J. Math. Anal. Appl. 387 (2012) 1188–1208

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Bivariate second-order linear partial differential equations and orthogonal polynomial solutions

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ARTICLE INFO

Article history:
Received 7 December 2010
Available online 12 October 2011
Submitted by D. Waterman

Keywords:
Second-order admissible potentially self-adjoint partial differential equations of hypergeometric type
Bivariate orthogonal polynomials
Rodrigues formula
Generalized Kampé de Fériet hypergeometric series
Appell polynomials
Connection problems

ABSTRACT

In this paper we construct the main algebraic and differential properties and the weight functions of orthogonal polynomial solutions of bivariate second-order linear partial differential equations, which are admissible potentially self-adjoint and of hypergeometric type. General formulae for all these properties are obtained explicitly in terms of the polynomial coefficients of the partial differential equation, using vector matrix notation. Moreover, Rodrigues representations for the polynomial eigensolutions and for their partial derivatives of any order are given. As illustration, these results are applied to a two parameter monic Appell polynomials. Finally, the non-monic case is briefly discussed.

1. Introduction

The theory of orthogonal polynomials in one variable is in permanent expansion due to its relationship with other areas of mathematics and also with several applications in physics and engineering. They provide a natural way to solve many types of important differential equations of mathematical physics, expanding solutions in appropriate Fourier series of orthogonal polynomial basis. They play therefore an important role in the study of wave mechanics, heat conduction, electromagnetic theory, quantum mechanics or mathematical statistics.

In this context, it is first relevant to study whether a (one variable) polynomial family, $\{p_n(x)\}_{n\in\mathbb{N}}$ $(x\in\mathbb{R})$, is orthogonal. This problem is solved in different ways but the Favard theorem [31], linking orthogonality and the fundamental three-term recurrence relation

$$xp_n(x)=\alpha_n p_{n+1}(x)+\beta_n p_n(x)+\gamma_n p_{n-1}(x),\qquad \gamma_n\neq 0, \tag{1}$$

provides certainly the most powerful characterization [7]. Here, if $p_n(x)=g_{n,n}x^n+g_{n,n-1}x^{n-1}+g_{n,n-2}x^{n-2}+\cdots$, then orthogonality of the p_n-family leads easily to the well-known expressions (see e.g. [7,15], [24, Eq. (1.4.17), p. 14] or

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.10.024

[25, p. 36]):

$$\alpha_n = \frac{g_{n,n}}{g_{n+1,n+1}}, \qquad \beta_n = \frac{g_{n,n-1} - \alpha_n g_{n+1,n}}{g_{n,n}}, \qquad \gamma_n = \frac{g_{n,n-2} - \beta_n g_{n,n-1} - \alpha_n g_{n+1,n-1}}{g_{n-1,n-1}}. \tag{2}$$

It is also important to provide ways of constructing efficiently these polynomials. For, several approaches are at hand. Besides the use of the recurrence (1) itself, we can mention (in a non-exhaustive way) the generating function methods, those based on a Rodrigues formula or the hypergeometric approach which gives nice and useful representations of the polynomials in terms of hypergeometric series.

As it is very well known, among all the one variable orthogonal polynomials, the four classical continuous families of Jacobi, Laguerre, Hermite and Bessel, are those sharing the widest set of properties. Besides the three-term recurrence (1) [7,25,31], they can be characterized in a number of ways, e.g. they are orthogonal polynomial solutions of the hypergeometric type differential equation [6,25] and the k-th derivatives of each family are again orthogonal and belong to the same family [1,25]. Moreover, the orthogonality weight functions satisfy Pearson-type equations [6,24] giving rise to Rodrigues formulae [1,25] for the corresponding orthogonal polynomials and for their derivatives of any order. Also, the orthogonal polynomials satisfy a number of algebraic and differential properties such as derivative representations [1,23] or also structure relations [1,7,17], among other properties. The list of references in this paragraph is not exhaustive but only indicative of the kind of references that could be examined on this topic.

In these classical settings, it is remarkable that the coefficients appearing in all the aforementioned algebraic and differential characterizations can be explicitly computed in terms of the polynomial coefficients $\sigma(x)$ and $\tau(x)$ of the hypergeometric-type differential equation [5,12,25,30,37–39]

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0, \quad \lambda_n = -n\tau' - \frac{1}{2}n(n-1)\sigma'',$$

satisfied by the classical families.

Our main contribution is to extend this remarkable property to the bivariate situation for polynomial solutions of admissible potentially self-adjoint partial differential equations of hypergeometric-type [20–22]. These polynomials play an important role in many applications, for instance in spectral/*hp*-finite element methods for solving partial differential equations [8,13].

One essential difference between polynomials in one variable and in several variables is the lack of an obvious basis in the latter [9]. One possibility to avoid this problem is to consider graded lexicographical order and use the matrix vector representation, first introduced by Kowalski [18,19] and afterwards considered by Xu [35,36]. In fact, using this point of view, in [2] the authors proved some structure and orthogonality relations for the successive partial derivatives of the vector orthogonal polynomials associated with a quasi-definite moment functional which satisfies a Pearson-type partial differential equation.

In this paper we deal with bivariate polynomials written in vector representation (and graded lexicographical order) which are solutions of admissible potentially self-adjoint linear second-order partial differential equation of hypergeometric type. In this context, we prove that (as it happens in the one variable hypergeometric type case) the coefficients characterizing the three-term recurrence relations, the first structure relations and the derivative representations fulfilled by the vector polynomials can be written explicitly in terms of the coefficients of the partial differential equation they satisfy. In the bivariate discrete case some results in this direction have been already given in [27–29].

The structure of the paper is as follows: In Section 2, after introducing basic definitions and notations, we present in Propositions 2.4 and 2.5 the general framework to be considered through the paper, i.e. admissible potentially self-adjoint second-order partial differential equations of hypergeometric type. In Section 3 we give the partial differential equations for the partial derivatives of the eigensolutions and we construct the corresponding weight functions for the orthogonal polynomials. Then, the relations linking these weight functions are obtained and they allow us to deduce a Rodrigues formula for the orthogonal polynomial solutions and for their partial derivatives of any order. In Sections 4 and 5, using vector matrix notation [9,18,19], general formulae for the main algebraic and differential properties (three-term recurrence relations, structure relations and derivative representations) are explicitly obtained in terms of the coefficients fully characterizing the partial differential equation, both in monic and non-monic cases. Finally, in Section 6 our results are applied to a two parameter monic Appell polynomials [4]. Another two non-monic different eigensolutions of the same partial differential equation [11,16,26], which are orthogonal on the same domain with respect to the same weight function $\varrho(x) = x^{\alpha-1}y^{\beta-1}$, with $\alpha > 0$ and $\beta > 0$, are also briefly analyzed.

2. Vector representation and admissible partial differential equations of hypergeometric type

Let $\mathbf{x} = (x, y) \in \mathbb{R}^2$, and let $\mathbf{x}^n$ ($n \in \mathbb{N}_0$) denote the column vector of the monomials $x^{n-k}y^k$, whose elements are arranged in graded lexicographical order (see [9, p. 32]):

$$\mathbf{x}^n = \left(x^{n-k}y^k\right), \quad 0 \leqslant k \leqslant n, \ n \in \mathbb{N}_0. \tag{3}$$

Let $\{P^n_{n-k,k}(x, y)\}$ be a sequence of polynomials in the space Π^2_n of all polynomials of total degree at most n in two variables, $\mathbf{x} = (x, y)$, with real coefficients. Such polynomials are finite sums of terms of the form $ax^{n-k}y^k$, where $a \in \mathbb{R}$.

From now on the (column) vector representation [18,19] will be adopted, so that $\mathbb{P}_n$ will denote the (column) polynomial vector

$$\mathbb{P}_n = \left(P^n_{n,0}(x,y), P^n_{n-1,1}(x,y), \ldots, P^n_{1,n-1}(x,y), P^n_{0,n}(x,y)\right)^{\mathrm{T}}. \tag{4}$$

Then, each polynomial vector $\mathbb{P}_n$ can be written in terms of the basis (3) as:

$$\mathbb{P}_n = G_{n,n}\mathbf{x}^n + G_{n,n-1}\mathbf{x}^{n-1} + \cdots + G_{n,0}\mathbf{x}^0, \tag{5}$$

where $G_{n,j}$ are matrices of size $(n+1)\times(j+1)$ and the leading matrix coefficient $G_{n,n}$ is a nonsingular square matrix of size $(n+1)\times(n+1)$.

Definition 2.1 *(Monic polynomial vector).* A polynomial vector $\widehat{\mathbb{P}}_n$ is said to be monic if its leading matrix coefficient $\widehat{G}_{n,n}$ is the identity matrix (of size $(n+1)\times(n+1)$); i.e.:

$$\widehat{\mathbb{P}}_n = \mathbf{x}^n + \widehat{G}_{n,n-1}\mathbf{x}^{n-1} + \cdots + \widehat{G}_{n,0}\mathbf{x}^0. \tag{6}$$

Then, each of its polynomial entries $\widehat{P}^n_{n-k,k}(x,y)$ are of the form:

$$\widehat{P}^n_{n-k,k}(x,y) = x^{n-k}y^k + \text{terms of lower total degree}. \tag{7}$$

In what follows the "hat" notation $\widehat{\mathbb{P}}_n$ will be used for monic polynomials.

Definition 2.2 *(Orthogonality).* Let $\mathcal{L}$ be a moment linear functional acting on Π^2_n. A sequence of polynomials $\{P^n_{n-k,k}(x,y)\} \subset \Pi^2_n$ $(n \in \mathbb{N}_0)$, is said to be orthogonal with respect to $\mathcal{L}$ or, equivalently, $\{\mathbb{P}_n\}_{n\geqslant 0}$ (as defined by Eqs. (4)–(5)) is a vector orthogonal polynomial family with respect to $\mathcal{L}$, if for each $n \in \mathbb{N}_0$ there exists an invertible matrix H_n of size $n+1$ such that:

$$\mathcal{L}\left[\left(\mathbf{x}^m\mathbb{P}^T_n\right)\right] = 0 \in \mathcal{M}^{(m+1,n+1)}, \quad n > m, \tag{8}$$

$$\mathcal{L}\left[\left(\mathbf{x}^n\mathbb{P}^T_n\right)\right] = H_n \in \mathcal{M}^{(n+1,n+1)}. \tag{9}$$

If there exists an integral representation of this orthogonality functional $\mathcal{L}$, then its action can be written in terms of a weight function $\varrho := \varrho(x,y)$ over a certain domain $D \subset \mathbb{R}^2$:

$$\mathcal{L}(P) = \iint_D P(x,y)\varrho(x,y)\,dx\,dy, \quad P \in \Pi^2_n, \tag{10}$$

which is defined in the set Π^2_n provided that all the above integrals exist. Then, the family $\{\mathbb{P}_n\}_{n\geqslant 0}$ is said to be orthogonal with respect to ϱ in the domain D.

In this multivariate context, Bochner [6] posed the problem of identifying those families of polynomials which are eigenfunctions of a second-order linear partial differential operator. Krall and Sheffer [20] started to study eigenfunctions which are orthogonal over a domain giving conditions of admissibility and a first attempt of classifying admissible equations. Engelis [10] gave a detailed list of second-order linear partial differential equations for which orthogonal polynomials in two variables are solutions. This question was afterwards studied and systematically described by Suetin [33]. In this paper, we analyze polynomial eigenfunctions of admissible potentially self-adjoint partial differential equations of hypergeometric type.

In order to present this study, we consider a bivariate class of linear partial differential equations, introduced as "the basic class" by Lyskova [22] for the multivariate case (see also [3]) and called here hypergeometric type equations:

$$\begin{aligned}&\left(a_1x^2+b_1x+c_1\right)\partial_{xx}u(x,y) + 2(a_3xy+b_3x+c_3y+d_3)\partial_{xy}u(x,y)\\&\quad+\left(a_2y^2+b_2y+c_2\right)\partial_{yy}u(x,y) + (e_1x+f_1)\partial_x u(x,y) + (e_2y+f_2)\partial_y u(x,y) + \lambda u(x,y) = 0,\end{aligned} \tag{11}$$

where a_j, b_j, c_j, d_j, e_j, f_j and λ are real numbers. The solutions of this equation have the remarkable property that all their partial derivatives of any order are also solutions of an equation of the same form.

Moreover, we shall also consider admissible partial differential equations.

Definition 2.3. A second-order partial differential equation is admissible if and only if [20,33] for any non-negative integer n there exists a number λ_n such that Eq. (11) with $\lambda := \lambda_n$ has $n+1$ linearly independent solutions which are polynomials of total degree n and has no non-trivial solutions in the set of polynomials of total degree less than n.

The following characterization has been proved in [22,33].

Proposition 2.4. *Eq.* (11) *is an admissible second-order partial differential equation of hypergeometric type if and only if it can be written in the form*

$$\begin{aligned}&\big(ax^2+b_1x+c_1\big)\partial_{xx}u(x,y)+2(axy+b_3x+c_3y+d_3)\partial_{xy}u(x,y)\\&\quad+\big(ay^2+b_2y+c_2\big)\partial_{yy}u(x,y)+(ex+f_1)\partial_xu(x,y)+(ey+f_2)\partial_yu(x,y)+\lambda_nu(x,y)=0,\end{aligned}\tag{12}$$

where $\lambda_n=-n((n-1)a+e)$ and the coefficients a,b_j,c_j,d_j,e,f_j are arbitrary fixed real numbers, but the numbers a and e are such that the condition

$$\varpi_k:=ak+e\neq 0\tag{13}$$

holds true for any non-negative integer k.

In the conditions of Proposition 2.4, the results of Suetin [33, Chapter 5] can be used to define an orthogonality weight function over a certain domain of $\mathbb{R}^2$ which is related with the partial differential equation (12). These results are summarized in the following proposition.

Proposition 2.5. *Let $\alpha(x,y)$ be the discriminant of Eq.* (12)*, i.e.*:

$$\alpha(x,y)=\big(c_1+x(b_1+ax)\big)\big(c_2+y(b_2+ay)\big)-\big(d_3+b_3x+(c_3+ax)y\big)^2,\tag{14}$$

and $D\subset\mathbb{R}^2$ be the domain:

$$D=\big\{(x,y)\in\mathbb{R}^2\colon\ \alpha(x,y)\neq 0\big\}.\tag{15}$$

Define the two functions [33, Eq. (15), p. 132]

$$\begin{aligned}\beta(x,y)&=(-b_1-c_3+f_1-3ax+ex)\big(ay^2+b_2y+c_2\big)\\&\qquad-(-b_2-b_3+f_2-3ay+ey)(axy+b_3x+c_3y+d_3),\end{aligned}\tag{16}$$

$$\begin{aligned}\gamma(x,y)&=-\big(c_1+x(b_1+ax)\big)(b_2+b_3-f_2+3ay-ey)\\&\qquad+(b_1+c_3-f_1+3ax-ex)\big(d_3+b_3x+(c_3+ax)y\big).\end{aligned}\tag{17}$$

Assume that in D the following condition holds true:

$$\frac{\partial}{\partial x}\left(\frac{\gamma(x,y)}{\alpha(x,y)}\right)=\frac{\partial}{\partial y}\left(\frac{\beta(x,y)}{\alpha(x,y)}\right).\tag{18}$$

Consider the weight function given by [33, Eq. (22), p. 134]

$$\varrho(x,y)=\exp\left\{\int_{y_0}^{y}\frac{\gamma(x,y)}{\alpha(x,y)}\,dy+\int_{x_0}^{x}\left[\left(\frac{\beta(x,y)}{\alpha(x,y)}\right)_{y=y_0}\right]dx\right\},\tag{19}$$

which determines (*up to a multiplicative constant*) *the functional*

$$\mathcal{L}(P)=\iint_D P(x,y)\varrho(x,y)\,dx\,dy,\qquad P\in\Pi_n^2,\tag{20}$$

defined in the set Π_n^2 provided that all such integrals exist.

Then, there exists a unique monic vector polynomial family $\{\widehat{\mathbb{P}}_n\}_{n\geqslant 0}$ solution of (12) *and orthogonal with respect to ϱ in D, i.e. satisfying*

$$\iint_D \mathbf{x}^m\widehat{\mathbb{P}}_n^T\varrho(x,y)\,dx\,dy=\begin{cases}0\in\mathcal{M}^{(m+1,n+1)}, & \text{if } n>m,\\ H_n\in\mathcal{M}^{(n+1,n+1)}, & \text{if } m=n,\end{cases}\tag{21}$$

where H_n (*of size $(n+1)\times(n+1)$*) *is nonsingular.*

Remark 1. From the admissible second-order partial differential equation of hypergeometric type (12) it is possible to introduce the linear operator $\mathcal{D}$ by

$$\begin{aligned}\mathcal{D}u&=\big(ax^2+b_1x+c_1\big)\partial_{xx}u+2(axy+b_3x+c_3y+d_3)\partial_{xy}u\\&\quad+\big(ay^2+b_2y+c_2\big)\partial_{yy}u+(ex+f_1)\partial_xu+(ey+f_2)\partial_yu,\end{aligned}\tag{22}$$

which is potentially self-adjoint if and only if (18) holds true [33, Theorem 1, p. 133].

So, as it has been mentioned in the introduction, we deal with bivariate orthogonal polynomial families $\{\mathbb{P}_n\}_{n\geqslant 0}$ characterized in Proposition 2.5, which are solutions of admissible potentially self-adjoint second-order partial differential equations of hypergeometric type described in Proposition 2.4 and Remark 1.

3. Weight functions and Rodrigues formula for the polynomials and for their partial derivatives

By differentiating (12) $(r+s)$ times, it turns out that

$$z^{(r,s)}(x,y)=\frac{\partial^{r+s}u}{\partial x^r\partial y^s}(x,y),\quad r,s=0,1,2,\ldots,\tag{23}$$

satisfies an admissible second-order partial differential equation of hypergeometric type

$$\begin{aligned}&\left(ax^2+b_1x+c_1\right)\partial_{xx}z^{(r,s)}(x,y)+2(axy+b_3x+c_3y+d_3)\partial_{xy}z^{(r,s)}(x,y)\\&\quad+\left(ay^2+b_2y+c_2\right)\partial_{yy}z^{(r,s)}(x,y)+\tau_x^{(r,s)}(x)\partial_x z^{(r,s)}(x,y)\\&\quad+\tau_y^{(r,s)}(y)\partial_y z^{(r,s)}(x,y)+\mu_{r+s}z^{(r,s)}(x,y)=0,\end{aligned}\tag{24}$$

where

$$\tau_x^{(r,s)}(x)=\big(e+2a(r+s)\big)x+f_1+rb_1+2sc_3,\tag{25}$$

$$\tau_y^{(r,s)}(y)=\big(e+2a(r+s)\big)y+f_2+2rb_3+sb_2,\tag{26}$$

$$\mu_{r+s}=\lambda_n+(r+s)e+(r+s)(r+s-1)a,\tag{27}$$

and a,b_j,c_j,d_3,e,f_j are arbitrary fixed real numbers satisfying condition (13). We should mention that Eq. (24) and the above relations have been obtained in the multivariate case for hypergeometric type equations (not necessarily admissible) in [22, Theorem 1].

From the admissible second-order partial differential equation of hypergeometric type (24), let us introduce the linear operator

$$\begin{aligned}\mathcal{D}^{(r,s)}z=&\left(ax^2+b_1x+c_1\right)\partial_{xx}z+2(axy+b_3x+c_3y+d_3)\partial_{xy}z\\&+\left(ay^2+b_2y+c_2\right)\partial_{yy}z+\tau_x^{(r,s)}(x)\partial_x z+\tau_y^{(r,s)}(y)\partial_y z.\end{aligned}\tag{28}$$

In a similar way as in Proposition 2.5, for (24) we introduce

$$\beta^{(r,s)}(x,y)=\beta(x,y)+r\frac{\partial\alpha}{\partial x}(x,y)+s\theta(x,y),\tag{29}$$

$$\gamma^{(r,s)}(x,y)=\gamma(x,y)+r\omega(x,y)+s\frac{\partial\alpha}{\partial y}(x,y),\tag{30}$$

where the polynomials $\beta(x,y)$ and $\gamma(x,y)$ have been defined in (16) and (17). Applying condition (18) to the operator (28), we obtain that this operator $\mathcal{D}^{(r,s)}$ is potentially self-adjoint in a domain D, if and only if

$$\frac{\partial}{\partial x}\left[\frac{\gamma(x,y)}{\alpha(x,y)}+r\frac{\omega(x,y)}{\alpha(x,y)}+\frac{s}{\alpha(x,y)}\frac{\partial\alpha(x,y)}{\partial y}\right]=\frac{\partial}{\partial y}\left[\frac{\beta(x,y)}{\alpha(x,y)}+s\frac{\theta(x,y)}{\alpha(x,y)}+\frac{r}{\alpha(x,y)}\frac{\partial\alpha(x,y)}{\partial x}\right],$$

where $\alpha(x,y)$ was defined in (14),

$$\omega(x,y)=2A\frac{\partial B}{\partial x}-B\frac{\partial A}{\partial x},$$

$$\theta(x,y)=2C\frac{\partial B}{\partial y}-B\frac{\partial C}{\partial y},$$

and

$$A=A(x,y)=ax^2+b_1x+c_1,$$

$$B=B(x,y)=axy+b_3x+c_3y+d_3,$$

$$C=C(x,y)=ay^2+b_2y+c_2.$$

Note that $\alpha(x,y)=AC-B^2$. Therefore, if we assume that the operator $\mathcal{D}$ defined in (22) is potentially self-adjoint, then the operator $\mathcal{D}^{(r,s)}$ defined in (28) is potentially self-adjoint if and only if

$$r\frac{\partial}{\partial x}\left(\frac{\omega(x,y)}{\alpha(x,y)}\right)+(s-r)\frac{\partial}{\partial x}\left(\frac{1}{\alpha(x,y)}\frac{\partial\alpha(x,y)}{\partial y}\right)-s\frac{\partial}{\partial y}\left(\frac{\theta(x,y)}{\alpha(x,y)}\right)=0.\tag{31}$$

If the operator $\mathcal{D}^{(r,s)}$ defined in (28) is potentially self-adjoint in a domain D, then there exists in this domain a positive and twice continuously differentiable function $\varrho^{(r,s)}(x,y)$ which is the solution of the system of differential equations (Pearson type equations) [22, Eqs. (7) and (8)] and [33, p. 132]

$$\begin{cases} \dfrac{1}{\varrho^{(r,s)}(x,y)} \dfrac{\partial \varrho^{(r,s)}(x,y)}{\partial x} = \dfrac{\beta^{(r,s)}(x,y)}{\alpha(x,y)}, \\ \dfrac{1}{\varrho^{(r,s)}(x,y)} \dfrac{\partial \varrho^{(r,s)}(x,y)}{\partial y} = \dfrac{\gamma^{(r,s)}(x,y)}{\alpha(x,y)}, \end{cases} \tag{32}$$

where $\beta^{(r,s)}(x,y)$ and $\gamma^{(r,s)}(x,y)$ are given in (29) and (30) respectively.

From the Pearson type equations (32) we obtain the following expression for the orthogonality weight function of the admissible potentially self-adjoint second-order partial differential equations of hypergeometric type (24)

$$\varrho^{(r,s)}(x,y) = \exp\left\{ \int_{y_0}^{y} \frac{\gamma^{(r,s)}(x,y)}{\alpha(x,y)}\,dy + \int_{x_0}^{x} \left[\left(\frac{\beta^{(r,s)}(x,y)}{\alpha(x,y)} \right)_{y=y_0} \right] dx \right\}, \tag{33}$$

up to a multiplicative constant.

3.1. Relation between weight functions

Now, we can establish the connection between $\varrho^{(r,s)}(x,y)$ and $\varrho^{(0,0)}(x,y) \equiv \varrho(x,y)$, given in (33) and (19) respectively. From Eq. (33), after straightforward computations, we obtain that

$$\varrho^{(r,s)}(x,y) = \phi^{(r,s)}(x,y)\varrho(x,y), \quad r,s = 0,1,2,\ldots, \tag{34}$$

up to a multiplicative constant, where $\phi^{(r,s)}(x,y)$ is a polynomial whose explicit expression depends on the coefficients of the partial differential equation (12). After solving the non-linear system of equations (31) for any r and s, we can reduce the solutions of the system to the following ten cases:

(i) If $b_1 = 2c_3$ and $b_2 = 2b_3$, we have

$$\phi^{(r,s)}(x,y) = \left[\alpha(x,y)\right]^{r+s},$$

where

$$\alpha(x,y) = -\big(d_3 + b_3 x + (c_3 + ax)y\big)^2 + \big(c_1 + x(2c_3 + ax)\big)\big(c_2 + y(2b_3 + ay)\big),$$

assuming that $af_2 = eb_3$, $af_1 = ec_3$ and $f_2 c_3 = f_1 b_3$.

(ii) If $c_3 \neq 0$, $d_3 \neq 0$, $b_3 \neq 0$, $a = b_3 c_3 / d_3$, $c_1 = (b_1 - c_3)d_3/b_3$, and $c_2 = (b_2 - b_3)d_3/c_3$, we have

$$\phi^{(r,s)}(x,y) = \frac{[\alpha(x,y)]^{r+s}}{(d_3 + c_3 y)^r (d_3 + b_3 x)^s},$$

where

$$\alpha(x,y) = -\frac{1}{b_3 c_3 d_3}\Big((d_3 + b_3 x)(d_3 + c_3 y)\big(-b_1(b_2 d_3 - b_3 d_3 + b_3 c_3 y) + c_3\big(b_2(d_3 - b_3 x) + 2b_3(b_3 x + c_3 y)\big)\big)\Big).$$

(iii) If $a = b_1 = c_1 = c_3 = 0$ and $b_2 = b_3$ we obtain

$$\phi^{(r,s)}(x,y) = \left[\alpha(x,y)\right]^r, \quad \alpha(x,y) = (d_3 + b_3 x)^2.$$

(iv) If $a = b_2 = b_3 = c_2 = 0$ and $b_1 = c_3$ we have

$$\phi^{(r,s)}(x,y) = \left[\alpha(x,y)\right]^s, \quad \alpha(x,y) = (d_3 + c_3 y)^2.$$

(v) If $a = b_3 = c_3 = d_3 = 0$, we have

$$\phi^{(r,s)}(x,y) = \frac{[\alpha(x,y)]^{r+s}}{(c_1 + b_1 x)^s (c_2 + b_2 y)^r}, \quad \alpha(x,y) = (c_1 + b_1 x)(c_2 + b_2 y).$$

(vi) If $a \neq 0$, $b_3 = c_2 = d_3 = 0$, and $c_1 = (b_1 - c_3)c_3/a$, we obtain

$$\phi^{(r,s)}(x, y) = \frac{[\alpha(x, y)]^{r+s}}{y^r (c_3 + ax)^s},$$

where

$$\alpha(x, y) = \frac{(c_3 + ax)y(b_2(b_1 - c_3 + ax) + a(b_1 - 2c_3)y)}{a}.$$

(vii) If $c_3 \neq 0$, $a = b_3 = 0$, $b_1 = c_3$, and $c_2 = b_2 d_3/c_3$, we obtain

$$\phi^{(r,s)}(x, y) = \frac{[\alpha(x, y)]^{r+s}}{(d_3 + c_3 y)^r},$$

where

$$\alpha(x, y) = \frac{(d_3 + c_3 y)(b_2(c_1 + c_3 x) - c_3(d_3 + c_3 y))}{c_3}.$$

(viii) If $b_3 \neq 0$, $a = c_3 = 0$, $b_2 = b_3$, and $c_1 = b_1 d_3/b_3$, we obtain

$$\phi^{(r,s)}(x, y) = \frac{[\alpha(x, y)]^{r+s}}{(d_3 + b_3 x)^s},$$

where

$$\alpha(x, y) = \frac{(d_3 + b_3 x)(-b_3(d_3 + b_3 x) + b_1(c_2 + b_3 y))}{b_3}.$$

(ix) If $a \neq 0$, $c_1 = c_3 = d_3 = 0$, and $c_2 = (b_2 - b_3)b_3/a$, we obtain

$$\phi^{(r,s)}(x, y) = \frac{[\alpha(x, y)]^{r+s}}{x^s (b_3 + ay)^r},$$

where

$$\alpha(x, y) = \frac{x(b_3 + ay)(a(b_2 - 2b_3)x + b_1(b_2 - b_3 + ay))}{a}.$$

(x) If $c_1 = c_2 = d_3 = b_3 = c_3 = 0$, we obtain

$$\phi^{(r,s)}(x, y) = \frac{[\alpha(x, y)]^{r+s}}{x^s y^r},$$

where

$$\alpha(x, y) = xy\big(ab_2 x + b_1(b_2 + ay)\big).$$

Remark 2. Observe that in all the above cases the polynomial $\phi^{(r,s)}(x, y)$ defined in (34) can be factorized as

$$\phi^{(r,s)}(x, y) = \big[\phi^{(1,0)}(x, y)\big]^r \big[\phi^{(0,1)}(x, y)\big]^s, \tag{35}$$

for any r and s.

Remark 3. It is important to notice here that different cases could give rise to the same $\phi^{(r,s)}(x, y)$ associated with the same partial differential equation. This situation appears in the example studied in detail in Section 6 where the coefficients of the partial differential equation (74) fulfill the conditions of cases (vi), (ix) and (x).

Remark 4. We must now mention that in [22, Theorem 3], Lyskova presented Eq. (34), but in a non-explicit form.

3.2. Rodrigues formula

One of the main problems in the theory of orthogonal polynomials in several variables is to obtain explicit expressions for the orthogonal polynomial solutions of the partial differential equation. In this direction we could mention the works of Engelis [10], who derived the Rodrigues formula for some classes of orthogonal polynomials in two variables, and Suetin [33, Theorem 3, p. 151], who showed that this Rodrigues representation is one of the ways of constructing explicitly orthogonal polynomial families in the potentially self-adjoint case.

As a consequence, in the context considered here (admissible potentially self-adjoint partial differential equations of hypergeometric type), we have the following explicit Rodrigues formula for the polynomial solutions of (12) of total degree $n+m$

$$P_{n,m}(x,y)=\frac{\aleph_{n,m}}{\varrho(x,y)}\frac{\partial^{n+m}}{\partial x^n\partial y^m}\bigl[\varrho(x,y)\bigl[\phi^{(1,0)}(x,y)\bigr]^n\bigl[\phi^{(0,1)}(x,y)\bigr]^m\bigr], \tag{36}$$

where $\aleph_{n,m}$ is a normalizing constant and the polynomials $\phi^{(1,0)}(x,y)$ and $\phi^{(0,1)}(x,y)$ have been introduced in (34).

Moreover, the partial differential equation (24) for the partial derivatives $P_{n,m}^{(r,s)}(x,y)$ of the polynomial solutions of Eq. (12) is also of hypergeometric type. So, we also have a Rodrigues representation for the partial derivatives of any order given by

$$P_{n,m}^{(r,s)}(x,y)=\frac{\aleph_{n,m,r,s}}{\varrho^{(r,s)}(x,y)}\frac{\partial^{n+m-r-s}}{\partial x^{n-r}\partial y^{m-s}}\bigl[\varrho^{(r,s)}(x,y)\bigl[\phi^{(1,0)}(x,y)\bigr]^{n-r}\bigl[\phi^{(0,1)}(x,y)\bigr]^{m-s}\bigr], \tag{37}$$

where $\varrho^{(r,s)}(x,y)$ is given in (33) and $\aleph_{n,m,r,s}$ is a normalizing constant. In this way we have obtained a natural extension to the bivariate case of the Rodrigues representation for classical orthogonal polynomials in one variable [24].

4. Explicit expressions for algebraic properties

Let us consider a vector polynomial family $\{\mathbb{P}_n\}_{n\in\mathbb{N}_0}$ solution of (12) orthogonal in the sense of Proposition 2.5, i.e. it is orthogonal with respect to a weight (19) and satisfies (21) in an appropriate domain $D\in\mathbb{R}^2$. With such conditions it can be proved that the family satisfies a number of algebraic and differential properties. Here we focus our attention in three of the most relevant: the three-term recurrence relations, the structure relations and the derivative representations. As already mentioned, our aim in this section is to give all of these relations in terms of the matrix coefficients in the expansions (5) of the $\{\mathbb{P}_n\}$-family elements.

For, let us first introduce the matrices $L_{n,j}$ of size $(n+1)\times(n+2)$ defined by

$$L_{n,1}\mathbf{x}^{n+1}=x\mathbf{x}^n,\qquad L_{n,2}\mathbf{x}^{n+1}=y\mathbf{x}^n, \tag{38}$$

so that,

$$L_{n,1}=\begin{pmatrix}1 & & \bigcirc & 0\\ & \ddots & & \vdots\\ \bigcirc & & 1 & 0\end{pmatrix}\quad\text{and}\quad L_{n,2}=\begin{pmatrix}0 & 1 & & \bigcirc\\ \vdots & & \ddots & \\ 0 & \bigcirc & & 1\end{pmatrix}. \tag{39}$$

Let us observe that

$$\begin{aligned}&x^2\mathbf{x}^n=L_{n,1}L_{n+1,1}\mathbf{x}^{n+2},\qquad y^2\mathbf{x}^n=L_{n,2}L_{n+1,2}\mathbf{x}^{n+2},\\ &L_{n,2}L_{n+1,1}=L_{n,1}L_{n+1,2},\end{aligned} \tag{40}$$

and for $j=1,2$,

$$L_{n,j}L_{n,j}^{\mathrm{T}}=I_{n+1}, \tag{41}$$

where I_{n+1} denotes the identity matrix of size $n+1$.

Moreover, for $n\geqslant 1$,

$$\begin{cases}\partial_x\mathbf{x}^n=\mathbb{E}_{n,1}\mathbf{x}^{n-1},\\ \partial_y\mathbf{x}^n=\mathbb{E}_{n,2}\mathbf{x}^{n-1},\end{cases} \tag{42}$$

where the matrices $\mathbb{E}_{n,j}$ of size $(n+1)\times n$ are given by

$$\mathbb{E}_{n,1}=\begin{pmatrix}n & & & \bigcirc\\ & n-1 & & \\ & & \ddots & \\ & \bigcirc & & 1\\ 0 & \cdots & 0 & 0\end{pmatrix}\quad\text{and}\quad \mathbb{E}_{n,2}=\begin{pmatrix}0 & \cdots & & 0\\ 1 & & & \bigcirc\\ & 2 & & \\ & & \ddots & \\ & \bigcirc & & n\end{pmatrix}. \tag{43}$$

4.1. The three-term recurrence relations

The existence of a recurrence relation for a vector orthogonal polynomial family can be established in more general settings than those considered here. More precisely, the following existence theorem is proved in [9].

Theorem 4.1. *Let $\mathcal{L}$ be the positive definite moment functional as defined in* (10) *and* $\{\mathbb{P}_n\}_{n\geqslant 0}$ *be an orthogonal family with respect to $\mathcal{L}$. Then, for $n \geqslant 0$, there exist unique matrices $A_{n,j}$ of size $(n+1)\times(n+2)$, $B_{n,j}$ of size $(n+1)\times(n+1)$,and $C_{n,j}$ of size $(n+1)\times n$, such that*

$$x_j\mathbb{P}_n = A_{n,j}\mathbb{P}_{n+1} + B_{n,j}\mathbb{P}_n + C_{n,j}\mathbb{P}_{n-1}, \quad j = 1, 2, \tag{44}$$

with the initial conditions $\mathbb{P}_{-1} = 0$ and $\mathbb{P}_0 = 1$. Here the notation $x_1 = x$, $x_2 = y$ is used.

Now it is possible to generalize the well-known expressions (2) for the one variable case to the bivariate case. This is done in the following proposition which is proved with the help of the auxiliary matrices $L_{n,j}$ defined in (38)–(39).

Theorem 4.2. *The explicit expressions of the matrices $A_{n,j}$, $B_{n,j}$ and $C_{n,j}$ $(j = 1, 2)$ appearing in* (44) *in terms of the values of the leading coefficients $G_{n,n}$, $G_{n,n-1}$ and $G_{n,n-2}$ in the expansions* (5) *are given by*

$$\begin{cases} A_{n,j} = G_{n,n}L_{n,j}G^{-1}_{n+1,n+1}, \quad n \geqslant 0, \\ B_{0,j} = -A_{0,j}G_{1,0}, \\ B_{n,j} = (G_{n,n-1}L_{n-1,j} - A_{n,j}G_{n+1,n})G^{-1}_{n,n}, \quad n \geqslant 1, \\ C_{1,j} = -(A_{1,j}G_{2,0} + B_{1,j}G_{1,0}), \\ C_{n,j} = (G_{n,n-2}L_{n-2,j} - A_{n,j}G_{n+1,n-1} - B_{n,j}G_{n,n-1})G^{-1}_{n-1,n-1}, \quad n \geqslant 2. \end{cases} \tag{45}$$

Proof. In Eq. (44), substitute $\mathbb{P}_n$ as given in (5), equate the coefficients of $\mathbf{x}^k$ for $k = n$, $n-1$, $n-2$ and solve the corresponding linear system. □

The above result is valid for any orthogonal polynomial sequence (21). From now on we shall consider algebraic properties involving the partial derivatives of the orthogonal polynomials. To set up sufficient conditions for the existence of this type of relations is not enough with the orthogonality of the vector polynomial family. For these relations to exist, the orthogonal polynomial family has to be a solution of an admissible potentially self-adjoint second-order partial differential equation of hypergeometric type (12). With these assumptions, we present explicit expressions for the three-term recurrence relation of the first partial derivatives, structure relations and derivative representations.

4.2. The three-term recurrence relations for the first partial derivatives

Let $\mathcal{L}$ be the positive definite moment functional as defined in (20). Let $\mathbb{P}_{n+1}$ be a vector column polynomial of size $n+2$ which contains $n+2$ linearly independent solutions of total degree $n+1$, orthogonal with respect to $\mathcal{L}$, of the admissible second-order partial differential equation of hypergeometric type (12). Then, for $j = 1, 2$ and the notation $x_1 = x$, $x_2 = y$,

$$\frac{\partial}{\partial x_j}\mathbb{P}_{n+1} = \left(\frac{\partial}{\partial x_j}P^{n+1}_{n+1,0}(x,y), \frac{\partial}{\partial x_j}P^{n+1}_{n,1}(x,y), \ldots, \frac{\partial}{\partial x_j}P^{n+1}_{1,n}(x,y), \frac{\partial}{\partial x_j}P^{n+1}_{0,n+1}(x,y)\right)^{\mathrm{T}} \tag{46}$$

is a column vector of size $n+2$ and total degree n. Thus, we have

$$\frac{\partial}{\partial x_j}\mathbb{P}_{n+1} = Gd^{(j)}_{n,n}\mathbf{x}^n + Gd^{(j)}_{n,n-1}\mathbf{x}^{n-1} + \cdots + Gd^{(j)}_{n,0}\mathbf{x}^0, \tag{47}$$

where

$$Gd^{(j)}_{n,k} = G_{n+1,k+1}\mathbb{E}_{k+1,j}, \quad 0 \leqslant k \leqslant n, \ j = 1, 2, \tag{48}$$

are matrices of size $(n+2)\times(k+1)$, where $G_{n+1,k+1}$ have been introduced in (5) and $\mathbb{E}_{k+1,j}$ are given in (43).

Since $x_j\frac{\partial}{\partial x_j}\mathbb{P}_{n+1}$ is a polynomial of total degree $n+1$, and the family $\frac{\partial}{\partial x_j}\mathbb{P}_n$ is orthogonal with respect to $\varrho^{(1,0)}$ or $\varrho^{(0,1)}$ $(j = 1, 2)$ defined in (33), then in the orthogonal expansion

$$x_j\frac{\partial}{\partial x_j}\mathbb{P}_{n+1} = \sum_{k=0}^{n+2}\Upsilon_{k,j}\frac{\partial}{\partial x_j}\mathbb{P}_k$$

we have that $\Upsilon_{k,j} = 0$ for $k < n$. So, there exist matrices $A_{n,j}^{(j)}$ of size $(n+2) \times (n+3)$, $B_{n,j}^{(j)}$ of size $(n+2) \times (n+2)$, and $C_{n,j}^{(j)}$ of size $(n+2) \times (n+1)$, giving the three-term recurrence relations

$$x_j \frac{\partial}{\partial x_j} \mathbb{P}_{n+1} = A_{n,j}^{(j)} \frac{\partial}{\partial x_j} \mathbb{P}_{n+2} + B_{n,j}^{(j)} \frac{\partial}{\partial x_j} \mathbb{P}_{n+1} + C_{n,j}^{(j)} \frac{\partial}{\partial x_j} \mathbb{P}_n, \quad j = 1, 2, \tag{49}$$

with the initial conditions $\frac{\partial}{\partial x_j} \mathbb{P}_0 = 0$ and $\frac{\partial}{\partial x_j} \mathbb{P}_1 = G_{1,1} \mathbb{E}_{1,j}$.

The explicit expressions of the matrices $A_{n,j}^{(j)}$, $B_{n,j}^{(j)}$ and $C_{n,j}^{(j)}$ can be obtained in terms of the values of the leading coefficients $Gd_{n,n}^{(j)}$, $Gd_{n,n-1}^{(j)}$ and $Gd_{n,n-2}^{(j)}$, defined in (48), in a similar way as in Theorem 4.2.

4.3. First structure relations

In the one variable case, the so-called first structure relation plays an important role since, e.g. it gives rise to lowering and rising operators. Next we present the extension to the bivariate case.

Theorem 4.3. *Let $\{\mathbb{P}_n\}_{n \in \mathbb{N}_0}$ be a vector orthogonal polynomial family satisfying Proposition 2.5. Then, for $n \geqslant 1$, there exist unique matrices $W_{n,j}$ of size $(n+1) \times (n+2)$, $S_{n,j}$ of size $(n+1) \times (n+1)$, and $T_{n,j}$ of size $(n+1) \times n$, such that*

$$\phi_j(x, y) \frac{\partial}{\partial x_j} \mathbb{P}_n = W_{n,j} \mathbb{P}_{n+1} + S_{n,j} \mathbb{P}_n + T_{n,j} \mathbb{P}_{n-1}, \quad j = 1, 2, \tag{50}$$

where

$$\phi_1(x, y) = \phi^{(1,0)}(x, y), \qquad \phi_2(x, y) = \phi^{(0,1)}(x, y), \tag{51}$$

and the polynomials $\phi^{(r,s)}(x, y)$ have been introduced in (34) *and* (35)*. With the notation*

$$\phi_j(x, y) = \alpha_j x^2 + \beta_j xy + \gamma_j y^2 + \delta_j x + \varepsilon_j y + \omega_j, \quad j = 1, 2, \tag{52}$$

the coefficients of the structure relations are explicitly given by

$$\begin{aligned}
W_{n,j} G_{n+1,n+1} &= G_{n,n} \mathbb{E}_{n,j} (\alpha_j L_{n-1,1} L_{n,1} + \beta_j L_{n-1,1} L_{n,2} + \gamma_j L_{n-1,2} L_{n,2}), \\
S_{n,j} G_{n,n} &= G_{n,n} \mathbb{E}_{n,j} (\delta_j L_{n-1,1} + \varepsilon_j L_{n-1,2}) - W_{n,j} G_{n+1,n} \\
&\quad + G_{n,n-1} \mathbb{E}_{n-1,j} (\alpha_j L_{n-2,1} L_{n-1,1} + \beta_j L_{n-2,2} L_{n-1,1} + \gamma_j L_{n-2,2} L_{n-1,2}), \\
T_{n,j} G_{n-1,n-1} &= \omega_j G_{n,n} \mathbb{E}_{n,j} - W_{n,j} G_{n+1,n-1} - S_{n,j} G_{n,n-1} + G_{n,n-1} \mathbb{E}_{n-1,j} (\delta_j L_{n-2,1} + \varepsilon_j L_{n-2,2}) \\
&\quad + G_{n,n-2} \mathbb{E}_{n-2,j} (\alpha_j L_{n-3,1} L_{n-2,1} + \beta_j L_{n-3,2} L_{n-2,1} + \gamma_j L_{n-3,2} L_{n-2,2}).
\end{aligned}$$

Proof. Since

$$\phi_j(x, y) \frac{\partial}{\partial x_j} \mathbb{P}_n$$

are polynomials of total degree $n+1$ we can write for $j = 1, 2$,

$$\phi_j(x, y) \frac{\partial}{\partial x_j} \mathbb{P}_n = \sum_{\ell=0}^{n+1} \Lambda_{n,\ell} \mathbb{P}_\ell, \quad n \geqslant 1. \tag{53}$$

As a consequence of the analysis done in Section 3 (in particular see Eq. (34)), the polynomials $\frac{\partial}{\partial x} \mathbb{P}_n$ are orthogonal with respect to $\phi_1(x, y) \varrho(x, y)$, and the polynomials $\frac{\partial}{\partial y} \mathbb{P}_n$ are orthogonal with respect to $\phi_2(x, y) \varrho(x, y)$. If we multiply from the right the latter equation by $\mathbb{P}_n^{\mathrm{T}}$ and apply the corresponding orthogonality we get $\Lambda_{n,\ell} = 0$ for $\ell < n-1$.

The explicit expressions for the matrix coefficients are obtained from Eq. (50), by substituting $\mathbb{P}_n$ as given in (5), equating the coefficients of $\mathbf{x}^k$ for $k = n, n-1, n-2$ and solving the corresponding linear system. □

Observe that if we know the explicit expression of the polynomials $\phi_j(x, y)$, $j = 1, 2$, by using the results given in Section 3.1, it is possible to obtain explicitly the matrices $W_{n,j}$, $S_{n,j}$ and $T_{n,j}$ in (50) in the following way: substitute $\mathbb{P}_n$ as given in (5) equate the coefficients of $\mathbf{x}^k$ for $k = n, n-1, n-2$ and solve the corresponding linear system, by using (38), (40) and (42).

4.4. Derivative representations (or second structure relations)

Next we present finite-type relations between the orthogonal polynomial sequence $\{\mathbb{P}_n\}$ and the sequence of the partial derivatives $\{\frac{\partial}{\partial x_j}\mathbb{P}_n\}$.

Theorem 4.4. *Let $\{\mathbb{P}_n\}_{n\in\mathbb{N}_0}$ be a vector orthogonal polynomial family satisfying Proposition 2.5. Then, for $n \geqslant 2$ we have*

$$\mathbb{P}_n = V_{n,j}\frac{\partial}{\partial x_j}\mathbb{P}_{n+1} + Y_{n,j}\frac{\partial}{\partial x_j}\mathbb{P}_n + Z_{n,j}\frac{\partial}{\partial x_j}\mathbb{P}_{n-1}, \tag{54}$$

where the matrices $V_{n,j}$ of size $(n+1)\times(n+2)$, $Y_{n,j}$ of size $(n+1)\times(n+1)$, and $Z_{n,j}$ of size $(n+1)\times n$ are given by

$$V_{n,j} = A_{n,j} - A^{(j)}_{n-1,j},$$
$$Y_{n,j} = B_{n,j} - B^{(j)}_{n-1,j},$$
$$Z_{n,j} = C_{n,j} - C^{(j)}_{n-1,j},$$

and the matrices $A_{n,j}$, $B_{n,j}$ and $C_{n,j}$ are given in (45) and the matrices $A^{(j)}_{n-1,j}$, $B^{(j)}_{n-1,j}$ and $C^{(j)}_{n-1,j}$ are introduced in (49).

Proof. The above result is a consequence of (44) and (49). □

5. Explicit expressions for algebraic properties in the monic case

As pointed out in the introduction, in this section we give the explicit expression of the coefficients $\widehat{G}_{n,n-1}$ and $\widehat{G}_{n,n-2}$ in (6) in terms of the coefficients a, b_j, c_j, d_3, e, f_j fully characterizing the already mentioned partial differential equation (12). After that, results already given in the previous section will allow us to express (for monic polynomials) the three algebraic and differential properties here considered (44), (50) and (54), in terms of the admissible partial differential equation coefficients in (12).

Proposition 5.1. *Let $\widehat{\mathbb{P}}_n$ $(n \in \mathbb{N}_0)$ be a monic vector polynomial, as given by the expansion (6), solution of an admissible hypergeometric type partial differential equation of the form (12). Then, the matrix coefficients $\widehat{G}_{n,n-1} \in \mathcal{M}^{(n+1,n)}$ and $\widehat{G}_{n,n-2} \in \mathcal{M}^{(n+1,n-1)}$ in (6) can be written in term of the coefficients a, b_j, c_j, d_3, e, f_j in (12) as:*

$$\widehat{G}_{n,n-1} = \begin{pmatrix} \tilde{g}_{1,1} & & & & \bigcirc \\ \tilde{g}_{2,1} & \tilde{g}_{2,2} & & & \\ & \ddots & \ddots & & \\ & & \tilde{g}_{n-1,n-2} & \tilde{g}_{n-1,n-1} & \\ & & & \tilde{g}_{n,n-1} & \tilde{g}_{n,n} \\ & \bigcirc & & 0 & \tilde{g}_{n+1,n} \end{pmatrix} \quad (n \geqslant 1), \tag{55}$$

where, for $1 \leqslant i \leqslant n$,

$$\tilde{g}_{i,i} = \frac{(n+1-i)((n-i)b_1 + 2(i-1)c_3 + f_1)}{\varpi_{2n-2}},$$
$$\tilde{g}_{i+1,i} = \frac{i((i-1)b_2 + 2(n-i)b_3 + f_2)}{\varpi_{2n-2}},$$

and

$$\widehat{G}_{n,n-2} = \begin{pmatrix} g_{1,1} & & & \bigcirc \\ g_{2,1} & g_{2,2} & & \\ g_{3,1} & g_{3,2} & g_{3,3} & \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ & g_{n-1,n-3} & g_{n-1,n-2} & g_{n-1,n-1} \\ \bigcirc & & g_{n,n-2} & g_{n,n-1} \\ & & 0 & g_{n+1,n-1} \end{pmatrix} \quad (n \geqslant 2), \tag{56}$$

where, for $1 \leqslant i \leqslant n-1$,

$$
\begin{aligned}
g_{i,i} &= \frac{(n-i)(n+1-i)}{2\varpi_{2n-2}\varpi_{2n-3}}\big(\varpi_{2n-2}c_1 + \big((n-i)b_1 + 2(i-1)c_3 + f_1\big)\big((n-i-1)b_1 + 2(i-1)c_3 + f_1\big)\big),\\
g_{i+1,i} &= \frac{i(n-i)}{\varpi_{2n-2}\varpi_{2n-3}}\big(f_1 f_2 + d_3\varpi_{2n-2} + b_3\big(2\big(n-2+2(i-2)(n-i-1)\big)c_3\\
&\quad + (2n-2i-1)f_1\big) + (2i-1)c_3 f_2 + (i-1)b_2\big((2i-1)c_3 + f_1\big)\\
&\quad + (n-1-i)\big((i-1)b_2 + (2n-2i-1)b_3 + f_2\big)\big),\\
g_{i+2,i} &= \frac{i(i+1)}{2\varpi_{2n-2}\varpi_{2n-3}}\big(\varpi_{2n-2}c_2 + \big((i-1)b_2 + 2(n-i-1)b_3 + f_2\big)\big(ib_2 + 2(n-i-1)b_3 + f_2\big)\big).
\end{aligned}
$$

In all of these expressions $\varpi_n = na + e \neq 0$ *as already shown in Eq.* (13)*.*

Proof. Plug into Eq. (12) the expansion (6) and then make equal zero the coefficients of the column vector of monomials $\mathbf{x}^k$ (defined in (3)) for $k = n,\ n-1,\ n-2$. $\square$

Now, having in mind that in the monic case $\widehat{G}_{n,n} = I_{n+1}$, from these results and Theorem 4.2, we can deduce the following corollaries.

Corollary 5.2 *(Three-term recurrence relations). For monic polynomials and* $j = 1, 2$*, the coefficients of the three-term recurrence relation* (44) *are given in terms of the coefficients of the second-order partial differential equation* (12) *by*

$$A_{n,j} = L_{n,j}, \tag{57}$$

$$B_{0,1} = \left(-\frac{f_1}{e}\right), \qquad B_{0,2} = \left(-\frac{f_2}{e}\right), \tag{58}$$

$$B_{n,j} = \widehat{G}_{n,n-1}L_{n-1,j} - L_{n,j}\widehat{G}_{n+1,n}, \quad n \geqslant 1, \tag{59}$$

$$C_{1,1} = \begin{pmatrix} \frac{-c_1e^2+f_1(b_1e-af_1)}{e^2(a+e)} \\ \frac{-d_3e^2+b_3ef_1+c_3ef_2-af_1f_2}{e^2(a+e)} \end{pmatrix}, \qquad C_{1,2} = \begin{pmatrix} \frac{-d_3f_1^2+b_3ef_1+c_3ef_2-af_1f_2}{e^2(a+e)} \\ \frac{-c_2e^2+f_2(b_2e-af_2)}{e^2(a+e)} \end{pmatrix}, \tag{60}$$

$$C_{n,j} = \widehat{G}_{n,n-2}L_{n-2,j} - L_{n,j}\widehat{G}_{n+1,n-1} - B_{n,j}\widehat{G}_{n,n-1}, \quad n \geqslant 2. \tag{61}$$

It has some interest to remark here that, as described in [9], since

$$\operatorname{rank}(L_{n,j}) = n+1 = \operatorname{rank}(C_{n+1,j}), \quad j = 1, 2,\ n \geqslant 0, \tag{62}$$

the columns of the joint matrices

$$L_n = \big(L_{n,1}^T, L_{n,2}^T\big)^T \quad \text{and} \quad C_n = \big(C_{n,1}^T, C_{n,2}^T\big)^T$$

of size $(2n+2)\times(n+2)$ and $(2n+2)\times n$ respectively, are linearly independent, i.e.

$$\operatorname{rank}(L_n) = n+2, \qquad \operatorname{rank}(C_n) = n. \tag{63}$$

Therefore, the matrix L_n has full rank so that there exists a unique matrix $D_n^\dagger$ of size $(n+2)\times(2n+2)$, called the generalized inverse of L_n:

$$D_n^\dagger = (D_{n,1}|D_{n,2}) = \big(L_n^T L_n\big)^{-1}L_n^T, \tag{64}$$

such that

$$D_n^\dagger L_n = I_{n+2}.$$

Moreover, using the left inverse $D_n^\dagger$ of the joint matrix L_n

$$
D_n^\dagger = \begin{pmatrix}
1 & & & & 0 & & & \\
& 1/2 & & \bigcirc & 1/2 & & \bigcirc & \\
& & \ddots & & & \ddots & & \\
r\ \bigcirc & & & 1/2 & 0 & & 1/2 & \\
& & & & 0 & & & 1
\end{pmatrix},
$$

we can write a recursive formula for the monic orthogonal polynomials

$$\widehat{\mathbb{P}}_{n+1} = D_n^{\dagger}\left[\begin{pmatrix} x \\ y \end{pmatrix} \otimes I_{n+1} - B_n\right]\widehat{\mathbb{P}}_n - D_n^{\dagger} C_n \widehat{\mathbb{P}}_{n-1}, \quad n \geqslant 0, \tag{65}$$

with the initial conditions $\widehat{\mathbb{P}}_{-1} = 0$, $\widehat{\mathbb{P}}_0 = 1$, where $\otimes$ denotes the Kronecker product and

$$B_n = \left(B_{n,1}^T, B_{n,2}^T\right)^T, \qquad C_n = \left(C_{n,1}^T, C_{n,2}^T\right)^T \tag{66}$$

are matrices of size $(2n+2) \times (n+1)$ and $(2n+2) \times n$, respectively. This recurrence (65) gives another representation of [9, (3.2.10)], already presented in the bivariate discrete case in [29].

Corollary 5.3 *(Structure relations). For monic polynomials, $n \geqslant 3$ and $j = 1, 2$, with the notation* (52) *for $\phi_j(x, y)$ given in* (51)*, the coefficients of the structure relations* (50) *are given in terms of the coefficients of the second-order partial differential equation* (12) *by*

$$W_{n,j} = \mathbb{E}_{n,j}(\alpha_j L_{n-1,1} L_{n,1} + \beta_j L_{n-1,2} L_{n,1} + \gamma_j L_{n-1,2} L_{n,2}), \tag{67}$$

$$\begin{aligned} S_{n,j} = {} & \mathbb{E}_{n,j}(\delta_j L_{n-1,1} + \varepsilon_j L_{n-1,2}) - W_{n,j}\widehat{G}_{n+1,n} \\ & + \widehat{G}_{n,n-1}\mathbb{E}_{n-1,j}(\alpha_j L_{n-2,1} L_{n-1,1} + \beta_j L_{n-2,2} L_{n-1,1} + \gamma_j L_{n-2,2} L_{n-1,2}), \end{aligned} \tag{68}$$

$$\begin{aligned} T_{n,j} = {} & \omega_j E_{n,j} + \widehat{G}_{n,n-1}\mathbb{E}_{n-1,j}(\delta_j L_{n-2,1} + \varepsilon_j L_{n-2,2}) - W_{n,j}\widehat{G}_{n+1,n-1} - S_{n,j}\widehat{G}_{n,n-1} \\ & + \widehat{G}_{n,n-2}\mathbb{E}_{n-2,j}(\alpha_j L_{n-3,1} L_{n-2,1} + \beta_j L_{n-3,2} L_{n-2,1} + \gamma_j L_{n-3,2} L_{n-2,2}). \end{aligned} \tag{69}$$

Corollary 5.4 *(Derivative representations). For monic polynomials, $n \geqslant 2$ and $j = 1, 2$, the coefficients of the derivative representations* (54) *are given in terms of the coefficients of the second-order partial differential equation* (12) *by*

$$V_{n,j} = (L_{n,j}\mathbb{E}_{n+1,j})^{-1}, \tag{70}$$

$$Y_{n,j} = (\widehat{G}_{n,n-1} - V_{n,j} L_{n,j} \widehat{G}_{n+1,n}\mathbb{E}_{n,j}) V_{n-1,j}, \tag{71}$$

$$Z_{n,j} = (\widehat{G}_{n,n-2} - V_{n,j} L_{n,j} \widehat{G}_{n+1,n-1}\mathbb{E}_{n-1,j} - Y_{n,j} L_{n-1,j}\widehat{G}_{n,n-1}\mathbb{E}_{n-1,j}) V_{n-2,j}. \tag{72}$$

The results obtained in Sections 4 and 5 can be applied to any polynomial solution of an admissible partial differential equation of hypergeometric type.

6. Example: Monic Appell polynomials

In order to illustrate how the results presented in the previous sections work in practice, we have chosen a partial differential equation having the well-known two-parameter monic Appell polynomials [4] as one of its solutions. For these polynomials, we shall give explicitly the matrices appearing in the three-term recurrence relations and structure relations. Of course, other examples of bivariate orthogonal polynomials can be treated in a similar way.

In 1882, Appell [4] introduced a three parameter family of polynomials of degree $n+m$ in terms of generalized Kampé de Fériet hypergeometric series [32] which in the monic case and reduced to two parameters can be written as [11, Eq. (12), p. 271]

$$\widehat{\mathrm{A}}_{n,m}^{(\alpha,\beta)}(x, y) = (-1)^{n+m} \frac{(\alpha)_n(\beta)_m}{(\alpha+\beta+n+m)_{n+m}} F^{1:1;1}_{0:1;1}\left(\begin{matrix} \alpha+\beta+n+m : -n; -m \\ - : \alpha; \beta \end{matrix} \,\middle|\, x, y \right), \tag{73}$$

where $\alpha > 0$ and $\beta > 0$ and $(\alpha)_n$ denotes the Pochhammer symbol.

6.1. Second-order partial differential equation

The admissible partial differential equation of hypergeometric type satisfied by $\widehat{\mathrm{A}}_{n,m}^{(\alpha,\beta)}(x, y)$ is [11, Eq. (15), p. 272]

$$\begin{aligned} & x(1-x)\frac{\partial^2 \widehat{\mathrm{A}}_{n,m}^{(\alpha,\beta)}}{\partial x^2} - 2xy\frac{\partial^2 \widehat{\mathrm{A}}_{n,m}^{(\alpha,\beta)}}{\partial x \partial y} + y(1-y)\frac{\partial^2 \widehat{\mathrm{A}}_{n,m}^{(\alpha,\beta)}}{\partial y^2} + \big(\alpha - (\alpha+\beta+1)x\big)\frac{\partial \widehat{\mathrm{A}}_{n,m}^{(\alpha,\beta)}}{\partial x} \\ & \quad + \big(\beta - (\alpha+\beta+1)y\big)\frac{\partial \widehat{\mathrm{A}}_{n,m}^{(\alpha,\beta)}}{\partial y} + (n+m)(\alpha+\beta+n+m)\widehat{\mathrm{A}}_{n,m}^{(\alpha,\beta)} = 0. \end{aligned} \tag{74}$$

From (24) we obtain that the partial derivatives

$$z^{(r,s)}(x, y) = \frac{\partial^{r+s}}{\partial x^r \partial y^s}\widehat{\mathrm{A}}_{n,m}^{(\alpha,\beta)}(x, y)$$

satisfy the admissible partial differential equation of hypergeometric type

$$\begin{aligned} x(1-x)\frac{\partial^2}{\partial x^2}z^{(r,s)}(x,y) - 2xy\frac{\partial^2}{\partial x\partial y}z^{(r,s)}(x,y) + y(1-y)\frac{\partial^2}{\partial y^2}z^{(r,s)}(x,y) \\ + \bigl(\alpha + r - \bigl(\alpha+\beta+1+2(r+s)\bigr)x\bigr)\frac{\partial}{\partial x}z^{(r,s)}(x,y) \\ + \bigl(\beta + s - \bigl(\alpha+\beta+1+2(r+s)\bigr)y\bigr)\frac{\partial}{\partial y}z^{(r,s)}(x,y) \\ + (n+m-r-s)(\alpha+\beta+n+m+r+s)z^{(r,s)}(x,y) = 0. \end{aligned}$$

It is easy to check that, from the above differential equation, the matrices $\widehat{G}_{n,n-1}$ and $\widehat{G}_{n,n-2}$ in the expansion

$$\begin{aligned} \widehat{\mathbb{A}}_n = \widehat{\mathbb{A}}_n^{(\alpha,\beta)}(x,y) = \bigl(\widehat{\mathrm{A}}_{n,0}^{(\alpha,\beta)}(x,y), \dots, \widehat{\mathrm{A}}_{n-i,i}^{(\alpha,\beta)}(x,y), \dots, \widehat{\mathrm{A}}_{0,n}^{(\alpha,\beta)}(x,y)\bigr)^{\mathrm{T}} \\ = \mathbf{x}^n + \widehat{G}_{n,n-1}\mathbf{x}^{n-1} + \dots + \widehat{G}_{n,0}\mathbf{x}^0 \end{aligned} \tag{75}$$

are explicitly given from the general expressions (55) and (56).

6.2. Orthogonality

From (19) we obtain the following weight function

$$\varrho(x,y) = x^{\alpha-1}y^{\beta-1}. \tag{76}$$

In this case the polynomial $\alpha(x,y)$ defined in (14) is given by $\alpha(x,y) = xy(1-x-y)$. From the positivity of $\varrho(x,y)$ and (15) it yields the following triangular domain

$$\mathcal{R} = \bigl\{(x,y)\colon x>0,\ y>0,\ x+y<1\bigr\}. \tag{77}$$

Therefore, the orthogonality property reads as

$$\iint_{\mathcal{R}} \widehat{\mathbb{A}}_n\widehat{\mathbb{A}}_m^T\varrho(x,y)\,dx\,dy = \Lambda_n\delta_{n,m}. \tag{78}$$

6.3. Three-term recurrence relations

For $n \geqslant 0$, monic Appell polynomials satisfy the three-term recurrence relations

$$\begin{aligned} x\widehat{\mathbb{A}}_n = L_{n,1}\widehat{\mathbb{A}}_{n+1} + B_{n,1}\widehat{\mathbb{A}}_n + C_{n,1}\widehat{\mathbb{A}}_{n-1}, \\ y\widehat{\mathbb{A}}_n = L_{n,2}\widehat{\mathbb{A}}_{n+1} + B_{n,2}\widehat{\mathbb{A}}_n + C_{n,2}\widehat{\mathbb{A}}_{n-1}, \end{aligned}$$

with the initial conditions $\widehat{\mathbb{A}}_0 = 1$ and $\widehat{\mathbb{A}}_{-1} = 0$, where $\widehat{\mathbb{A}}_n$ is defined in (75) and $L_{n,j}$ are defined in (39). Using (58) and (59) the recursion coefficients $B_{n,j}$ are given by

$$B_{n,1} = \begin{pmatrix} b_{0,0} & 0 & & & \bigcirc \\ b_{1,0} & b_{1,1} & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{n-1,n-2} & b_{n-1,n-1} & 0 \\ \bigcirc & & & b_{n,n-1} & b_{n,n} \end{pmatrix}, \tag{79}$$

where

$$\begin{aligned} b_{i,i} &= -\frac{(n-i)(\alpha+n-1-i)}{2n-1+\alpha+\beta} + \frac{(n+1-i)(\alpha+n-i)}{2n+1+\alpha+\beta}, \quad 0 \leqslant i \leqslant n, \\ b_{i+1,i} &= -\frac{2(i+1)(\beta+i)}{(2n-1+\alpha+\beta)(2n+1+\alpha+\beta)}, \quad 0 \leqslant i \leqslant n-1, \end{aligned}$$

and

$$B_{n,2} = \begin{pmatrix} \tilde{b}_{0,0} & \tilde{b}_{0,1} & & & \bigcirc \\ 0 & \tilde{b}_{1,1} & \tilde{b}_{1,2} & & \\ & \ddots & \ddots & \ddots & \\ & & & \tilde{b}_{n-1,n-1} & \tilde{b}_{n-1,n} \\ \bigcirc & & & 0 & \tilde{b}_{n,n} \end{pmatrix}, \tag{80}$$

with

$$\tilde{b}_{i,i} = 1 + \frac{i(2n-i+\alpha)}{2n-1+\alpha+\beta} - \frac{(i+1)(\alpha+2n+1-i)}{2n+1+\alpha+\beta}, \quad 0 \leqslant i \leqslant n,$$

$$\tilde{b}_{i,i+1} = -\frac{2(n-i)(\alpha+n-1-i)}{(2n-1+\alpha+\beta)(2n+1+\alpha+\beta)}, \quad 0 \leqslant i \leqslant n-1.$$

Moreover, using (60) and (61) we have

$$C_{n,1} = \begin{pmatrix} c_{0,0} & & & & \bigcirc \\ c_{1,0} & c_{1,1} & & & \\ c_{2,0} & c_{2,1} & c_{2,2} & & \\ \ddots & \ddots & \ddots & & \\ \bigcirc & c_{n-1,n-3} & c_{n-1,n-2} & c_{n-1,n-1} \\ & & c_{n,n-2} & c_{n,n-1} \end{pmatrix}, \tag{81}$$

where

$$c_{i,i} = \frac{(n-i)(\alpha+n-1-i)(n+i+\beta)(n-1+i+\alpha+\beta)}{(2n+\alpha+\beta)(2n-1+\alpha+\beta)^2(2n-2+\alpha+\beta)}, \quad 0 \leqslant i \leqslant n-1,$$

$$c_{i+1,i} = -\frac{(i+1)(\beta+i)(2(n-i-1)(n+i+\beta)+\alpha(2n+\alpha+\beta-2))}{(2n+\alpha+\beta)(2n-1+\alpha+\beta)^2(2n-2+\alpha+\beta)}, \quad 0 \leqslant i \leqslant n-1,$$

$$c_{i+2,i} = \frac{(i+2)(i+1)(\beta+i)(\beta+i+1)}{(2n+\alpha+\beta)(2n-1+\alpha+\beta)^2(2n-2+\alpha+\beta)}, \quad 0 \leqslant i \leqslant n-2,$$

and

$$C_{n,2} = \begin{pmatrix} \tilde{c}_{0,0} & \tilde{c}_{0,1} & & & \bigcirc \\ \tilde{c}_{1,0} & \tilde{c}_{1,1} & \tilde{c}_{1,2} & & \\ & \ddots & \ddots & \ddots & \\ & \tilde{c}_{n-2,n-3} & \tilde{c}_{n-2,n-2} & \tilde{c}_{n-2,n-1} & \\ & & \tilde{c}_{n-1,n-2} & \tilde{c}_{n-1,n-1} & \\ & \bigcirc & & \tilde{c}_{n,n-1} & \end{pmatrix}, \tag{82}$$

with

$$\tilde{c}_{i,i} = -\frac{(n-i)(\alpha+n-1-i)(\beta(2n-2+\beta)+\alpha(2i+\beta)+2i(2n-1-i))}{(2n+\alpha+\beta)(2n-1+\alpha+\beta)^2(2n-2+\alpha+\beta)},$$

$$\tilde{c}_{i+1,i} = \frac{(i+1)(\alpha+2n-1-i)(\beta+i)(\alpha+\beta+2n-2-i)}{(2n+\alpha+\beta)(2n-1+\alpha+\beta)^2(2n-2+\alpha+\beta)},$$

for $0 \leqslant i \leqslant n-1$ and

$$\tilde{c}_{i,i+1} = \frac{(n-i)(n-1-i)(\alpha+n-1-i)(\alpha+n-2-i)}{(2n+\alpha+\beta)(2n-1+\alpha+\beta)^2(2n-2+\alpha+\beta)},$$

for $0 \leqslant i \leqslant n-2$.

6.4. First structure relations

The partial differential equation (74) for monic Appell polynomials corresponds to cases (vi), (ix) and (x) in Section 3.1. Therefore, we obtain that

$$\phi^{(r,s)}(x,y) = \left[x(1-x-y)\right]^r \left[y(1-x-y)\right]^s. \tag{83}$$

As a consequence, the structure relations (50) satisfied by monic Appell polynomials defined in (75) are given by

$$x(1-x-y)\frac{\partial}{\partial x}\widehat{\mathbb{A}}_n = W_{n,1}\widehat{\mathbb{A}}_{n+1} + S_{n,1}\widehat{\mathbb{A}}_n + T_{n,1}\widehat{\mathbb{A}}_{n-1}, \tag{84}$$

$$y(1-x-y)\frac{\partial}{\partial y}\widehat{\mathbb{A}}_n = W_{n,2}\widehat{\mathbb{A}}_{n+1} + S_{n,2}\widehat{\mathbb{A}}_n + T_{n,2}\widehat{\mathbb{A}}_{n-1}, \tag{85}$$

for $n \geqslant 1$, where using (67) we get

$$W_{n,1}=\begin{pmatrix} w_{0,0} & w_{0,1} & 0 & & \bigcirc \\ 0 & w_{1,1} & w_{1,2} & & \\ & \ddots & \ddots & & \\ & & w_{n-1,n-1} & w_{n-1,n} & 0 \\ & \bigcirc & & w_{n,n} & w_{n,n+1} \end{pmatrix}, \tag{86}$$

with $w_{i,i}=-n+i=w_{i,i+1}$, $0\leqslant i\leqslant n$, and

$$W_{n,2}=\begin{pmatrix} \tilde{w}_{0,0} & \tilde{w}_{0,1} & 0 & & \bigcirc \\ 0 & \tilde{w}_{1,1} & \tilde{w}_{1,2} & & \\ & & \ddots & \ddots & \\ & & \tilde{w}_{n-1,n-1} & \tilde{w}_{n-1,n} & 0 \\ & \bigcirc & & \tilde{w}_{n,n} & \tilde{w}_{n,n+1} \end{pmatrix}, \tag{87}$$

with $\tilde{w}_{i,i}=-i=\tilde{w}_{i,i+1}$, $0\leqslant i\leqslant n$.

Moreover, from (68) we obtain

$$S_{n,1}=\begin{pmatrix} s_{0,0} & s_{0,1} & & \bigcirc \\ s_{1,0} & s_{1,1} & s_{1,2} & \\ & \ddots & \ddots & \\ & s_{n-1,n-2} & s_{n-1,n-1} & s_{n-1,n} \\ & \bigcirc & 0 & 0 \end{pmatrix}, \tag{88}$$

with

$$s_{i,i}=-\frac{(n-i)(-n+(2n-1)i-4i^2+(n-2-3i)\beta+\alpha(n-1+i+\alpha+\beta))}{(2n-1+\alpha+\beta)(2n+1+\alpha+\beta)},$$

$$s_{i,i+1}=-\frac{(n-i)(n-1-i+\alpha)(2i+1+\alpha+\beta)}{(2n-1+\alpha+\beta)(2n+1+\alpha+\beta)},$$

for $0\leqslant i\leqslant n-1$, and

$$s_{i+1,i}=\frac{2(i+1)(n-1-i)(\beta+i)}{(2n-1+\alpha+\beta)(2n+1+\alpha+\beta)},$$

for $0\leqslant i\leqslant n-2$. Also, using (68) we have

$$S_{n,2}=\begin{pmatrix} 0 & 0 & & & \bigcirc \\ \tilde{s}_{1,0} & \tilde{s}_{1,1} & \tilde{s}_{1,2} & & \\ 0 & \tilde{s}_{2,1} & \tilde{s}_{2,2} & & \\ & \ddots & \ddots & \ddots & \\ & & \tilde{s}_{n-1,n-2} & \tilde{s}_{n-1,n-1} & \tilde{s}_{n-1,n} \\ & \bigcirc & 0 & \tilde{s}_{n,n-1} & \tilde{s}_{n,n} \end{pmatrix}, \tag{89}$$

with

$$\tilde{s}_{i,i}=\frac{i(\beta-\beta^2-i+\beta i+4i^2-\alpha(-2+\beta+3i-2n)-2(-1+\beta+3i)n+2n^2)}{(2n+1+\alpha+\beta)(2n-1+\alpha+\beta)},$$

for $1\leqslant i\leqslant n$, and

$$\tilde{s}_{i+1,i}=-\frac{(1+i)(\beta+i)(-1+\alpha+\beta-2i+2n)}{(-1+\alpha+\beta+2n)(1+\alpha+\beta+2n)},$$

$$\tilde{s}_{i,i+1}=\frac{2i(-i+n)(-1+\alpha-i+n)}{(-1+\alpha+\beta+2n)(1+\alpha+\beta+2n)},$$

for $0\leqslant i\leqslant n-1$. Furthermore, from (69) it holds

$$T_{n,1}=\begin{pmatrix} t_{0,0} & t_{0,1} & & & \bigcirc \\ t_{1,0} & t_{1,1} & t_{1,2} & & \\ t_{2,0} & t_{2,1} & t_{2,2} & t_{2,3} & \\ & \ddots & \ddots & \ddots & \\ & t_{n-2,n-4} & t_{n-2,n-3} & t_{n-2,n-2} & t_{n-2,n-1} \\ & & t_{n-1,n-3} & t_{n-1,n-2} & t_{n-1,n-1} \\ & \bigcirc & & 0 & 0 \end{pmatrix}, \tag{90}$$

where

$$\begin{aligned}
t_{i,i} &= \frac{(n-i)(n-1+\alpha-i)}{(-2+\alpha+\beta+2n)(-1+\alpha+\beta+2n)^2(\alpha+\beta+2n)} \\
&\quad \times \big(\beta^2(1+i)+i^2(1+3i)+\alpha\beta(1+n)+\alpha^2(-i+n) \\
&\quad + \beta\big(i(3+4i)+n(-2i+n)\big)+n\big(-\big((-2+i)i\big)+n(-1-i+n)\big) \\
&\quad + \alpha\big(i(2+i)+n(-1-2i+2n)\big)\big), \quad 0 \leqslant i \leqslant n-1, \\
t_{i,i+1} &= \frac{(n-i)(n-i-1)(\alpha+n-2-i)(\alpha+n-i-1)(\alpha+\beta+n+i)}{(2n+\alpha+\beta)(2n-1+\alpha+\beta)^2(2n-2+\alpha+\beta)}, \quad 0 \leqslant i \leqslant n-2, \\
t_{i+1,i} &= \frac{(\beta+i)(n-i-1)(i+1)}{(2n+\alpha+\beta)(2n-1+\alpha+\beta)^2(2n-2+\alpha+\beta)} \\
&\quad \times \big(\alpha(\alpha+\beta+n+i-1)+\beta(n-2i-3)+\big(-2+(2n-5)i-3i^2\big)\big), \quad 0 \leqslant i \leqslant n-2, \\
t_{i+2,i} &= -\frac{(\beta+i)(\beta+i+1)(n-i-2)(i+1)(i+2)}{(2n+\alpha+\beta)(2n-1+\alpha+\beta)^2(2n-2+\alpha+\beta)}, \quad 0 \leqslant i \leqslant n-3,
\end{aligned}$$

and using (69)

$$T_{n,2} = \begin{pmatrix}
0 & 0 & & & \bigcirc \\
\tilde{t}_{1,0} & \tilde{t}_{1,1} & \tilde{t}_{1,2} & & \\
\tilde{t}_{2,0} & \tilde{t}_{2,1} & \tilde{t}_{2,2} & \tilde{t}_{2,3} & \\
 & \ddots & \ddots & \ddots & \\
 & \tilde{t}_{n-2,n-4} & \tilde{t}_{n-2,n-3} & \tilde{t}_{n-2,n-2} & \tilde{t}_{n-2,n-1} \\
 & & \tilde{t}_{n-1,n-3} & \tilde{t}_{n-1,n-2} & \tilde{t}_{n-1,n-1} \\
 & \bigcirc & & \tilde{t}_{n,n-2} & \tilde{t}_{n,n-1}
\end{pmatrix}, \tag{91}$$

where

$$\begin{aligned}
\tilde{t}_{i,i} &= \frac{i(-i+n)(-1+\alpha-i+n)}{(-2+\alpha+\beta+2n)(-1+\alpha+\beta+2n)^2(\alpha+\beta+2n)} \\
&\quad \times \big(\alpha\beta+\beta^2-i(1+3i-4n)-\beta(2+i-2n)+\alpha(-1+2i-n)-n(1+n)\big), \quad 1 \leqslant i \leqslant n-1, \\
\tilde{t}_{i,i+1} &= -\frac{i(-1-i+n)(-i+n)(-2+\alpha-i+n)(-1+\alpha-i+n)}{(-2+\alpha+\beta+2n)(-1+\alpha+\beta+2n)^2(\alpha+\beta+2n)}, \quad 1 \leqslant i \leqslant n-2, \\
\tilde{t}_{i+1,i} &= \frac{(1+i)(\beta+i)}{(-2+\alpha+\beta+2n)(-1+\alpha+\beta+2n)^2(\alpha+\beta+2n)} \\
&\quad \times \big(-3(1+i)^3+(1+n)(\alpha+n)(\alpha+\beta+2n)+(1+i)^2(1+4\alpha+\beta+8n) \\
&\quad - (1+i)\big(\alpha(3+\alpha)-(-2+\beta)\beta+4n+6\alpha n+6n^2\big)\big), \quad 0 \leqslant i \leqslant n-1, \\
\tilde{t}_{i+2,i} &= \frac{(1+i)(2+i)(\beta+i)(1+\beta+i)(-2+\alpha+\beta-i+2n)}{(-2+\alpha+\beta+2n)(-1+\alpha+\beta+2n)^2(\alpha+\beta+2n)}, \quad 0 \leqslant i \leqslant n-2.
\end{aligned}$$

6.5. Derivative representations or second structure relations

The monic Appell polynomials defined in (75) satisfy the derivative representations

$$\widehat{\mathbb{A}}_n = V_{n,1}\mathbb{Q}_n^{(j)} + Y_{n,1}\mathbb{Q}_{n-1}^{(j)} + Z_{n,1}\mathbb{Q}_{n-2}^{(j)}, \quad n \geqslant 2, \ j=1,2, \tag{92}$$

where $\mathbb{Q}_n^{(1)} = L_{n,1}\frac{\partial}{\partial x}\widehat{\mathbb{A}}_{n+1}$ and $\mathbb{Q}_n^{(2)} = L_{n,2}\frac{\partial}{\partial y}\widehat{\mathbb{A}}_{n+1}$. In this example the matrices $V_{n,j}$ defined in (70) are given by

$$V_{n,1} = \begin{pmatrix}
v_{0,0} & & & & \bigcirc \\
 & v_{1,1} & & & \\
 & & \ddots & & \\
 & & & v_{n-1,n-1} & \\
 & \bigcirc & & & v_{n,n}
\end{pmatrix}, \quad \text{with } v_{i,i} = \frac{1}{n+1-i}, \ 0 \leqslant i \leqslant n, \tag{93}$$

and

$$V_{n,2}=\begin{pmatrix} \tilde{v}_{0,0} & & & & \bigcirc \\ & \tilde{v}_{1,1} & & & \\ & & \ddots & & \\ & & & \tilde{v}_{n-1,n-1} & \\ & \bigcirc & & 0 & \tilde{v}_{n,n} \end{pmatrix}, \quad \text{with } \tilde{v}_{i,i}=\frac{1}{i+1},\ 0\leqslant i\leqslant n. \tag{94}$$

Moreover, from (71) we have

$$Y_{n,1}=\begin{pmatrix} y_{0,0} & & & & \bigcirc \\ y_{1,0} & y_{1,1} & & & \\ & \ddots & \ddots & & \\ & & y_{n-1,n-2} & y_{n-1,n-1} & \\ & \bigcirc & 0 & y_{n,n-1} & \end{pmatrix}, \tag{95}$$

where

$$\begin{aligned} y_{i,i}&=\frac{2i+1-\alpha+\beta}{(2n+1+\alpha+\beta)(2n-1+\alpha+\beta)},\\ y_{i+1,i}&=-\frac{2(i+1)(\beta+i)}{(n-i)(2n+1+\alpha+\beta)(2n-1+\alpha+\beta)}, \quad 0\leqslant i\leqslant n-1, \end{aligned}$$

and

$$Y_{n,2}=\begin{pmatrix} \tilde{y}_{0,0} & & & & \bigcirc \\ \tilde{y}_{1,0} & \tilde{y}_{1,1} & & & \\ & \ddots & \ddots & & \\ & & \tilde{y}_{n-2,n-3} & \tilde{y}_{n-2,n-2} & \\ & & & \tilde{y}_{n-1,n-2} & \tilde{y}_{n-1,n-1} \\ & \bigcirc & & 0 & \tilde{y}_{n,n-1} \end{pmatrix}, \tag{96}$$

where

$$\begin{aligned} \tilde{y}_{i,i}&=-\frac{2(n-i)(n-1-i+\alpha)}{(1+i)(2n+1+\alpha+\beta)(2n-1+\alpha+\beta)},\\ \tilde{y}_{i+1,i}&=\frac{2n-1-2i+\alpha-\beta}{(2n+1+\alpha+\beta)(2n-1+\alpha+\beta)}, \quad 0\leqslant i\leqslant n-1. \end{aligned}$$

Also, from (72) we have

$$Z_{n,1}=\begin{pmatrix} z_{0,0} & & & & \bigcirc \\ z_{1,0} & z_{1,1} & & & \\ z_{2,0} & z_{2,1} & z_{2,2} & & \\ \ddots & \ddots & \ddots & & \\ & z_{n-2,n-4} & z_{n-2,n-3} & z_{n-2,n-2} & \\ & & z_{n-1,n-3} & z_{n-1,n-2} & \\ & \bigcirc & 0 & z_{n,n-2} & \end{pmatrix}, \tag{97}$$

where for $0\leqslant i\leqslant n-2$,

$$\begin{aligned} z_{i,i}&=-\frac{(n-i)(n-1-i+\alpha)(n+i+\beta)}{(2n+\alpha+\beta)(2n-1+\alpha+\beta)^2(2n-2+\alpha+\beta)},\\ z_{i+1,i}&=\frac{(i+1)(-2(i+1)+\alpha-\beta)(\beta+i)}{(2n+\alpha+\beta)(2n-1+\alpha+\beta)^2(2n-2+\alpha+\beta)},\\ z_{i+2,i}&=\frac{(i+1)(i+2)(\beta+i)(\beta+i+1)}{(n-1-i)(2n+\alpha+\beta)(2n-1+\alpha+\beta)^2(2n-2+\alpha+\beta)}, \end{aligned}$$

and

$$Z_{n,2}=\begin{pmatrix} \tilde{z}_{0,0} & & & \bigcirc \\ \tilde{z}_{1,0} & \tilde{z}_{1,1} & & \\ \tilde{z}_{2,0} & \tilde{z}_{2,1} & \tilde{z}_{2,2} & \\ \ddots & \ddots & \ddots & \\ & \tilde{z}_{n-2,n-4} & \tilde{z}_{n-2,n-3} & \tilde{z}_{n-2,n-2} \\ & & \tilde{z}_{n-1,n-3} & \tilde{z}_{n-1,n-2} \\ & \bigcirc & 0 & \tilde{z}_{n,n-2} \end{pmatrix}, \tag{98}$$

where for $0 \leqslant i \leqslant n-2$,

$$\tilde{z}_{i,i}=\frac{(-1-i+n)(-i+n)(-2+\alpha-i+n)(-1+\alpha-i+n)}{(1+i)(-2+\alpha+\beta+2n)(-1+\alpha+\beta+2n)^2(\alpha+\beta+2n)},$$

$$\tilde{z}_{i+1,i}=-\left(\frac{(-1-i+n)(-2+\alpha-i+n)(\alpha-\beta+2(-1-i+n))}{(-2+\alpha+\beta+2n)(-1+\alpha+\beta+2n)^2(\alpha+\beta+2n)}\right),$$

$$\tilde{z}_{i+2,i}=-\frac{(2+i)(1+\beta+i)(-2+\alpha-i+2n)}{(-2+\alpha+\beta+2n)(-1+\alpha+\beta+2n)^2(\alpha+\beta+2n)}.$$

6.6. Non-monic orthogonal solutions of (74)

For any non-monic orthogonal polynomial solution of the partial differential equation (74) it is possible to obtain the main differential and algebraic properties by using the results given in Section 4. In this section we give some relations for two concrete non-monic solutions of (74), orthogonal with respect to (76) in the domain (77).

On one hand, also in 1882, Appell considered a family of non-monic polynomials solution of the partial differential equation (74). This orthogonal family can be obtained from the Rodrigues formula (36) (see [11, Eq. (11), p. 271]) using the weight (83)

$$F_{n,m}^{(\alpha,\beta)}(x,y)=\frac{x^{1-\alpha}y^{1-\beta}}{(\alpha)_n(\beta)_m}\frac{\partial^{n+m}}{\partial x^n\partial y^m}\left[x^{n+\alpha-1}y^{m+\beta-1}(1-x-y)^{n+m}\right]. \tag{99}$$

These polynomials can also be obtained from the classical Appell's orthogonal polynomials defined in [33, Eq. (6), p. 63] by taking $\gamma=\alpha+\beta$.

Clearly, both, monic Appell polynomials defined in (73) and the non-monic family (99), form a biorthogonal system in the domain (77) with respect to the weight function (76) [11, Eq. (17), p. 272]

$$\iint_{\mathcal{R}} x^{\alpha-1}y^{\beta-1}F_{n,m}^{(\alpha,\beta)}(x,y)\widehat{\mathrm{A}}_{k,l}^{(\alpha,\beta)}(x,y)\,dx\,dy=\delta_{nk}\delta_{ml}\Lambda_{n,m},\quad \alpha,\beta>0.$$

Then, if we denote

$$\begin{aligned}\mathbb{F}_n=\mathbb{F}_n^{(\alpha,\beta)}(x,y)&=\left(\mathrm{F}_{n,0}^{(\alpha,\beta)}(x,y),\dots,\mathrm{F}_{n-i,i}^{(\alpha,\beta)}(x,y),\dots,\mathrm{F}_{0,n}^{(\alpha,\beta)}(x,y)\right)^{\mathrm{T}}\\&=G_{n,n}^F\mathbf{x}^n+G_{n,n-1}^F\mathbf{x}^{n-1}+G_{n,n-2}^F\mathbf{x}^{n-2}+\cdots+G_{n,0}^F\mathbf{x}^0,\end{aligned} \tag{100}$$

we have the following formula linking both solutions in column polynomial vector form

$$\mathbb{F}_n=G_{n,n}^F\widehat{\mathbb{A}}_n, \tag{101}$$

where $\widehat{\mathbb{A}}_n$ is defined in (75) and the entries of the matrix $G_{n,n}^F=(g_{i,j}^F(n))$ of size $(n+1)\times(n+1)$ have the following explicit form

$$g_{i,j}^F(n)=(-1)^n\binom{n}{j}\frac{(\alpha+n-i)_{n-j}(\beta+i)_j}{(\alpha)_{n-j}(\beta)_j},\quad 0\leqslant i,j\leqslant n.$$

Once the matrix $G_{n,n}^F$ is known, applying the formulae given in Section 4 it is possible to obtain the coefficients of the thee-term recurrence relations, structure relations and derivative representations for this non-monic family (99).

On the other hand, let us consider the non-monic family defined in [9, p. 86] with $k\to m$, $\alpha\to\alpha-1/2$, $\beta\to\beta-1/2$ and $\gamma=1/2$:

$$K_{n,m}^{(\alpha,\beta)}(x,y)=P_n^{(2m+\beta,\alpha-1)}(2x-1)(1-x)^mP_m^{(0,\beta-1)}\left(\frac{2y}{1-x}-1\right),\quad \alpha,\beta>0, \tag{102}$$

where $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials [14]. This family was previously considered by Proriol in [26]. As already mentioned, this family is another polynomial solution of the partial differential equation (74) orthogonal with respect to (76) in the domain (77). If we denote

$$\mathbb{K}_n = \mathbb{K}_n^{(\alpha,\beta)}(x,y) = \left(\mathrm{K}_{n,0}^{(\alpha,\beta)}(x,y), \ldots, \mathrm{K}_{n-i,i}^{(\alpha,\beta)}(x,y), \ldots, \mathrm{K}_{0,n}^{(\alpha,\beta)}(x,y)\right)^{\mathrm{T}}$$
$$= G_{n,n}^K \mathbf{x}^n + G_{n,n-1}^K \mathbf{x}^{n-1} + G_{n,n-2}^K \mathbf{x}^{n-2} + \cdots + G_{n,0}^K \mathbf{x}^0,$$

then we have

$$\mathbb{K}_n = G_{n,n}^K \widehat{\mathbb{A}}_n, \tag{103}$$

where the entries of the matrix $G_{n,n}^K = (g_{i,j}^K(n))$ of size $(n+1) \times (n+1)$ have the following explicit form

$$g_{i,j}^K(n) = \begin{cases} 0, & i < j, \\ \frac{(\alpha+\beta+n+i)_{n-i}(\beta+j)_i}{(n-i)!j!(i-j)!}, & i \geqslant j, \end{cases}$$

for $0 \leqslant i, j \leqslant n$.

Relations (101) and (103) between monic (73) and non-monic (99) Appell polynomials, and between monic Appell (73) and the polynomials defined in (102) solve the following connection problems

$$F_{n-\ell,\ell}^{(\alpha,\beta)}(x,y) = \sum_{j=0}^{n} g_{\ell,j}^F(n) \widehat{\mathrm{A}}_{n-j,j}^{(\alpha,\beta)}(x,y), \quad 0 \leqslant \ell \leqslant n,$$
$$K_{n-\ell,\ell}^{(\alpha,\beta)}(x,y) = \sum_{j=0}^{n} g_{\ell,j}^K(n) \widehat{\mathrm{A}}_{n-j,j}^{(\alpha,\beta)}(x,y), \quad 0 \leqslant \ell \leqslant n,$$

between the corresponding scalar orthogonal polynomial families.

Finally, we would like to mention here that our goal is not to exploit all possible situations covered by our approach, but to emphasize its systematic character, which allow one to implement it in any computer algebra system, here *Mathematica* [34] symbolic language has been used.

Acknowledgments

This work was partially supported by Ministerio de Educación y Ciencia of Spain under grant MTM2006-07186, by Ministerio de Ciencia e Innovación of Spain under grants MTM2009-14668-C02-01, MTM2009-14668-C02-02, and MTM2008-06689-C02, and cofinanced by the European Community fund FEDER. Alejandro Zarzo acknowledges partial financial support from UPM. André Ronveaux also thanks the Departamento de Matemática Aplicada II of Universidade de Vigo for the kind invitations and financial support.

The authors wish to thank the referees for their helpful and constructive comments that improved the presentation of this paper.

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