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Stable partitions with $\mathscr{W}$-preferences☆

Katarína Cechlárová*, Jana Hajduková

Institute of mathematics, Faculty of Science, P.J. Šafárik University, Jesenná 5, 041 54 Košice, Slovak Republic

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Abstract

Suppose that in a coalition formation game each participant has a preference list of the other participants and she prefers a set S to a set T if and only if she prefers the worst participant of S to the worst participant of T. We consider three definitions of stability. In the case of no indifferences stable partitions cannot contain very large sets and their existence can be decided polynomially. However, in the presence of ties one of the existence problems is NP-complete, the other is polynomial and the existence of a polynomial algorithm for the third one is still open.

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1. Introduction

In the stable roommates problem the participants are to be partitioned into pairs in such a way that no pair of participants who are not matched in a current matching would be strictly better off together. A polynomial algorithm for deciding the existence of a stable matching was derived by Irving in [5] for the case when participants' preferences contain no ties. Ronn [8] showed that the existence problem for the stable roommates with indifferences is NP-complete.

A possible generalization of the stable roommates problem is to allow groups of arbitrary cardinality to be formed. If preferences of participants over groups are derived from the worst participant of the group, we get $\mathscr{W}$-preferences, introduced in [3]. In

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* Corresponding author.
E-mail address: cechlarova@science.upjs.sk (K. Cechlárová).

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[3] it was already shown that the stable roommates problem and the $\mathscr{W}$-stable partition problem have some similarities. In the present paper we explore these similarities more deeply. In particular, for three different definitions of blocking inspired by [6] and [7], we show that all existence problems in the case without ties are polynomially solvable while the complexity of the same problems in the presence of ties exactly corresponds to their stable roommates counterparts.

$\mathscr{W}$-preferences are in a sense a dual notion to $\mathscr{B}$-preferences, where preferences of participants over groups are derived from the best participant of the group. Here, stable partitions in the case without ties always exist and one of them can be found polynomially, while in the presence of ties the existence problem is also NP-complete (see [1–3]).

The organization of the paper is as follows. In Section 2 we introduce the basic notions. Section 3 studies the properties of stable partitions. In Section 4 we modify Irving's Stable roommates algorithm for use in the $\mathscr{W}$-stable partitions context. Section 5 deals with problems with ties.

2. Basic definitions

$N = \{1, 2, \ldots, n\}$ is the set of participants. Each participant $i \in N$ is endowed with a reflexive, transitive and complete relation $\succcurlyeq_i$ on the set N, called the *preference relation*. If $j \succcurlyeq_i k$ and $k \succcurlyeq_i j$, we say that i is *indifferent* between j and k and write $j \sim_i k$. If $j \succcurlyeq_i k$ and not $j \sim_i k$, we write $j \succ_i k$ and say that i prefers j to k *strictly*. (Notice that $j \preccurlyeq_i k$ and $j \prec_i k$ are equivalent to $k \succcurlyeq_i j$ and $k \succ_i j$, respectively.) Participants j such that $j \succcurlyeq_i i$ are for i *acceptable*, the others are *unacceptable*. The preference relation of participant i will usually be represented as an ordered list $P(i)$ of acceptable participants (excluding i), with the understanding that participants appearing earlier are preferred to those appearing later in the list and brackets denote indifference. The n-tuple $\mathscr{P} = (P(1), P(2), \ldots, P(n))$ of preference relations is called a *preference profile*.

Let $\mathscr{N}_i$ stand for $\{S \subseteq N; i \in S\}$.

If $S \in \mathscr{N}_i$, $S \neq \{i\}$, then by $\mathscr{W}_i(S)$ we denote an arbitrary participant $j \in S - \{i\}$ such that $j \preccurlyeq_i k$ for all $k \in S - \{i\}$ and by $\mathscr{B}_i(S)$ any participant $j \in S - \{i\}$ such that $k \preccurlyeq_i j$ for all $k \in S - \{i\}$. For $S = \{i\}$ we set $\mathscr{W}_i(S) = \mathscr{B}_i(S) = i$. Obviously, if $S, T \in \mathscr{N}_i$ are such that $S \subseteq T$ and $S \neq \{i\}$ then $\mathscr{W}_i(T) \preccurlyeq_i \mathscr{W}_i(S)$ and $\mathscr{B}_i(T) \succcurlyeq_i \mathscr{B}_i(S)$.

Definition 1. Let $S, T \in \mathscr{N}_i$. Then participant i $\mathscr{W}$-prefers a set S to a set T, $S \succcurlyeq_i T$, if $\mathscr{W}_i(S) \succcurlyeq_i \mathscr{W}_i(T)$.

Clearly, participant i $\mathscr{W}$-prefers $\{i\}$ to any set containing at least one unacceptable participant and $\mathscr{W}$-prefers any set containing only acceptable participants to $\{i\}$.

Definition 2. We say that $\mathscr{M} = \{M_1, M_2, \ldots, M_r\}$ is a partition of N if $M_p \cap M_q = \emptyset$ for all distinct p, q and $\bigcup_{p=1}^{r} M_p = N$. The set in $\mathscr{M}$ containing participant i will be denoted by $M(i)$.

Definition 3. A nonempty set $Z \subseteq N$ $\mathscr{W}$-blocks a partition $\mathscr{M}$, if

$$(\forall i \in Z) Z \succ_i M(i).$$

A partition for which no $\mathscr{W}$-blocking set exists, is said to be $\mathscr{W}$-stable.

Definition 4. A nonempty set $Z \subseteq N$ weakly $\mathscr{W}$-blocks a partition $\mathscr{M}$, if

$$(\forall i \in Z) Z \succcurlyeq_i M(i) \quad \text{and} \quad (\exists j \in Z) Z \succ_j M(j).$$

A partition for which no weakly $\mathscr{W}$-blocking set exists, is said to be strongly $\mathscr{W}$-stable.

Definition 5. A nonempty set $Z \subseteq N$ sub-$\mathscr{W}$-blocks a partition $\mathscr{M}$, if

$$(\forall i \in Z) Z \succcurlyeq_i M(i) \quad \text{and} \quad Z \neq M(i).$$

A partition for which no sub-$\mathscr{W}$-blocking set exists, is said to be super $\mathscr{W}$-stable.

Obviously, each super $\mathscr{W}$-stable partition is strongly $\mathscr{W}$-stable and each strongly $\mathscr{W}$-stable partition is $\mathscr{W}$-stable. The converse implications are not true.

In this paper we shall deal with the following problems:

Problem WS. Given a set N and a preference profile $\mathscr{P}$ on N with all preferences strict, does there exist a $\mathscr{W}$-stable partition?

Problem StronglyWS. Given a set N and a preference profile $\mathscr{P}$ on N with all preferences strict, does there exist a strongly $\mathscr{W}$-stable partition?

Problem SuperWS. Given a set N and a preference profile $\mathscr{P}$ on N with all preferences strict, does there exist a super $\mathscr{W}$-stable partition?

The above problems, when preferences of participants are allowed to contain ties, will be denoted by *WST*, *StronglyWST* and *SuperWST*, respectively.

3. Properties of $\mathscr{W}$-stable partitions

The following observation is trivial.

Lemma 6. *If in a partition $\mathscr{M}$, some participant is in a common set with an unacceptable participant, then $\mathscr{M}$ is neither $\mathscr{W}$-stable nor strongly $\mathscr{W}$-stable nor super $\mathscr{W}$-stable.*

The following lemma will simplify some proofs:

Lemma 7. *Any partition $\mathscr{M}$ that is not super $\mathscr{W}$-stable, not strongly $\mathscr{W}$-stable or not $\mathscr{W}$-stable is sub-$\mathscr{W}$-blocked, weakly $\mathscr{W}$-blocked or $\mathscr{W}$-blocked, respectively, by a singleton or a two-element set.*

Proof. Suppose that a partition $\mathscr{M}$ is not strongly $\mathscr{W}$-stable. Then there exists a weakly $\mathscr{W}$-blocking set Z, such that $\forall i \in Z$: $\mathscr{W}_i(Z) \succcurlyeq_i \mathscr{W}_i(M(i))$ and $\exists j \in Z$: $\mathscr{W}_j(Z) \succ_j$

$\mathscr{W}_j(M(j))$. If $|Z| \leqslant 2$, then the assertion follows, otherwise denote by k an arbitrary participant in Z different from j and let us consider the set $Z' = \{j,k\} \subset Z$. Obviously $\mathscr{W}_j(Z') \succcurlyeq_j \mathscr{W}_j(Z) \succ_j \mathscr{W}_j(M(j))$ and $\mathscr{W}_k(Z') \succcurlyeq_k \mathscr{W}_k(Z) \succcurlyeq_k \mathscr{W}_k(M(k))$, hence $\{j,k\}$ weakly $\mathscr{W}$-blocks $\mathscr{M}$. For a $\mathscr{W}$-blocking or sub-$\mathscr{W}$-blocking set Z with $|Z| > 2$ an arbitrary two-element subset $Z' \subset Z$ also $\mathscr{W}$-blocks or sub-$\mathscr{W}$-blocks, respectively, the given partition $\mathscr{M}$. □

Since testing a given partition for stability is, due to Lemma 7, polynomial, we have

Corollary 8. *All the considered problems belong to class NP.*

Now we shall study the structure of stable partitions.

Theorem 9. *Let a preference profile $\mathscr{P}$ on a set of participants N be given and let $\mathscr{M}$ be a strongly $\mathscr{W}$-stable partition of N. Then*

(i) *if $\mathscr{P}$ contains no indifferences then $|M| \leqslant 2$ for each set $M \in \mathscr{M}$;*
(ii) *if $M \in \mathscr{M}$ with $|M| > 2$ then $\mathscr{B}_i(M) \sim_i \mathscr{W}_i(M)$ for each participant $i \in M$.*

Proof. Let $\mathscr{M}$ be a strongly $\mathscr{W}$-stable partition.

(i) Let $\mathscr{P}$ contain no indifferences and let $M \in \mathscr{M}$ be a partition set such that $|M| > 2$. Let $i \in M$ be arbitrary, denote $j = \mathscr{B}_i(M)$. We show that the set $Z = \{i,j\} = \{i, \mathscr{B}_i(M)\} \subset M$ weakly $\mathscr{W}$-blocks $\mathscr{M}$.

Due to Lemma 6, participants i, j are mutually acceptable. Moreover, $\mathscr{W}_j(Z) = i \succcurlyeq_j \mathscr{W}_j(M)$ and $\mathscr{W}_i(Z) = j = \mathscr{B}_i(M) \succ_i \mathscr{W}_i(M)$, since M contains at least three participants and preferences are strict. Hence participant i strictly prefers $\{i,j\}$ to M and participant j is not worse off in Z than in $\mathscr{M}$. Therefore Z weakly $\mathscr{W}$-blocks $\mathscr{M}$, a contradiction.

(ii) Let us suppose that $\mathscr{M}$ contains a set M, $|M| > 2$ and there exists a participant $i \in M$ with $j = \mathscr{B}_i(M) \succ_i \mathscr{W}_i(M)$. Then $Z = \{i,j\}$ weakly $\mathscr{W}$-blocks $\mathscr{M}$, since $\mathscr{W}_i(Z) = j = \mathscr{B}_i(M) \succ_i \mathscr{W}_i(M)$ and $\mathscr{W}_j(Z) = i \succcurlyeq_j \mathscr{W}_j(M)$. A contradiction. □

Theorem 10. *Let a preference profile $\mathscr{P}$ on a set of participants N be given and let $\mathscr{M}$ be a super $\mathscr{W}$-stable partition of N. Then $|M| \leqslant 2$ for each $M \in \mathscr{M}$.*

Proof. Similar to the proof of Theorem 9(i), but for the participant j we do not need strict preference $\mathscr{W}_i(Z) \succ_i \mathscr{W}_i(\mathscr{M})$, hence we do not need the assumption of strict preferences. □

Theorem 11. *Let a preference profile $\mathscr{P}$ with strict preferences on a set of participants N be given. Then a partition $\mathscr{M}$ is super $\mathscr{W}$-stable if and only if it is strongly $\mathscr{W}$-stable.*

Proof. The "only if" implication is trivial.

For the "if" direction suppose that a partition $\mathscr{M}$ is strongly $\mathscr{W}$-stable but not super $\mathscr{W}$-stable. Hence there exists a sub-$\mathscr{W}$-blocking set Z, which is not weakly

$\mathscr{W}$-blocking, which means

$$(\forall i\in Z)\ Z\sim_i M(i),\ \text{or, equivalently,}\ (\forall i\in Z)\ \mathscr{W}_i(Z)\sim_i \mathscr{W}_i(M(i)) \qquad (1)$$

Due to Lemma 7 we can suppose $|Z|\leqslant 2$. If $Z=\{i\}$ then, since $\mathscr{M}$ is strongly $\mathscr{W}$-stable and preferences are strict, (1) implies $i=\mathscr{W}_i(M(i))$ and hence $M(i)=\{i\}$. So Z cannot be sub-$\mathscr{W}$-blocking.

Now suppose that $Z=\{i,j\}$. Since preferences are strict and $\mathscr{M}$ is strongly $\mathscr{W}$-stable, Theorem 9 implies that $|M(i)|\leqslant 2$ and $|M(j)|\leqslant 2$. Let us distinguish two cases.

Case I: One of the sets $M(i),M(j)$ is singleton, say $M(i)=\{i\}$. Then, since preferences are strict and Z is not weakly $\mathscr{W}$-blocking, we have $j\sim_i i$, so $i=j$ and $Z=\{i\}$, a contradiction.

Case II: Both $M(i),M(j)$ are two-element, say $M(i)=\{i,k\},M(j)=\{j,l\}$. Now we must have $j\sim_i k$ and $i\sim_j l$, hence $j=k$ and $i=l$, so $Z=M(i)=M(j)=\{i,j\}$, a contradiction. □

Theorem 12. *Let a preference profile $\mathscr{P}$ on a set of participants N be given and let $\mathscr{M}$ be a $\mathscr{W}$-stable partition of N. Then*

(i) *if $\mathscr{P}$ contains no indifferences then $|M|\leqslant 3$ for each set $M\in\mathscr{M}$;*
(ii) *if $M\in\mathscr{M}$ with $|M|>2$ then $i\sim_j \mathscr{W}_j(M)$ or $j\sim_i \mathscr{W}_i(M)$ for each pair of participants $i,j\in M, i\neq j$.*

Proof. Let $\mathscr{M}$ be a $\mathscr{W}$-stable partition.

(i) Let all the preferences be strict and $\mathscr{M}$ contain a set $M=\{i,j,k,l,\dots\}$. Without loss of generality we can assume that participants in M are ordered according to their position in the preference list of participant i, i.e. $j\succ_i k\succ_i l\succ_i \dots$.

If the set $\{i,j\}$ is not $\mathscr{W}$-blocking, we must have $i=\mathscr{W}_j(M)$, and in particular $k\succ_j i$. If the set $\{i,k\}$ is not $\mathscr{W}$-blocking, since $k\succ_i l$, we must have $i=\mathscr{W}_k(M)$, hence $j\succ_k i$. But then $Z=\{j,k\}$ is $\mathscr{W}$-blocking, since both j,k have strictly improved compared to M, as they got rid of i.

(ii) Let us suppose that $\mathscr{M}$ contains a set M, $|M|\geqslant 3$, and there exists a pair of participants $i,j\in M; i\neq j$ such that $i\succ_j \mathscr{W}_j(M)$ and simultaneously $j\succ_i \mathscr{W}_i(M)$. Then $Z=\{i,j\}$ $\mathscr{W}$-blocks $\mathscr{M}$, a contradiction. □

Proof of Theorem 12 implies the following

Corollary 13. *Let $\mathscr{M}$ be a $\mathscr{W}$-stable partition for a preference profile $\mathscr{P}$ with all preferences strict. If $\{p,r,s\}\in\mathscr{M}$ then participants p,r,s can be labelled in such a way that $r\succ_p s, s\succ_r p$ and $p\succ_s r$.*

We shall use this Corollary later, in our analysis of Irving's algorithm. One of its implications is that after reducing the preference lists of participants p,r,s to contain only themselves, we get the following preference profile, denoted by $\mathscr{P}_1$:

$$P(p) = r, s,$$

$$P(r) = s, p,$$

$$P(s) = p, r.$$

So a three-element set $\{p, r, s,\}$ in a $\mathscr{W}$-stable partition can be considered as a special case of a stable profile (a definition will be introduced later).

Example 1. Let us have the following preference profile $\mathscr{P}_2$ for five participants:

$$P(a) = b, e, c, d,$$

$$P(b) = c, a, d, e,$$

$$P(c) = d, b, e, a,$$

$$P(d) = e, c, a, b,$$

$$P(e) = a, d, b, c.$$

We want to prove that for $\mathscr{P}_2$ no $\mathscr{W}$-stable partition exists. Suppose that there exists a $\mathscr{W}$-stable partition $\mathscr{M}$. Since everybody is acceptable for everybody, a $\mathscr{W}$-stable partition can contain at most one singleton. If $\{a\} \in \mathscr{M}$, then since a is the first choice of e, participant e will for sure prefer $\{a, e\}$ to $M(e)$ and as a also prefers $\{a, e\}$ to $\{a\}$, we have a blocking set. As each participant is the first choice of some other participant, a similar argument implies that $\mathscr{M}$ contains no singleton.

Therefore it is sufficient to suppose that $\mathscr{M}$ contains one two-element and one three-element set. Without loss of generality (since participants are indistinguishable with respect to cyclic permutations) we can assume that participant a is in a three-element set with participants i and j, where $i, j \in \{b, c, d, e\}, i \neq j$ and $i \succ_a j$. Corollary 13 implies that $j \succ_i a$ and $a \succ_j i$. There are only three possibilities for participant i: $i = b, i = e$ or $i = c$.

If $i = b$, then since participant b prefers to participant a only participant c, we must have $j = c$, but $a \prec_c b$, a contradiction of Corollary 13.

Case $i = e$ is not possible because there is no participant whom participant e prefers to a.

If $i = c$, then since the only participant to whom a prefers c is d, we must have $j = d$, but $a \prec_d c$, again a contradiction.

4. Finding a $\mathscr{W}$-stable partition in the no-ties case

Cechlárová and Romero-Medina in [3] showed that in the case of strict preferences, the StronglyWS problem is very close to the stable roommates problem and proposed how to use Irving's stable roommates algorithm to find a strongly $\mathscr{W}$-stable partition. For the WS problem, if the stable roommates problem has a solution for a given preference profile, then this solution is a $\mathscr{W}$-stable partition. However, there may exist

$\mathscr{W}$-stable partitions even in the case when no stable roommates solution exists. To find $\mathscr{W}$-stable partitions correctly we will analyze Irving's stable roommates algorithm even further and show how its modification can be used for finding $\mathscr{W}$-stable partitions. In our analysis of Irving's algorithm we shall follow its presentation in [4], Chapter 4.

The main idea of Irving's algorithm is to successively delete pairs from the given preference profile to get *reduced* preference profiles. Pairs deleted in the first phase of the algorithm cannot be stable pairs (i.e. belong to some stable matching) and hence cannot be in a common set in a $\mathscr{W}$-stable partition. In the second phase the so-called rotations are eliminated. If the given preference profile contains a stable roommates matching then there is a stable roommates matching (and hence a $\mathscr{W}$-stable partition) in the preference profile obtained after eliminating a rotation too, on assumption that this elimination does not empty any preference list. Therefore we have to consider the case when elimination of a rotation causes some preference lists to become empty. Before deriving some results, we shall recall the notions of a rotation and its elimination. For a reduced preference profile $\mathscr{T}$ and a participant x, $f_T(x), s_T(x), \ell_T(x)$ denote the first, second and the last entry in the preference list of participant x in profile $\mathscr{T}$. Since our first phase is identical to that of Irving, we shall always suppose that we deal with a *stable* reduced preference profile, i.e. one which fulfills the properties listed in Lemma 4.2.2. of [4], p. 168, namely,

(1) $f_T(x) = y$ if and only if $y = \ell_T(x)$
(2) a pair $\{x, y\}$ is absent from $\mathscr{T}$ if and only if x prefers $\ell_{\mathscr{T}}(x)$ to y or y prefers $\ell_{\mathscr{T}}(y)$ to x.

This means that no pair that is not present in the reduced profile can form a stable pair nor block a partition present in the current profile.

Definition 14. A sequence $\rho = (x_0, y_0)(x_1, y_1)\dots(x_{r-1}, y_{r-1})$ is said to be a rotation exposed in a preference profile $\mathscr{T}$, if

$$y_i = f_T(x_i) \quad \text{and} \quad y_{i+1} = s_T(x_i) \quad \text{for all } i = 0, 1, \dots, r-1,$$

where indices are taken modulo r, when necessary. The sets $X = \{x_0, x_1, \dots, x_{r-1}\}$, and $Y = \{y_0, y_1, \dots, y_{r-1}\}$ are called the X-set and the Y-set of the rotation and denoted by X_ρ and Y_ρ, respectively.

Definition 15. To cancel a pair $\{x, y\}$ from a preference profile $\mathscr{T}$ means to cancel participant x from the list of participant y and symmetrically, participant y from the preference list of participant x.

To eliminate a rotation $\rho = (x_0, y_0)(x_1, y_1)\dots(x_{r-1}, y_{r-1})$ from a preference profile $\mathscr{T}$ means to cancel each pair $\{y_i, z\}$ such that y_i prefers x_{i-1} to z for all $i = 0, 1, \dots, r-1$. The obtained profile will be denoted by $\mathscr{T}/\rho$.

Theorem 16. *Let $\rho = (x_0, y_0)(x_1, y_1)\dots(x_{r-1}, y_{r-1})$ be a rotation exposed in a preference profile $\mathscr{T}$. If $\mathscr{T}/\rho$ contains an empty preference list then r is odd and there* *exists an integer k, coprime with r, $k \neq 0$ and $k \neq r-1$, such that $y_m = x_{m+k}$ and, in $\mathscr{T}$, $P_{\mathscr{T}}(x_m) = x_{m+k}, x_{m+k+1}$ for all $m = 0, 1, \ldots, r-1$, indices taken modulo r.*

Proof. Lemma 4.2.7. of [4] implies that when a rotation ρ is eliminated no person who is not in $X_\rho \cap Y_\rho$ can obtain an empty preference list. So suppose that participant x_i with preference list $P_{\mathscr{T}}(x_i) = y_i, y_{i+1}, \ldots$ has an empty preference list after the elimination of ρ. This means that during the elimination also the pair $\{x_i, y_{i+1}\}$ has been cancelled. This could only be because x_i is y_j for some j and y_j prefers x_{j-1} to y_{i+1}. Since y_{i+1} is the second entry in $P_{\mathscr{T}}(x_i) = P_{\mathscr{T}}(y_j)$, this means that x_{j-1} must be the first entry in $P_{\mathscr{T}}(x_i)$, or that $x_{j-1} = y_i$. Now let us compare the preference lists of participants x_{j-1} and y_i. We know that $\ell_T(y_i) = x_i$ and $s_T(x_{j-1}) = y_j = x_i$. Hence the preference list of x_{j-1} contains just two entries and it is $P_{\mathscr{T}}(x_{j-1}) = P_{\mathscr{T}}(y_i) = y_{j-1}, y_j = x_{i-1}, x_i$. This implies further equality $x_{i-1} = y_{j-1}$. Now by induction we get

$$x_m = y_{m+j-i} \quad \text{and} \quad x_m = y_{m+i-j+1} \tag{2}$$

and

$$P_{\mathscr{T}}(x_m) = x_{m+i-j}, x_{m+i-j+1}$$

for $m = 0, 1, \ldots, r-1$, indices taken modulo r.

Further, Eqs. (2) for $m = 0$ imply that

$$0 \equiv j - i \,(\mathrm{mod}\, r) \quad \text{and} \quad 0 \equiv i - j + 1 \,(\mathrm{mod}\, r) \tag{3}$$

which in turn gives $2(i-j)+1 \equiv 0 \,(\mathrm{mod}\, r)$. Hence r is odd and $k = i - j$ and r are coprime. Clearly $k \neq 0$ and $k \neq r-1$, because equations $y_m = x_m$ and $y_m = x_{m+r-1} = x_{m-1}$ are in a contradiction of the definition of a rotation.

This proves the assertions of the Theorem. □

A rotation with properties described in Theorem 16 will be called an r-rotation, specifically, depending on r, a 3-rotation, 5-rotation, etc. Clearly, in an r-rotation ρ we have $X_\rho = Y_\rho$, each $x \in X_\rho$ has just two entries in $P_{\mathscr{T}}(x)$ and the participants with empty preference lists in $\mathscr{T}/\rho$ are just the participants in X_ρ.

We can now state and prove three theorems that form the basis of the second phase of our modified algorithm. The first one is very similar to Theorem 4.2.1 from [4].

Theorem 17. *If there is a $\mathscr{W}$-stable partition embedded in a profile $\mathscr{T}$, and ρ is a rotation exposed in $\mathscr{T}$ such that profile $\mathscr{T}/\rho$ contains no empty lists, then there is a $\mathscr{W}$-stable partition embedded in $\mathscr{T}/\rho$.*

Proof. Let $\mathscr{M}$ be a $\mathscr{W}$-stable partition embedded in $\mathscr{T}$, and suppose that a rotation $\rho = (x_0, y_0)(x_1, y_1)\ldots(x_{r-1}, y_{r-1})$ exposed in $\mathscr{T}$ is not an r-rotation. Now consider two cases.

(i) There exists $i, 0 \leqslant i \leqslant r-1$, such that participants x_i and y_i are not in a common set in $\mathscr{M}$. Then, since an embedded $\mathscr{W}$-stable partition $\mathscr{M}$ is a special case of a subprofile, it follows from Corollary 4.2.1. in [4] that partition $\mathscr{M}$ is embedded in $\mathscr{T}/\rho$.

(ii) Otherwise, if participants x_i and y_i are in a common set in $\mathcal{M}$ for all $i, 0 \leqslant i \leqslant r-1$, we first show that $|M(x_i)| = 2$ for all participants x_i. Let us suppose that $M = M(x_i) = \{x_i, y_i, z\} \in \mathcal{M}$ for some i, $0 \leqslant i \leqslant r-1$. We have $y_{i+1} \succcurlyeq_{x_i} z$ and since $M(y_{i+1})$ contains x_{i+1}, who is the last choice of y_{i+1}, if $y_{i+1} \succ_{x_i} z$ then $Z = \{x_i, y_{i+1}\}$ $\mathcal{W}$-blocks the partition $\mathcal{M}$. Therefore $z \succcurlyeq_{x_i} y_{i+1}$ and since preferences are strict, $z = y_{i+1}$. Hence $M = \{x_i, y_i, y_{i+1}\}$. But since $x_{i+1} \in M(y_{i+1})$ and $x_{i+1} \neq x_i$, we have $M = \{x_i, y_i = x_{i+1}, y_{i+1}\}$. If we use the same argument as we did for participant x_i, we get that participant x_{i+1} can be in M only if $x_i = y_{i+2}$. But then also participant x_{i+2} must be in M and this is only possible if $y_{i+1} = x_{i+2}$. Again, so as the pair $\{x_{i+2}, y_{i+3}\}$ is not $\mathcal{W}$-blocking, we must have $x_{i+1} = y_{i+3}$. This means we have $x_i = y_{i+2}, x_{i+1} = y_i = y_{i+3}$ and $x_{i+2} = y_{i+1}$ and so rotation $\rho = \ldots(x_i, x_{i+1})(x_{i+1}, x_{i+2})(x_{i+2}, x_i)(\ldots, x_{i+1})\ldots$ is a 3-rotation, a contradiction.

We have just shown that in case (ii) partition $\mathcal{M}$ contains all the pairs $\{x_i, y_i\}$. Now the pairs $\{x_i, y_i\}$ are replaced by $\{x_i, y_{i+1}\}$ for each $i = 0, 1, \ldots, r-1$ and similarly as in the proof of Theorem 4.2.1. in [4], it is shown that we get a matching; these pairs remain undeleted, and the new matching is indeed a $\mathcal{W}$-stable partition. □

Theorem 18. *If in a stable profile $\mathcal{T}$ an r-rotation exists with $r \geqslant 5$ then there is no $\mathcal{W}$-stable partition embedded in $\mathcal{T}$.*

Proof. Let $\mathcal{M}$ be a $\mathcal{W}$-stable partition embedded in $\mathcal{T}$ and let ρ be an r-rotation with $r \geqslant 5$. Since each participant $x_m \in X_\rho$ has only two acceptable participants in $\mathcal{T}$ (namely x_{m+k} and x_{m+k+1} from X_ρ, see the formulation of Theorem 16), no participant from $X_\rho = Y_\rho$ can be in a common set in $\mathcal{M}$ with a participant that does not belong to X_ρ. In other words, all participants from X_ρ have in their partition sets in $\mathcal{M}$ only participants belonging to X_ρ. No participant $x_i \in X_\rho$ can be alone in $\mathcal{M}$, since he is the first choice of participant x_{i-k} and x_{i-k} is acceptable to x_i, so the pair $\{x_i, x_{i-k}\}$ would block any partition in which $\{x_i\} \in \mathcal{M}$. Since the cardinality of X_ρ is odd, $\mathcal{M}$ must contain at least one three-element set, say $M = \{x_m, x_{m+k}, x_{m+k+1}\}$. The preference lists of participants in M are

$$P(x_m) = x_{m+k}, x_{m+k+1},$$

$$P(x_{m+k}) = x_{m+2k}, x_{m+2k+1},$$

$$P(x_{m+k+1}) = x_{m+2k+1}, x_{m+2k+2}$$

and when we set $p = x_m$, $r = x_{m+k}$, $s = x_{m+k+1}$, Corollary 13 implies the following system of congruences modulo r:

$$m + k \equiv m + 2k + 2$$

$$m + k + 1 \equiv m + 2k$$

$$m + 2k + 1 \equiv m$$

which is equivalent to $k \equiv -2 \equiv 1$ and $2k \equiv -1 \pmod r$. This is only possible if $r = 3$. A contradiction. □

```
begin T:=phase-1 profile; M := ∅;
    while some list in T has more than one entry
    and no list in T is empty do
    begin find a rotation ρ exposed in T;
        if ρ is a 3-rotation (x0,x1)(x1,x2)(x2,x0)
        then begin T := T − {P(x0), P(x1), P(x2)};
                   M := M ∪ {x0, x1, x2} end;
        else T := T/ρ; {eliminate ρ}
    end;
    if some list in T is empty then report instance insolvable
    else output T and M;
end.
```

Fig. 1. Phase 2 of the $\mathscr{W}$-stable algorithm.

Theorem 19. *If $\mathscr{M}$ is a $\mathscr{W}$-stable partition embedded in a profile $\mathscr{T}$, and if $\rho = (x_0,x_1)(x_1,x_2)(x_2,x_0)$ is a 3-rotation exposed in $\mathscr{T}$, then*

(i) *partition $\mathscr{M}$ contains the set $\{x_0,x_1,x_2\}$ and*
(ii) *partition $\mathscr{M}' = \mathscr{M} - \{\{x_0,x_1,x_2\}\}$ is $\mathscr{W}$-stable in profile $\mathscr{T}'$ obtained from $\mathscr{T}$ by deleting participants x_0,x_1,x_2.*

Proof. (i) Let $\mathscr{M}$ be a $\mathscr{W}$-stable partition present in profile $\mathscr{T}$. Then the only possible members of the sets $M(x_0), M(x_1)$ and $M(x_2)$ are participants x_0,x_1 and x_2. But if any participant $i \in \{x_0,x_1,x_2\}$ were alone, since he is the first choice of some other participant $j \in \{x_0,x_1,x_2\}$, the set $\{i,j\}$ would be $\mathscr{W}$-blocking. Hence each $\mathscr{W}$-stable partition present in profile $\mathscr{T}$ contains $\{x_0,x_1,x_2\}$.

(ii) Trivial. □

Summarizing, Phase 2 of the $\mathscr{W}$-stable partitions algorithm is formally written in Fig. 1. (Notice that when a 3-rotation is encountered, the three participants involved are deleted together with their preference lists, so no empty preference list arises.) Its output are the three-element sets corresponding to 3-rotations identified during the algorithm and the final profile $\mathscr{T}$, which, in a similar way to the matching case, gives the two-element sets of the partition, by the condition $\{x,y\} \in \mathscr{M}$ if and only if $P_{\mathscr{T}}(x) = y$, which in turns happens if and only if $P_{\mathscr{T}}(y) = x$.

5. Problems with ties

The similarity of the $\mathscr{W}$-stable partition problem to the stable roommates problem extends also to the case with ties.

First we show that the WST problem is NP-complete. For this, we shall only slightly modify Ronn's construction [8] used to show that the stable roommates problem with ties is NP-complete. First we describe Ronn's polynomial transformation from the restricted 3-SAT problem to the stable roommates problem in greater detail.

The restricted 3-SAT problem (which is NP-complete, see e.g. [4, p. 210]) is to decide the satisfiability of a boolean formula in conjunctive normal form with each clause containing at most three literals and each literal appearing at most twice in the formula. Moreover, we shall suppose that no clause contains a literal as well as its negation.

Ronn constructed the following preference profile, denoted by $\mathscr{P}_3$, for each restricted boolean formula $\mathscr{B}$ with n boolean variables $x_1, x_2, \ldots, x_n$ and m clauses $C_1, C_2, \ldots, C_m$:

$$P(u_i) = f_1(C_i), f_2(C_i), f_3(C_i), v_i, w_i, \ldots$$

$$P(v_i) = w_i, u_i, \ldots$$

$$P(w_i) = u_i, v_i, \ldots$$

$$P(p_{jk}) = (q_{jk}, r_{jk}), \ldots$$

$$P(q_{jk}) = p_{jk}, r_{j,3-k}, u(q_{jk}), g'\text{s in arbitrary order}, \ldots$$

$$P(r_{jk}) = p_{jk}, q_{j,3-k}, u(r_{jk}), g'\text{s in arbitrary order}, \ldots$$

$$P(g_l) = q'\text{s and } r'\text{s in arbitrary order}, \ldots$$

The interpretation of individual participants is the following:

- Participants u_i, v_i, w_i for $i = 1, 2, \ldots, m$ represent clause C_i. Participants $p_{j1}, p_{j2}, q_{j1}, q_{j2}, r_{j1}, r_{j2}$ for $j = 1, 2, \ldots, n$ represent variable x_j, where participants q_{j1}, q_{j2} correspond to the first and the second occurrence of literal x_j in the formula and participants r_{j1}, r_{j2} correspond to the first and the second occurrence of literal $\bar{x}_j$ in the formula, respectively. Participants g_l for $l = 1, 2, \ldots, 2n - m$ are called "garbage collectors".
- Functions $f_1(C_i), f_2(C_i), f_3(C_i)$ denote the participants, corresponding to the first, second and third literal in clause C_i, respectively. If a clause C_i contains only two literals, participant $f_3(C_i)$ does not occur in the preference list of u_i.
- $u(q_{jk}) = u_i$ if the kth occurrence of literal x_j appears in clause C_i, similarly $u(r_{jk}) = u_i$ if the kth occurrence of literal $\bar{x}_j$ appears in clause C_i. If this literal is not present in the formula at all, these inputs are simply missing.
- The dots indicate that the rest of the participants may appear in arbitrary order.

Theorem 20 (Ronn). *$\mathscr{P}_3$ admits a stable roommates solution if and only if boolean formula $\mathscr{B}$ is satisfiable.*

To prove the NP-completeness of the problem WST, we first define the preference profile for a given boolean formula. Like Ronn, for each variable x_j, $j = 1, 2, \ldots, n$, we shall have participants p_{jk}, q_{jk}, r_{jk} for $k = 1, 2$; garbage collectors g_l for $l = 1, 2, \ldots, 2n - m$, but for a clause C_i we shall have instead of u_i, v_i and w_i five clause participants

a_i, b_i, c_i, d_i, e_i with the following preferences:

$$P(a_i) = f_1(C_i), f_2(C_i), f_3(C_i), b_i, e_i, c_i, d_i, \ldots$$
$$P(b_i) = c_i, a_i, d_i, e_i, \ldots$$
$$P(c_i) = d_i, b_i, e_i, a_i, \ldots$$
$$P(d_i) = e_i, c_i, a_i, b_i, \ldots$$
$$P(e_i) = a_i, d_i, b_i, c_i, \ldots$$

Preferences of participants p_{jk}, q_{jk}, r_{jk} and g_l are the same as in $\mathscr{P}_3$, only the value of $u(q_{jk})$ and $u(r_{jk})$ is now participant a_i of clause C_i that contains the corresponding literal. We shall call the derived preference profile $\mathscr{P}_4$.

We want to prove that $\mathscr{P}_4$ admits a $\mathscr{W}$-stable partition if and only if the original boolean formula $\mathscr{B}$ is satisfiable.

Let us have a satisfying truth assignment for formula $\mathscr{B}$. We can use a similar construction of a $\mathscr{W}$-stable partition $\mathscr{M}$ for $\mathscr{P}_4$ as Ronn did for the profile $\mathscr{P}_3$:

(i) For each $i = 1, 2, \ldots, m$ select the participant $f_l(C_i)$ which corresponds to the first true literal in C_i and create sets $\{a_i, f_l(C_i)\}, \{b_i, c_i\}$ and $\{d_i, e_i\}$.
(ii) For each $j = 1, 2, \ldots, n$: if x_j is true, then create sets $\{p_{j1}, r_{j1}\}$ and $\{p_{j2}, r_{j2}\}$; if x_j is false, then create sets $\{p_{j1}, q_{j1}\}$ and $\{p_{j2}, q_{j2}\}$.
(iii) The number of participants r_{jk}, q_{jk} who are matched neither to some a_i nor to p_{jk} (called leftovers) is equal to the number of garbage collectors, so they are matched according to some stable marriage matching involving leftovers and garbage collectors.

Lemma 21. *The constructed partition $\mathscr{M}$ is $\mathscr{W}$-stable.*

Proof. We will show that no participant can be in a $\mathscr{W}$-blocking set:

(1) Since, in $\mathscr{M}$, all participants p_{jk} are in their favourite sets $\{p_{jk}, q_{jk}\}$ or $\{p_{jk}, r_{jk}\}$, no $\mathscr{W}$-blocking set can contain a participant p_{jk}; nor a participant q_{jk} for which x_j is false, nor a participant r_{jk} for which x_j is true.
(2) Participants a_i prefer to their partition sets $\{a_i, f_l(C_i)\}$ only sets $\{a_i, f_s(C_i)\}$ for $s < l$. Since $f_s(C_i)$ can be either a participant q_{jk} for which x_j is false, or a participant r_{jk} for which x_j is true and these participants are already excluded, a $\mathscr{W}$-blocking set cannot contain a participant a_i. Participants b_i and d_i have in their sets their favourites, so none of them can be in a $\mathscr{W}$-blocking set. Participants c_i and e_i prefer to their partition sets only sets with excluded participants d_i and b_i, so they will not $\mathscr{W}$-block the constructed partition $\mathscr{M}$ either.
(3) Participant q_{jk}, who is in $\mathscr{M}$ with some participant a_i, prefers to a_i only participants p_{jk} and $r_{j,3-k}$, but both of them have in $\mathscr{M}$ their first choice sets, see (1). For the same reason also participants r_{jk} paired with some a_i are excluded from $\mathscr{W}$-blocking.
(4) Leftovers and garbage collectors will also find no partner for a $\mathscr{W}$-blocking set, due to (iii) and (1),(2),(3). □

Hence, we have shown that based on a satisfying truth assignment, a $\mathscr{W}$-stable partition can be defined. For the converse implication, namely defining a satisfying truth assignment on the basis of a $\mathscr{W}$-stable partition, we need first to derive some properties of W-stable partitions for $\mathscr{P}_4$.

Lemma 22. *If $\mathscr{M}$ is a $\mathscr{W}$-stable partition for $\mathscr{P}_4$, then for each $i = 1, 2, \ldots, m$ there exists some $l = 1, 2, 3$ such that $\{a_i, f_l(C_i)\}, \{b_i, c_i\}, \{d_i, e_i\} \in \mathscr{M}$.*

Proof. Let us have a $\mathscr{W}$-stable partition $\mathscr{M}$ for preference profile $\mathscr{P}_4$. Since everybody is acceptable for everybody and the number of participants is even, $\mathscr{M}$ cannot contain singletons. Fix i and for a participant $u \in \{a_i, b_i, c_i, d_i, e_i\}$ let us call a participant v inferior for u if $\mathscr{W}_u(\{a_i, b_i, c_i, d_i, e_i\}) \succ_u v$.

Participant a_i is the first choice of participant e_i, e_i is the first choice of d_i, d_i is the first choice of c_i and c_i is the first choice of b_i. Therefore participants a_i, c_i, d_i, e_i cannot be in $\mathscr{M}$ in a common set with inferior participants, because then sets $\{a_i, e_i\}, \{e_i, d_i\}$, $\{d_i, c_i\}$, or $\{c_i, b_i\}$ respectively, will $\mathscr{W}$-block $\mathscr{M}$.

Suppose that participant b_i is in $\mathscr{M}$ with one of his inferior participants. Then participant a_i can be in $\mathscr{M}$ only with some $f_l(C_i)$, because otherwise $\{a_i, b_i\}$ will $\mathscr{W}$-block $\mathscr{M}$. But then $\{c_i, d_i, e_i\} \in \mathscr{M}$ and $\{d_i, e_i\}$ $\mathscr{W}$-blocks $\mathscr{M}$, a contradiction.

Hence we have shown, that no participant a_i, b_i, c_i, d_i, e_i can be in a $\mathscr{W}$-stable partition with an inferior participant. Example 1 implies that we can create no $\mathscr{W}$-stable partition, where participants a_i, b_i, c_i, d_i, e_i are together in one two-element and one three-element set. Hence the only possibility is that participant a_i is in partition $\mathscr{M}$ with one or two participants from $\{f_1(C_i), f_2(C_i), f_3(C_i)\}$ and participants b_i, c_i, d_i, e_i form two two-element sets.

Now we show that $|M(a_i)| = 2$. To get a contradiction, suppose that $M(a_i) = \{a_i, \alpha, \beta\}$, where α is a participant q_{jk} or r_{jk} and β is a participant $q_{j'k'}$ and $r_{j'k'}$ and $j \neq j'$ (because no clause contains the same literal twice nor a literal as well as its negation). But a_i prefers α before β if the literal corresponding to α is written before the literal corresponding to β in clause C_i and α prefers a_i before β due to the definition of the preference lists of q's and r's. Hence $\{a_i, \alpha\}$ is a blocking set.

So it follows that for every $i = 1, 2, \ldots, m$ the partition $\mathscr{M}$ must contain $\{a_i, f_l(C_i)\}$ for some $l = 1, 2, 3$ and sets $\{b_i, c_i\}$ and $\{d_i, e_i\}$. □

Lemma 23. *If in a $\mathscr{W}$-stable partition participant q_{jk} for some $k = 1, 2$ and $j = 1, 2, \ldots, n$ is in a common set with some participant a_i, then neither r_{j1} nor r_{j2} can be in a common set with any participant a_s.*

Proof. Let us suppose that in a $\mathscr{W}$-stable partition $\mathscr{M}$ there is a participant q_{jk} in a common set with a participant a_i. Without loss of generality we can assume that $k = 1$. If participant r_{j1} is in a common set with some participant a_s, then the set $\{p_{j1}, q_{j1}\}$ $\mathscr{W}$-blocks $\mathscr{M}$. If participant r_{j2} is in a common set with some participant a_s, then the set $\{q_{j1}, r_{j2}\}$ $\mathscr{W}$-blocks the partition $\mathscr{M}$. □

Lemma 24. *If in a $\mathscr{W}$-stable partition participant r_{jk} for some $k=1,2$ and $j=1,2,\ldots,n$ is in a common set with some participant a_i, then neither q_{j1} nor q_{j2} can be in a common set with any participant a_s.*

Proof. Similar to the proof of the previous Lemma. □

Hence based on a $\mathscr{W}$-stable partition we can define the boolean values of the variables in the following way:

- x_j will be true, if at least one of the participants q_{j1}, q_{j2} is in a common set with a participant a_i
- x_j will be false, if at least one of the participants r_{j1}, r_{j2} is in a common set with a participant a_i
- otherwise x_j can be arbitrary

In the light of what has been said, this definition causes no conflict and since for each clause C_i participant a_i is in a common set with a participant corresponding to a true literal contained in C_i, formula $\mathscr{B}$ is satisfied.

To summarize, we have just proved

Theorem 25. *Problem WST is NP-complete.*

On the other hand, Theorem 10 implies that also in the case with ties each set in a super $\mathscr{W}$-stable partition has cardinality at most 2, so a super $\mathscr{W}$-stable partition is a super stable matching. Since Irving and Manlove in [7] derived a polynomial algorithm to find a super stable matching, this algorithm also finds a super $\mathscr{W}$-stable partition.

Theorem 26. *There exists a polynomial algorithm to decide the existence of super $\mathscr{W}$-stable partition also in the case with ties.*

However, the polynomiality of the strongly stable roommates is an open problem and this is also the case for strongly $\mathscr{W}$-stable partitions.

6. Conclusion

In this paper we have introduced the $\mathscr{W}$-preferences of players and studied partitions that are stable according to three different definitions. It is quite striking that even if we allow sets of arbitrary cardinality, the $\mathscr{W}$-preferences lead to solutions very similar to the stable roommates case, in the case with no indifferences as well as in the case with ties.

Notice that for a similar problem, where the preferences of players are derived from the best player of a set, the situation is analogous: there exists a polynomial algorithm deciding the existence of a stable partition in the case with no ties (see [1]) and the existence problem is NP-complete in the case with ties (see [3]).

One might argue that it is unrealistic to expect that people would decide their preferences over sets only according to the best or the worst participant of a set, respectively. However, the authors are not aware of any attempt to model a more complex, but still tractable, preference criterion over sets.

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