*$. If $m = 1$ and $r = 1$, $F$ is the identity map. If $m = 1$ and $r = 0$, $F$ is $i \ast p^*$, where

$$p: A \rightarrow A/A_+ = A_0 \quad \text{and} \quad i: A_0 \rightarrow A$$

induce the functors

$$p^*: \mathbf{P}(A) \rightarrow \mathbf{P}(A_0) \quad \text{and} \quad i^*: \mathbf{P}(A_0) \rightarrow \mathbf{P}(A).$$

On $K$-theory, the functors $F$, $p^*$, and $i^*$ induce maps which we abusively write as $K_i F: K_i(A) \rightarrow K_i(A)$, $p_*^*: K_i(A) \rightarrow K_i(A_0)$, and $i^*: K_i(A_0) \rightarrow K_i(A)$. Since $p: A_0 \rightarrow A_0$ is the identity, we obtain a direct sum decomposition $K_i(A) = K_i(A_0) \oplus K_i(A, A_+)$. 

**Lemma/Definition 1.5.** The induced functor $K_i F: K_i(A) \rightarrow K_i(A)$ respects the direct sum decomposition $K_i(A) = K_i(A_0) \oplus K_i(A, A_+)$ and is multiplication by $m$ on the summand $K_i(A_0)$. The map $(1 - rtm)^*: K_i(A, A_+) \rightarrow K_i(A, A_+)$ is defined to be the restriction of $K_i F$ to the summand $K_i(A, A_+)$. 

**Proof.** Let $F_0: \mathbf{P}(A_0) \rightarrow \mathbf{P}(A_0)$ by $F_0(M) = M \otimes A_0^m = M \oplus \cdots \oplus M$. Since $P \cong A^m$ as left $A_0$-modules we have $F_i = i F_0$. Since $(A/A_+) \otimes_A P \cong P \otimes_A (A/A_+) \cong A_0^m$ as $A_0$-bimodules, we have $F_0 p \cong p F$. This implies that $K_i F$ respects the decomposition of $K_i(A)$, and is $K_i F_0$ on $K_i(A_0)$. The fact that $K_i F_0$ is multiplication by $m$ is standard.

Before moving on, we should clear up an apparent notational problem, namely the case $r = 0$. For clarity, let us write $P_m$ and $F_m$ for the $A$-bimodule and functor constructed for $r = 0$, and our chosen integer $m$. 

**Lemma 1.6.** If $r = 0$, the map $(1 - 0tm)^*: K_i(A, A_+) \rightarrow K_i(A, A_+)$ induced from $F_m$ is the zero map for all $m$. 

**Proof.** We have already observed that $F_1 = i \ast p^*$, so the case $m = 1$ follows from Lemma 1.5. Inductively, note that the subbimodule $e_m A$ of $P_{m+1}$ is isomorphic to $P_1$, and that the quotient bimodule is $P_m$. This yields a short exact sequence of functors $\mathbf{P}(A) \rightarrow \mathbf{P}(A)$,

$$0 \rightarrow F_1 \rightarrow F_{m+1} \rightarrow F_m \rightarrow 0.$$ 

By the additivity theorem [Q, p. 106], $K_i F_{m+1} = K_i F_m + K_i F_1$. Hence we have $(1 - 0tm + 1)^* = (1 - 0tm)^* + (1 - 0t)^* = 0.$
2. The Case $A = A_0[x]$.

In this section, $A$ will denote the polynomial ring $A_0[x]$ with $x$ in degree one, and we will write $NK_i(A_0)$ for $K_i(A_0[x], x)$. It is a result of Bloch and Stienstra that the groups $NK_i(A_0)$ are continuous $W(R)$-modules; in this section we shall write $\omega \circ v$ for the Bloch-Stienstra product of $\omega \in W(R)$ and $v \in NK_i(A_0)$. We will show that the map $\left(1 - rt^m\right)^*$ of the last section produces the same endomorphism of $NK_i(A_0)$ as the Bloch-Stienstra map $\left(1 - rt^m\right) \circ$. This will prove that the maps $\left(1 - rt^m\right)^*$ satisfy the axioms (1.2) for every graded $R$-algebra $A$, since the $\left(1 - rt^m\right) \circ$ satisfy (1.2) for $A = A_0[x]$. These axioms imply that the $K_i(A, A_+)$ are naturally continuous $W(R)$-modules.

Under the Bloch-Stienstra module structure on $NK_i(A_0)$, multiplication by $\left(1 - rt^m\right)$ is induced from the functor

$$P(A_0[x]) \xrightarrow{i^*} P(A_0[y]) \xrightarrow{\rho^*} P(A_0[y]) \xrightarrow{i^*} P(A_0[x]),$$

where $i : A_0[y] \to A_0[x]$ and $\rho : A_0[y] \to A_0[y]$ are the $A_0$-algebra maps given by $i(y) = x^m$, $\rho(y) = ry$. (The fact that this yields a $W(R)$-module structure on $NK_i(A_0)$ is asserted on p. 316 of [B2] and proven in [S2]. A discussion may be found in [W1].)

By 1.3, the functor $i^* \rho^* i_*$ is determined by the $A$-bimodule $Q = i^* \rho^* i_*(A)$. As a right $A$-module, $Q$ is free on basis $\{f_0, \ldots, f_{m-1}\}$. Making the convention that $f_{j+m}$ means $f_j(rx^m)$, the left $A$-module structure on $Q$ is given by the formula:

$$(a_0 x^i) f_j = f_{j+i} a_0 \quad \text{for } a_0 \text{ in } A_0.$$

Fixing $m \geq 1$, let us write $P_r$ and $Q_r$ for the $A$-bimodules corresponding to $\left(1 - rt^m\right)^*$ and $\left(1 - rt^m\right) \circ$, respectively. Write $F_r$ and $G_r$ for the respective functors $P(A) \to P(A)$ they induce. With respect to the bases $\{e_0, \ldots, e_{m-1}\}$ and $\{f_0, \ldots, f_{m-1}\}$ of $P_r$ and $Q_r$, left multiplication by $x$ is represented by the respective matrices

$$\begin{pmatrix}
0 & rx \\
x & 0 & \ddots & \vdots \\
& \ddots & \ddots & 0 \\
& & x & 0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
0 & rx^m \\
& & & \\
& & 1 & 0 \\
& & & & 0
\end{pmatrix}.$$

Define the right module map $\eta_A : P_r \to Q_r$ by the formula $\eta_A(e_j) = f_j x^{m-1-j}$, $0 \leq j \leq m-1$. For example, $\eta_A(e_0) = f_0 x^{m-1}$ and $\eta_A(e_{m-1}) = f_{m-1}$. Since $x \eta_A(e_j) = \eta_A(x e_j)$ for all $j$, it follows that $\eta_A$ is a left $A$-module map as well. By 1.3, $\eta_A$ induces a natural transformation $\eta : F_r \to G_r$. 
PROPOSITION 2.1. The two functors \( F_r = j^*_\sigma^* j^* \) and \( G_r = i^* \rho^* i_* \) induce the same maps \( K_r(A) \to K_r(A) \) and \( NK_r(A_0) \to NK_r(A_0) \).

Proof. Let \( C_r \) be the cokernel of the injection \( \eta_A : P_r \to Q_r \). As a right \( A \)-module, it has finite homological dimension. The functor \( H_r(M) = M \otimes_A C_r \) maps \( P(A) \) to the exact category \( H(A) \) of finitely generated right \( A \)-modules with finite homological dimension. There is a short exact sequence

\[
0 \to F_r \to G_r \to H_r \to 0
\]

of exact functors from \( P(A) \) to \( H(A) \). By the additivity theorem [Q, p. 106],

\[
K_r H_r = K_r G_r - K_r F_r : K_r(A) \to K_r(H(A)) = K_r(A).
\]

However, it is easy to see that the \( A \)-bimodule \( C_r \) is independent of the choice of \( r \). Taking \( r = 0 \), Lemma 1.6 yields the desired equation,

\[
K_r H_r = K_r H_0 = K_r G_0 - K_r F_0 = 0 - 0 = 0.
\]

COROLLARY 2.2. The \( W(R) \)-module structure on \( NK_r(A_0) \) given in Section 1 agrees with the Bloch–Stienstra module structure.

3. BLOCH’S PAIRING

There is another way to define a \( W(R) \)-module structure on the groups \( K_r(A, A_{+}) \), implicitly due to Bloch [B2, p. 315]. Bloch begins with the biexact functor

\[
\text{P}(R[t]) \otimes \text{Nil}(A) \to \text{P}(A), \quad \text{M} \otimes (N, v) = \text{M} \otimes_{R[t]} N,
\]

where \( N \) is considered to be a left \( R[t] \)-module with \( t \) acting via the nilpotent endomorphism \( v \). This produces a map from \( K_q(R[t]) \otimes K_p \text{Nil}(A) \) to \( K_{p+q}(A) \). Identifying \( W(R) \) with \( K_1(R[t], t) \) and \( K_{p+1}(A[x], x) \) with the kernel of the forgetful map \( K_p \text{Nil}(A) \to K_p(A) \) we obtain a pairing

\[
W(R) \otimes K_{p+1}(A[x], x) \to K_{p+1}(A).
\]

Now suppose that \( A \) is graded. Using the injection \( \phi^* : K_r(A, A_{+}) \to K_r(A[x], x) \) of Section 1, we obtain pairings for each \( i \geq 1 \),

\[
W(R) \otimes K_r(A, A_{+}) \to K_r(A, A_{+}).
\]

This is Bloch’s pairing.
PROPOSITION 3.1. Bloch's pairing agrees with the action of $W(R)$ on $K_*(A, A_+)$ that we defined above. In particular, Bloch's pairing makes $K_*(A, A_+)$ into a continuous $W(R)$-module for every graded $R$-algebra $A$.

Proof. By the trick of section 1 involving $\phi: A \to A[x]$, it is enough to prove the result for $A = A_0[x]$. In this case, Stienstra showed in [S2, (9.23)] that Bloch's pairing agrees with the Bloch–Stienstra pairing we cited in Section 2. We are done by Corollary 2.2.

Remark 3.2. Bloch defined his pairing only for $A = R[x]$ and $R[\varepsilon]/(\varepsilon^n)$, but the construction in [B1, B2] extends word for word to graded $A$. In [B1, B2], Bloch asserted, but did not prove, that his pairing made these rings into $W(R)$-modules. In [B1, (II.2.1.4)], Bloch proved that $\lim_n K_i(R[\varepsilon]/(\varepsilon^n), \varepsilon)$ was a $W(R)$-module by verifying axioms (1.2). Using the presentation for $K_2$ of a radical ideal, Stienstra showed in [S0] that the $K_2(R[\varepsilon]/\varepsilon^n, \varepsilon)$ were $W(R)$-modules. We can now see that Bloch's assertion was correct.

4. STRUCTURAL RESULTS

In this section, we collect several results that are useful in calculations. First note that the group $K_*(A, A_+)$ is a graded module over the graded ring $K_*(R)$. We have

PROPOSITION 4.1 (Product formula). For $\gamma \in K_m(R)$, $\nu \in K_n(A, A_+)$, and $\omega(t) \in W(R)$ we have the formula in $K_{m+n}(A, A_+)$,

$$\omega(t) \ast \{\gamma, \nu\} = \{\gamma, \omega(t) \ast \nu\}.$$ 

Proof. By additivity, we can assume that $\omega(t)$ is $1 - rt^n$. Since the $K_*(R)$-module structure arises from the biexact pairing $\otimes: P(R) \times P(A) \to P(A)$, sending $(L, M)$ to $L \otimes_R M$, the product formula follows from the equation $(L \otimes_R M) \otimes_A P \cong L \otimes_R (M \otimes_A P)$, i.e., from commutativity up to natural isomorphism of the diagram:

$$
\begin{array}{ccc}
P(R) \times P(A) & \xrightarrow{1 \times (\otimes_A P)} & P(R) \times P(A) \\
\otimes & & \otimes \\
P(A) & \xrightarrow{\otimes_A P} & P(A).
\end{array}
$$

Next, we consider the effect of changing the grading on $A$. For our pur-
poses, a grading on $A$ is a decomposition $A = \Pi A_i$; we say that $A$ is regraded by a factor of $n$ if we give it the decomposition $A = \Pi B_j$, where

$$B_j = \begin{cases} A_i & \text{if } j = ni, \\ 0 & \text{if } j \equiv 0 \pmod{n}. \end{cases}$$

**Proposition (4.2) (Change of grading).** If $A$ is regraded by a factor of $m$, the resulting $W(R)$-module structure $*'$ on $K_i(A, A_\pm)$ is the pullback of the original $W(R)$-module structure $*$ along the Frobenius ring map $F_m: W(R) \to W(R)$. That is, we have

$$\omega *' v = (F_m \omega) * v \quad \text{for } \omega \in W(R) \text{ and } v \in K_i(A, A_\pm).$$

**Proof.** For clarity, let us write $B$ for the graded ring $A = \bigoplus B_i$. We grade $A[y]$ and $B[x]$ with $A, B$ in degree 0 and $x, y$ in degree 1, and let $\phi$ be the graded map of Section 1. Finally, let $i$ be the ungraded $A$-algebra map $A[y] \to B[x]$ given by $i(y) = x^m$. We have a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\phi} & A[y] \\
\downarrow & & \downarrow i \\
B & \xrightarrow{\phi} & B[x].
\end{array}
$$

The induced map $i^*: K_i(A[y], y) \to K_i(B[x], x)$ is called $V_m$ in [B2] and [S2], where they show that $V_m((F_m \omega) * v) = \omega * (V_m v)$ for every $\omega \in W(R)$ and $v \in K_i(A[y], y)$. (Beware the typo in (2.7.1) of [B2].) This establishes the commutativity of the right-hand face in the following cube:
The front and back faces commute by naturality. The top and bottom faces commute by the above discussion, the $\phi^*$ being split injections. Thus the left-hand face also commutes, which was to be shown.

**Remark 4.2.1.** If $R$ contains the rational numbers, the injection $\lambda_i : R \to W(R)$ is invariant under the Frobenius, i.e., $F_n(\lambda_i(r)) = \lambda_i(r)$. Hence regrading $A$ does not affect the $R$-module structure on $K_i(A, A_+)$. It does change the filtration on $K_i(A, A_+)$, however, as can be seen in Example 0.3.

**Proposition 4.3 (Morita invariance).** If $A$ is Morita equivalent over $R$ to $B$, then the natural isomorphism $NK_i(A) \cong NK_i(B)$ is an isomorphism of $W(R)$-modules.

**Proof.** If $L$ is an $R[\pi]$-bimodule and $M$ is an $A-B$ bimodule, then there is an isomorphism $L \otimes_R M \cong M \otimes_R L$ of $A[x] - B[x]$ bimodules. For example, the Morita equivalence $A \approx B$ of $R$-algebras is induced by a functor $\otimes N: \mod-A \to \mod-B$, where $N$ is an $A-B$ bimodule, and the $A[x] - B[x]$ bimodule $R[x] \otimes_R N \cong N \otimes_R R[x]$ induces a Morita equivalence $A[x] \approx B[x]$. On the other hand, if $P$ is the $R[x]$-bimodule such that $P$ induces $(1 - r\iota^n)*$ on $NK_i(R)$, then $P \otimes_R A \approx A \otimes_R P$ induces $(1 - r\iota^n)*$ on $NK_i(A)$, and similarly for $P \otimes_R B$. There is an $A[x] - B[x]$ bimodule isomorphism

\[
(P \otimes_R A) \otimes_{A[x]} (R[x] \otimes_R N) \cong P \otimes_R N \cong N \otimes_R P \cong (N \otimes_R R[x]) \otimes_{B[x]} (B \otimes_R P).
\]

This establishes commutativity up to natural transformation of

\[
P(A[x]) \xrightarrow{N \otimes_R R[x]} \to P(B[x]) \quad \text{and} \quad P(A[x]) \xrightarrow{R[x] \otimes N} P(B[x]).
\]

On the $K$-theory level, this implies that the Morita isomorphism $NK_i(A) \cong NK_i(B)$ commutes with $(1 - r\iota^n)*$, whence the result.

If we want to discuss Morita invariance of graded $R$-algebras, we have to discuss graded Morita equivalences. Rather than pursue this tangential issue, we content ourselves with a special case. If $A$ is a graded $R$-algebra, so is the matrix ring $M_n(A) = M_n(A_0) \oplus M_n(A_1) \oplus \cdots$, and the corresponding isomorphisms $K_i(A) \cong K_i(M_n(A))$ and $K_i(A_0) \cong K_i(M_n(A_0))$ induce a natural isomorphism $K_i(A, A_+) \cong K_i(M_n(A), M_n(A_+))$. 
Corollary (4.4). The natural isomorphism $K_0(A, A_+)$ is an isomorphism of $W(R)$-modules.

Proof. The maps $A \to \phi A[x] \to M_n(A[x])$ and $A \to M_n(A) \to A_{\phi}$ agree, so the Morita isomorphism $K_0(A[x]) \cong K_0(M_n(A[x]))$ sends the summands $\phi K(A)$ and $\phi K(M_nA)$ to $\phi K(M_n(A[x])$. Hence it sends $\phi K(A, A_+)$ to $\phi K(M_nA, M_nA_+)$, so we can deduce this result from 4.3.

5. Examples

In this section, we give some formulas to illustrate the $W(R)$-module structure. The action of $W(R)$ on $K_0(A, A_+)$ is completely determined by the action of the Witt vectors $(1 - \tau^m)$, so we concentrate on their effect.

The action on $K_0(A, A_+)$ is clear from the construction in Section 1: if $M$ is a projective $A$-module with $M/A_+M \cong (A/n)^n$, then $[M]/n$ is an element of $K_0(A, A_+)$ and $(1 - \tau^m) *[M]/n = [M \otimes_A P]/n = mn$.

The action on $K_1(A, A_+)$ is more complicated, but can be written down directly from the left action of $GL_n(A)$ on $A^n \otimes_A P \cong A^{mn}$. Here is one special case:

Lemma 5.1. If $v$ is a nilpotent $n \times n$ matrix with entries in $A$, $(i \neq 0)$, the action of $W(R)$ on the corresponding element $(1 - v)$ of $K_1(A, A_+)$ is given by $(1 - \tau^m) * (1 - v) = (1 - r^{\gcd(m, i)})$, $d = \gcd(m, i)$.

Proof. Using the embedding $\phi^*$ of $K_1(A, A_+)$ in $K_1(A[x], x)$, we can assume that $A = A_0[x]$ and $v = \alpha x^i$ for $\alpha$ a nilpotent matrix with entries in $A_0$. By Morita invariance, we can replace $A_0$ by $M_\phi(A_0)$ to assume $\alpha$ is in $A_0$. Replacing $R$ and $A_0$ by $R[x]$, we can assume that $A = R[x]$. But the formula is well-known in this case (see, e.g., [Bl, II.2.3]).

Next, consider the case in which $R$ contains the rational numbers. In this case, there is a ring map $\lambda_\phi: R \to W(R)$ sending $r$ to $(1 - t)^\gamma = \sum (-t)^\gamma$. The abelian groups $K_0(A, A_+)$ become $R$-modules in this way. Our next result describes the $R$-module structure on $K_1(A, A_+)$ when $A_+$ is nilpotent.

Proposition 5.2. Suppose that $A$ contains the rational numbers and that $A_+$ is nilpotent. Then every element of $K_1(A, A_+)$ is represented by a unit $(1 - f)$, $f \in A_+$, and the $R$-module structure is given by $r * (1 - f) = (1 - f^\gamma)$. If we let $[A, A_+]$ denote the subgroup of $A$ generated by all $af - fa$, $a \in A$ and $f \in A_+$, then there is an $R$-module isomorphism

$$\exp: A_+/[A, A_+] \to K_1(A, A_+)$$
Proof. It is well known that every element of $K_1(A, A_+)$ is represented by a unit $1 - f$ of $A$ with $f$ in $A_+$. In the Appendix, we show that $\ln$ and $\exp$ induce an isomorphism of $A_+/[A, A_+]$ with $K_1(A, A_+)$, so if $(1 - f) = \Pi(1 - f_i)$ then $(1 - f)'$ and $\Pi(1 - f_i)'$ represent the same element of $K_1(A, A_+)$. Therefore, to see that the module structure is given by $r \cdot (1 - f) = (1 - f)'$, we can factor $1 - f$ into terms $(1 - f_i)$ with $f_i$ in $A_i$. Replacing $A$ by $R[f_i]$, we can assume $A$ commutative. Via the $R$-module injection $\phi^*: K_1(A, A_+) \to K_1(A[x], x)$ we can assume $A = R[x]$. The result in this special case is well-known (see, e.g., [S0, II.5.10; B1, II.3.5; W1, p. 4803]).

Example 5.3. Let $A = R[\varepsilon, x]/(\varepsilon^n)$, where for convenience $R$ is a field. There are several ways to grade $A$, and each gives a different $W(R)$-module structure on $NK_*(R[\varepsilon], \varepsilon)$. To illustrate this, consider the product $(1 - r\varepsilon^m) \cdot (1 - r\varepsilon^n)$. Set $v = \varepsilon x^i$, $d = \gcd(m, i)$ and $e = \gcd(m, j)$. When $A_0 = R[x]$ and deg$(\varepsilon) = 1$, the product is $(1 - r^i\varepsilon^m/d)^d$, when $A_0 = R[\varepsilon]$ and deg$(x) = 1$, the product is $(1 - r^i\varepsilon^m/e)^e$. This clarifies the remark on p. 480 of [W1] that there are different $W(R)$-module structures on $NK_*(R[\varepsilon], \varepsilon)$. In fact, they arise from different gradings of $A = R[\varepsilon, x]$.

We now turn to the action of $W(R)$ on the relative $K_2$ group. We will assume that $A_+$ is a nilpotent ideal and that $A$ is commutative, so that we know that $K_2(A, A_+)$ is additively generated by symbols $\langle a, s \rangle$ and $\langle a, b \rangle$, where $s \in A_0$, $a \in A_1$, and $b \in A_j$ ($i, j \neq 0$). First, we describe the $R$-module structure in characteristic zero. To do this, we shall adopt the convention that the expression $(1 - (1 - ax)^r)/x$ means the polynomial

$$a \sum_{k=0}^{\infty} \frac{r}{k+1} (-ax)^k = ra - \left(\frac{r}{2}\right) a^2 x + \left(\frac{r}{3}\right) a^3 x^2 - \cdots.$$ 

Proposition 5.4. Suppose that $A$ is commutative, that $R$ contains the rational numbers, and that $A_+$ is nilpotent. Then the $R$-module structure on $K_2(A, A_+)$ is given by the formulas ($s \in A_0$, $a \in A_1$, and $b \in A_j$ ($i, j \neq 0$)),

$$r \cdot \langle a, s \rangle = \left\langle \frac{1 - (1 - as)^r}{s}, s \right\rangle$$
$$r \cdot \langle a, b \rangle = \left(\frac{i}{i+j}\right) \left\langle \frac{1 - (1 - ab)^r}{b}, b \right\rangle + \left(\frac{j}{i+j}\right) \left\langle \frac{a, 1 - (1 - ab)^r}{a} \right\rangle$$

$$= \left\langle \frac{1 - (1 - ab)^r}{b}, b \right\rangle + \left(\frac{j}{i+j}\right) \left\langle ab, \frac{1 - (1 - ab)^r}{ab} \right\rangle$$

Proof. To compute $r \cdot \langle a, s \rangle$ we can reduce to the generic case $R = Q[r, s]$, $A = R[a]/(a^n)$. For this $A$, $K_2(A, A_+)$ embeds in $K_2(A[s^{-1}])$,
\( A_+ [s^{-1}] \), so we can assume that \( s \) is a unit of \( R \). But then the product formula yields
\[
 r \ast (a, s) = r \ast \{1 - as, s\} = \{r \ast (1 - as), s\} = \{(1 - as)^r, s\}
\]
\[
= \left( \frac{1 - (1 - as)^r}{s} \right), s\).
\]
To compute \( r \ast \langle a, b \rangle \), we apply \( \phi^* \) to get
\[
r \ast \langle ax^i, bx^j \rangle = r \ast (\langle abx^i, x^j \rangle + \langle ax^{i+j}, b \rangle)
\]
\[
= \left( \frac{j}{i+j} \right) r \ast \langle ab, x^{i+j} \rangle + r \ast \langle ax^{i+j}, b \rangle
\]
\[
= \left( \frac{j}{i+j} \right) \langle ab, (1 - abx^{i+j})^r \rangle + \left( \frac{(1 - abx^{i+j})^r}{b}, b \right)
\]
\[
= \left( \frac{j}{i+j} \right) \langle a, \frac{(1 - abx^{i+j})^r}{a} \rangle + \left( \frac{i}{i+j} \right) \left( \frac{(1 - abx^{i+j})^r}{b}, b \right)
\]
\[
= \left( \frac{j}{i+j} \right) \langle ax^i, \frac{(1 - abx^{i+j})^r}{ax^i} \rangle + \left( \frac{i}{i+j} \right) \left( \frac{(1 - abx^{i+j})^r}{bx^j}, bx^j \right)
\]
\[
- \left( \frac{j}{i+j} \right) \left( \frac{(1 - abx^{i+j})^r}{x^i}, x^j \right) - \left( \frac{i}{i+j} \right) \left( \frac{(1 - abx^{i+j})^r}{x^i}, x^j \right)
\]
\[
= \left( \frac{j}{i+j} \right) \phi^* \left( \langle a, (1 - ab)^r \rangle \right) + \left( \frac{i}{i+j} \right) \phi^* \left( \langle \frac{1 - ab)^r}{b}, b \rangle \right)
\]
Since \( \phi^* \) is an injection, we deduce the formula for \( r \ast \langle a, b \rangle \).

Here are the general formulas for the module structure on \( K_2 \) when \( A_+ \) is nilpotent:

**Proposition 5.5.** Let \( A \) be a commutative graded \( R \)-algebra with \( A_0 = R \) and \( A_+ \) nilpotent. The \( W(R) \)-module structure on \( K_2(A, A_+) \) is completely determined by the formulas:

(a) \( (1 - r^m) \ast \langle a, s \rangle = d \langle a^{m/d}, r^{i/d}, s^{m/d - 1}, s \rangle \), where \( r, s \in R, a \in A, \) and \( d = \gcd(m, i) \)

(b) \( (1 - r^m) \ast \langle a, b \rangle = (um + iv) \langle a^k b^{k-1} r^n, b \rangle - ju \langle a^{k-1} b^k r^n, a \rangle + jv \langle (ab)^k r^n - 1, r \rangle + j(d - 1) \langle -(ab)^k r^n, -1 \rangle \), where \( a \in A_+, b \in A_+, r \in R, d = \gcd(i + j, m), k = m/d, n = (i + j)/d \) and \( u \) and \( v \) are integers such that \( d = um + v(i + j) \).

**Proof.** We compute as in \([W1, (4.4)]\), using the formulas (and symbols) on p. 62 of \([S0]\) (which may be derived from Sect. 2 of the published
version [S1]). These formulas give the \( W(R) \)-module structure on
\( K_2(A[x], x) \). One of these formulas is
\[
(1 - rt^m) * \langle ax^i, s \rangle = d \langle a^{m/d} r^{i/d} s^{m/d - 1} x^{im/d}, s \rangle, \quad d = \gcd(m, i).
\]
Now this is just \( \phi * \) applied to formula (a), so 5.5(a) holds in \( K_2(A, A_+) \).
The second formula from [S0] is (in the notation of (b)),
\[
(1 - rt^m) * \langle cx^{i+j-1}, x \rangle
= um \langle c^k r^nx^{mn-1}, x \rangle + u \langle c_k r^{n-1} x^{mn}, r \rangle
- v \langle c_k r^{n-1} x^{mn}, c \rangle + (d - 1) \langle -c_k r^n x^{mn}, -1 \rangle
= um \langle cx^{nd-1}, x \rangle + u \langle c_k r^{n-1} x^{mn}, r \rangle
- v \langle cx^{nd}, c \rangle + (d - 1) \langle -cx^{nd}, -1 \rangle,
\]
where \( z = r^n(cx^{nd})^{k-1} \). Now \( \phi^*(\langle a, b \rangle) = \langle ax^i, bx^j \rangle = j(\langle abx^{i+j-1}, x \rangle + \langle ax^{i+j}, b \rangle) \). Set \( c = ab \), so that \( c^k x^{mn} = \phi(c^k) \) and \( z = \phi(r^n c^{k-1}) \). The two cited formulas yield in \( K_2(A[x], x) \) that
\[
(1 - rt^m) * \langle ax^{i+j}, b \rangle = d \langle a^{k} r^{n} b^{k-1} x^{mn}, b \rangle = d \langle ax^{nd}, b \rangle;
(1 - rt^m) * \langle ax^i, bx^j \rangle = jum \langle czx^{nd-1}, x \rangle + \phi^*(\beta)
- jv \langle zx^{nd}, c \rangle + d \langle azx^{nd}, b \rangle,
\]
where \( \beta = jum \langle c_k r^{n-1}, r \rangle + j(d - 1) \langle -c_k r^n, -1 \rangle \). Since \( j \langle czx^{nd-1}, x \rangle = \langle czx', x' \rangle \) and \( iv = -um + d - jv \), we obtain
\[
(1 - rt^m) * \langle ax^i, bx^j \rangle - \phi^*(\beta)
= um(\langle azx', bx^j \rangle - \langle azx^{i+j}, b \rangle) + d \langle azx^{i+j}, b \rangle
- jv(\langle azx^{i+j}, b \rangle + \langle bx^{i+j}, a \rangle)
= um \langle azx', bx^j \rangle + iv \langle azx^{i+j}, b \rangle - jv \langle bx^{i+j}, a \rangle.
\]
The result now follows from the observation that
\[
\phi^*(i \langle a^k b^{k-1} r^n, b \rangle - j \langle a^{k-1} b^k r^n, a \rangle)
= i \langle azx', bx^j \rangle - j \langle bx^j, ax^i \rangle
= i \langle azx^{i+j}, b \rangle - j \langle bx^{i+j}, a \rangle
\]
because
\[
i \langle abzx^i, x' \rangle = ij \langle abzx^{i+j}, x' \rangle = j \langle abzx', x' \rangle.
\]
The complicated formula for \((1 - r t^m) \ast \langle a, b \rangle\) simplifies quite a bit when \(m\) and \((i + j)\) are units of \(R\), for then we can divide by these elements in \(K_2(A, A_+)\). This is because \(m\) and \((i + j)\) are units in the ring \(W(R)\). First, note that \(j(d - 1) \langle - (ab)^k r^n, -1 \rangle = 0\), because either \((d - 1)\) is even or else \(1/2\) is in \(A\). Second, note that \((1/k) \langle a, b^k \rangle = \langle ab^{k-1}, b \rangle\) for \(a\) or \(b\) in \(A_+\). Thus formula (5.5)(b) becomes

\[
(1 - r t^m) \ast \langle a, b \rangle
= d \langle a^k b^{k-1} r^n, b \rangle - j v \langle (ab)^k r^n, ab \rangle + j u \langle (ab)^k r^{n-1}, r \rangle
= (d/k) \langle a^k r^n, b^k \rangle - (jv/k) \langle r^n, (ab)^k \rangle + (ju/n) \langle (ab)^k, r^n \rangle
= (d/k) \langle a^k r^n, b^k \rangle + (j/nk) \langle (ab)^k, r^n \rangle
= (d/k - j/nk) \langle a^k r^n, b^k \rangle + (j/nk) \langle a^k, r^n b^k \rangle.
\]

In summary, we have derived the

**Simplification 5.6.** When \(m\) and \(i + j\) are units in \(R\) we have the simpler formula

\[
(1 - r t^m) \ast \langle a, b \rangle = \left( \frac{i}{nk} \right) \langle a^k r^n, b^k \rangle + \left( \frac{j}{nk} \right) \langle a^k, r^n b^k \rangle
\]

\((a \in A_1, b \in A_j, r \in R, d = \gcd(i + j, m), k = m/d, \text{ and } n = (i + j)/d)\).

We conclude with an application of these ideas to the paper [vdK-S] of Stienstra and van der Kallen. Let \(A = R[y_1, ..., y_r, ..., y_s]/I\), where \(I\) is an ideal generated by monomials of \(R[y_1, ..., y_r]\) and containing some power of each of \(\{y_1, ..., y_r\}\). We grade \(A\) by putting \(A_0 = R[y_{r+1}, ..., y_s]\) and letting \(y_1, ..., y_r\) belong to \(A_1\). For \(x = (x_1, ..., x_s)\) an \(s\)-tuple of nonnegative integers, write \(y^x\) for \(\Pi y_i^{x_i}\). If \(y^x\) belongs to \(I\) and \(x_i \neq 0\), Stienstra and van der Kallen define group maps

\[
\Gamma_{x,i}: (1 + x R[x])^* \to K_2(A, A_+),
\]

\[
\Gamma_{x,i}(1 - xf(x)) = \langle f(y^x)(y^x/y_i), y_i \rangle,
\]

and use these maps to completely describe \(K_2(A, A_+)\) when \(R\) is a perfect field of characteristic \(p\). (See [vdK-S, (2.6)].) Our observation is

**Theorem 5.7.** Given \(x\), let \(e = \deg(y^x) = x_1 + \cdots + x_s\), and identify \((1 + x R[x])^*\) with the ideal \(V_{\ast} W(R)\) of \(W(R)\) via \(x = t^e\). Then

(a) If \(i > r\), the map \(\Gamma_{x,i}\) is a \(W(R)\)-module homomorphism.

(b) If \(i \leq r\) and \(R\) is a perfect field of characteristic \(p \neq 0\), the map \(\Gamma_{x,i}\) is a \(W(R)\)-module homomorphism.
Proof: The ideal $V, W(R)$ is generated by $1 - x$, so it is enough to check that $(1 - r'^m) \ast \Gamma_{2,i}(1 - x) = \Gamma_{2,i}(1 - r'^m) \ast (1 - x))$. Write $d = \gcd(m, e)$, so that the right-hand side is $d \langle r'^{d'\frac{m}{d'}}/y_i, y_i \rangle$. If $i > r$ then $y_i \in A_0$, and part (a) follows immediately from formula (5.5)(a).

If $i \leq r$ the formula is more complicated. Set $e = nd$, $m = kd$ and choose $u, v$ so that $1 = uk + vn$. Formula (5.5)(b) then reads

$$(1 - r'^m) \ast \Gamma_{2,i}(1 - x) = d \langle r^n y^{kx}/y_i, y_i \rangle + v \langle y^x, r^n y^{(k - 1)x} \rangle$$
$$+ u \langle y^x, r^a r^{n - 1}, r \rangle + (d - 1) \langle -r^n y^{kx}, -1 \rangle.$$

Since $\text{char}(R) \neq 0$, the last term is zero. Since $R$ is perfect we can extract $p$th roots of $r$, and therefore $\langle y^{kx} r^{n - 1}, r \rangle$ is $p$-divisible. As $K_2(A, A_\ast)$ is a $p$-group, this must also be zero. The theorem will now follow once we show that $\langle y^x, r^n y^{(k - 1)x} \rangle = 0$ for all $k$. If $p \nmid k$, this term equals $(1/k) \langle y^{kx}, r^n \rangle = 0$. We now proceed by induction on $k$, using [S1, p. 414],

$$0 - p \langle y^x, s y^{(k - 1)x} \rangle - \langle y^x, s^p y^{p(k - 1)x} r^{(p - 1)x} \rangle$$
$$= \langle y^x, s^p y^{p(k - 1)x} \rangle.$$

Since $r^n = s^p$ for some $s$ in $R$, this establishes the result.

APPENDIX

In this Appendix, we give a proof that $K_1(A, I)$ carries a natural module structure in characteristic 0 whenever $I$ is nilpotent.

**Theorem A.1.** Let $A$ be a ring containing $Q$, and $I$ a nilpotent ideal of $A$. Then there is a natural isomorphism $K_1(A, I) \cong I/\langle A, I \rangle$, where $[A, I]$ is the subgroup of $I$ generated by all $[a, x] = ax - xa$ with $a \in A$ and $x \in I$.

Before giving our proof, we note that Goodwillie has proven that for any nilpotent ideal there is an isomorphism

$$K_1(A, I; Q) \cong HC_{i-1}(A \otimes Q, I \otimes Q).$$

Here $HC$ denotes cyclic homology over $Q$, and our indexing convention is such that $HC_i(A) \to HC_i(A/I) \to HC_{i-1}(A, I) \to HC_{i-1}(A)$ is exact. When $A$ contains $Q$, we know from [W2, 1.4] that $K_1(A, I; Q) = K_1(A, I)$. This yields a more general result:

**Theorem A.2.** Let $A$ be a ring containing $Q$, and $I$ a nilpotent ideal of $A$. There is a natural isomorphism $K_i(A, I) \cong HC_{i-1}(A, I)$ for all $i$. In par-
ticular, these groups are modules over the center of $A$. The case $i = 1$ yields the isomorphism $K_1(A, I) \cong H^0_c(A, I) \cong I/[A, I]$ of Theorem A.1.

To be more explicit about the isomorphism, we use a more explicit description of $K_1(A, I)$, due to Vaserstein. For any radical ideal $I$ in any ring $A$, let $W(A, I)$ denote the subgroup of $1 + I$ generated by all $(1 + ax)(1 + xa)^{-1}$ with $a \in A$ and $x \in I$. Then

$$K_1(A, I) \cong (1 + I)/W(A, I).$$

(See Theorem 2.1 of [Sw].) We will show that the power series expansions for $\ln$ and $\exp$ provide the isomorphisms in Theorem A.1.

We shall also need the Campbell–Hausdorff formula, which may be found in [J, pp. 170–174]. It states that for $x, y$ in a complete radical ideal $I$ that there are $u, v$ in $I$ such that

$$\exp(x) \exp(y) = \exp(x + y + [u, x] + [v, y]).$$

The actual formula is explicit enough to see that if $y \in I^n$ then $u \in I^n$. In fact, $u = \frac{1}{2}y + \frac{1}{12}[x, y] + \cdots$ is in the closure of the ideal $AyA$.

As an application, consider the set map $\ln: (1 + I) \rightarrow I$, whose inverse is the set map $\exp$. The Campbell–Hausdorff formula shows that $\ln$ is not a group homomorphism. In fact, for $x, y$ in $I$ it yields

$$\ln((1 + x)(1 + y)) = \ln(1 + x) + \ln(1 + y) + [u, \ln(1 + x)]$$

$$+ [v, \ln(1 + y)].$$

We summarize this computation:

**Lemma A.3.** If $I$ is any complete radical ideal which is also a $\mathbb{Q}$-vector space, then $\ln$ induces a group epimorphism

$$1 + I \xrightarrow{\ln} I/[I, I].$$

**Corollary A.4.** If $I$ is a complete radical ideal in a ring $A$ which contains $\mathbb{Q}$, then $\ln$ induces a surjection $K_1(A, I) \rightarrow I/[A, I]$.

**Proof.** Fix $a \in A$ and $x \in I$, and set

$$y = x + \sum_{i=1}^{\infty} x(-ax)^i/(i+1) = x + \sum_{i=1}^{\infty} (-xa)^i x/(i+1).$$

Then modulo $[A, I]$ we have that

$$\ln((1 + ax)(1 + xa)^{-1}) = \ln(1 + ax) - \ln(1 + xa) = ay - ya = 0.$$
In trying to construct an inverse to the map of A.3, we are led to consider \( \exp([I, I]) \). Note that for every \( n \) the set \( \exp([I, I^n]) \) is a subgroup of \( 1 + I \) by Campbell–Hausdorff.

**Lemma A.5.** If \( a \in A \), \( x \in I \) and \( y \in I^n \), then

(i) \( \exp(x + y) \exp(-x) \exp(-y) \) is in \( \exp([I, I^n]) \).

(ii) \( \exp(xa) \exp(-ax) \) is in \( W(A, I) \).

**Proof.** There are \( u, v, u' \in I \) and \( v', v'' \in I^n \) such that

\[
\exp(x + y) \exp(-x) \exp(-y) = \exp\left(y + [u, x] + [v, y]\right) \exp(-y) = \exp\left([u, x] + [v, y] + [u', y] + [v', [u, x]] + [v'', [v, y]]\right).
\]

We claim that \([u, x]\) is in \([I, I^n]\). To see this, note that by the Campbell–Hausdorff formula we can write \( u = u_1 + u_2 \), where \( u_1 \in I^n \) and \([u_2, x] = 0\). This establishes (i). For (ii), note that

\[
z = x + \sum_{i=1}^{\infty} x(ax)^i/(i+1)! = x + \sum_{i=1}^{\infty} (xa)^i x/(i+1)!
\]

satisfies \( 1 + za = \exp(xa) \) and \( (1 + az)^{-1} = \exp(ax)^{-1} = \exp(-ax) \).

**Proposition A.6.** When \( I \) is a nilpotent ideal which is also a \( \mathbb{Q} \)-vector space, then

(i) \( \exp([I, I]) = W(\mathbb{Q} \oplus I, I) \).

(ii) \( \exp \) induces a (well-defined) group epimorphism:

\[
I \xrightarrow{\exp} (1 + I)/W(\mathbb{Q} \oplus I, I) = K_1(I).
\]

**Proof.** Since \([\mathbb{Q} \oplus I, I] = [I, I]\), A.4 implies that \( W(\mathbb{Q} \oplus I, I) \) lies in \( \exp([I, I^n]) \). We show by descending induction on \( n \) that \( \exp([I, I^n]) \) lies in \( W(\mathbb{Q} \oplus I, I) \), the case \( n = 0 \) being given. For \( x \in I \), \( y \in I^n \) we have

\[
\exp([x, y]) \exp(yx) \exp(-xy) = \exp(xy + [u, [x, y]] + [v, xy]) \exp(-xy) = \exp(w), \quad w \in [I, I^{n+1}].
\]

The result follows from A.5.
THEOREM A.7. Let $I$ be a nilpotent ideal in a ring $A$ containing $\mathbb{Q}$.

(i) $W(A, I) = \exp([A, I])$

(ii) $\exp$ and $\ln$ induce an isomorphism $K_i(A, I) \cong I/[[A, I]]$.

Proof. We need only show that $\exp([A, I])$ is contained in $W(A, I)$, since we can then cite A.4 and A.6. But modulo $W(Q \oplus I, I)$, Lemma A.5 shows that $\exp(\Sigma [a_i, x_i]) = \Pi \exp([a_i, x_i])$, which is in $W(A, I)$.

REFERENCES


