Tangent-impulse transfer from elliptic orbit to an excess velocity vector

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Abstract The two-body orbital transfer problem from an elliptic parking orbit to an excess velocity vector with the tangent impulse is studied. The direction of the impulse is constrained to be aligned with the velocity vector, then speed changes are enough to nullify the relative velocity. First, if one tangent impulse is used, the transfer orbit is obtained by solving a single-variable function about the true anomaly of the initial orbit. For the initial circular orbit, the closed-form solution is derived. For the initial elliptic orbit, the discontinuous point is solved, then the initial true anomaly is obtained by a numerical iterative approach; moreover, an alternative method is proposed to avoid the singularity. There is only one solution for one-tangent-impulse escape trajectory. Then, based on the one-tangent-impulse solution, the minimum-energy multi-tangent-impulse escape trajectory is obtained by a numerical optimization algorithm, e.g., the genetic method. Finally, several examples are provided to validate the proposed method. The numerical results show that the minimum-energy multi-tangent-impulse escape trajectory is the same as the one-tangent-impulse trajectory.

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1. Introduction

The two-body orbital transfer problem from a parking orbit to a given excess velocity vector is a fundamental one in space exploration. The minimum-energy trajectory optimization for this problem has been studied for many years. For the initial circular orbit, an approximate analytical solution was obtained for the minimum-energy three- and four-impulse transfers between a given circular orbit and a given hyperbolic velocity vector at infinity.\textsuperscript{1} For the transfer from the initial circular orbit to an excess velocity vector, the two-impulse escape is never simultaneously of lower cost than either the one- or three-impulse.\textsuperscript{2} Recently, Ocampo et al.\textsuperscript{3,4} studied the one- and three-impulse escape trajectories, which can be constructed to serve as initial guesses for determining constrained optimal multi-impulsive escape trajectories. For the initial elliptic orbit, the optimal three-impulse transfer between an elliptic orbit and an escape asymptote was solved.\textsuperscript{5} Moreover, a simple numerical technique was proposed to minimize the time-of-flight for the multi-impulse transfer from an arbitrary elliptic orbit to a hyperbolic escape asymptote.\textsuperscript{6}
The existing methods for the impulse escape trajectory optimization problem do not include any constraint on the impulse direction.\(^7,^8\) If the impulse direction is aligned with the velocity vector, the impulse is called “tangent impulse” and a speed change will finish the impulse maneuver to nullify the relative velocity. The tangent orbit problem has also existed for many years. The classical Hohmann transfer is a cotangent transfer and is the minimum-energy one among all the two-impulse transfers between coplanar circular orbits and between coplanar coaxial elliptic orbits.\(^9\) The cotangent transfer problem between coplanar noncoaxial elliptic orbits has aroused considerable interest in recent years. The numerical solution was obtained based on the orbital hodograph theory.\(^10\) Moreover, the closed-form solution was obtained by using the geometric characteristics\(^11\) and by the flight-direction angle,\(^12\) respectively. The latter reference also gave the closed-form solution for the solution-existence condition. In addition, Zhang and Zhou\(^13\) studied the tangent orbit technique in 3D based on a new definition of orbit “tangency” condition for noncoplanar orbits. Different from the orbital transfer problem, the orbital rendezvous problem requires the same time-of-flight for both the chaser and the target. Zhang et al.\(^14\) solved the two-impulse rendezvous problem between two coplanar elliptic orbits with only the second impulse and both impulses being tangent,\(^15\) respectively. Furthermore, Zhang et al.\(^16\) solved the two-impulse rendezvous problem between coplanar elliptic and hyperbolic orbits.

This paper studies the coplanar orbit escape problem from an elliptic orbit to an excess velocity vector only with the tangent impulse. For a given initial orbit and a given excess velocity vector, the one-tangent-impulse transfer trajectory is obtained by solving a single-variable piecewise function. Then the optimal multi-tangent-impulse escape trajectory is obtained by the genetic method.

2. Orbital elements of transfer orbit

Assume that the spacecraft moves in a given initial orbit, a tangent impulse \(\Delta r\) with magnitude \(\lambda\) is imposed at \(P_1\) (see Fig. 1), where the position vector relative to the Earth’s center \(F_1\) is \(r_0\) and the velocity vector is \(v_0\), then the velocity vector of the transfer orbit (or the final hyperbolic orbit) at \(P_1\) is

\[
\mathbf{v}_1 = \mathbf{v}_0 + \Delta \mathbf{r} = \left(\mathbf{v}_0 + \frac{\lambda}{\mu} \right) \mathbf{r}_0 / \mu
\]  

where \(\mathbf{r}_j = \|\mathbf{r}_j\|, \mathbf{v}_j = \|\mathbf{v}_j\|, j = 0, f, 0\) and \(f\) denote the initial and final orbits, respectively. A relationship between the magnitudes of velocity and position vectors is

\[
\mathbf{v}_j = \sqrt{\frac{\mu}{\mathbf{r}_j}}
\]  

where \(\mu\) is the standard gravitational parameter, and \(a\) the semimajor axis. Then, the magnitude of the velocity can be written as a function of the initial true anomaly,

\[
\mathbf{v}_0 = \sqrt{\frac{\mu}{\mathbf{r}_0}} \left(1 + e_0^2 + 2e_0 \cos \varphi_0\right)
\]  

where \(\varphi\) is the true anomaly, \(e_0\) the true anomaly at infinity of the excess hyperbolic orbit, \(e\) the eccentricity, and \(p\) the semilatus rectum.

From the energy equation, the semimajor axis of the final orbit is

\[
a_f = -\frac{\mu}{\mathbf{v}_0^2}
\]  

where \(\mathbf{v}_0^2\) is the excess velocity vector. Substituting Eq. (4) into Eq. (2) gives

\[
\mathbf{v}_f = \sqrt{\left(\mathbf{v}_0^2 + \frac{2\mu}{\mathbf{r}_0}\right)^2 - \frac{2\mu}{\mathbf{r}_0} \left(1 + e_0 \cos \varphi_0\right)}
\]  

Thus, for a given initial true anomaly \(\varphi_0\) of the impulse point, the magnitude of the tangent impulse is

\[
\lambda = \mathbf{v}_f - \mathbf{v}_0
\]  

\[
= \sqrt{\left(\mathbf{v}_0^2 + \frac{2\mu}{\mathbf{r}_0}\right)^2 - \frac{2\mu}{\mathbf{r}_0} \left(1 + e_0 \cos \varphi_0\right) - \frac{\mu}{\mathbf{r}_0} \left(1 + e_0^2 + 2e_0 \cos \varphi_0\right)}
\]  

The eccentricity vector of the final orbit is

\[
e_f = \frac{1}{\mu} \left(\mathbf{v}_0^2 - \frac{2\mu}{\mathbf{r}_0} \mathbf{r}_0 - \mathbf{r}_0 \cdot \mathbf{v}_f \mathbf{r}_f\right)
\]  

By using the following expression

\[
r_0 \cdot \mathbf{v}_f = \left(1 + \frac{\lambda}{\mathbf{v}_0}\right) \mathbf{r}_0 \cdot \mathbf{v}_0 = \left(1 + \frac{\lambda}{\mathbf{v}_0}\right) \mathbf{r}_0 \cdot \mathbf{v}_0 = \sqrt{\mu p_0} \left(1 + \frac{\lambda}{\mathbf{v}_0}\right) \frac{e_0 \sin \varphi_0}{1 + e_0 \cos \varphi_0}
\]  

the eccentricity vector in Eq. (7) can be written as

\[
e_f = \frac{1}{\mu} \left\{ \left(\mathbf{v}_0^2 + \frac{2\mu}{\mathbf{r}_0} \left(1 + e_0 \cos \varphi_0\right)\right) \mathbf{r}_0 - \sqrt{\mu p_0} \left(1 + \frac{\lambda}{\mathbf{v}_0}\right)^2 \frac{e_0 \sin \varphi_0}{1 + e_0 \cos \varphi_0} \mathbf{r}_0\right\}
\]  

whose magnitude is the eccentricity \(e_f\) of the final orbit.

The normalized eccentricity vector is

\[
\hat{e}_f = \frac{e_f}{e_f} = \begin{bmatrix}
\cos \omega_f \cos \Omega_f - \sin \omega_f \sin \Omega_f \cos I_f \\
\cos \omega_f \sin \Omega_f + \sin \omega_f \cos \Omega_f \cos I_f \\
\sin \omega_f \sin I_f
\end{bmatrix}
\]  

![Fig. 1](image-url) Transfer to an excess velocity vector by a tangent impulse, example 1.
where \( I \) denotes the orbit inclination, \( \omega \) the argument of perigee, and \( \Omega \) the right ascension of ascending node. With a coplanar tangent impulse, the orbit plane will not change, which indicates that \( I_f = I_0 \) and \( \Omega_f = \Omega_0 \). In addition, the argument of perigee can be obtained as

\[
\omega_f = \begin{cases} \text{atan2}(\hat{e}_{\phi_i}/\sin l_0, \hat{e}_{\phi_i} \cos \Omega_0 + \hat{e}_{\phi_i} \sin \Omega_0) & \sin l_0 \neq 0 \\ \text{atan2}(\hat{e}_{\phi_i}, \hat{e}_{\phi_i}) & \sin l_0 = 0 \end{cases}
\]

(11)

where \( \hat{e}_{\phi_i}, \hat{e}_{\phi_i}, \) and \( \hat{e}_{\phi_i} \) are three components of \( \hat{e}_i \) in the Earth centered inertial (ECI) frame, where the origin is the center of the Earth, the \( k_1 \) axis is in the vernal equinox direction, the \( k_3 \) axis is the Earth’s rotation axis and perpendicular to equatorial plane, and the \( k_2 \) axis is in the equatorial plane and completes the right-hand system. The four-quadrant inverse tangent function \( x = \tan^{-1}(c_1, c_2) \) is defined as the angle satisfying both \( \sin x = c_1 \) and \( \cos x = c_2 \). The value range of \( x \) is \((-\pi, \pi] \).

The angular momentum of the initial/final orbit is

\[
h_i = r_i \times v_i
\]

and \( h_f = \|h_f\|_f \), then the true anomaly of \( P_1 \) on the final orbit can be obtained from

\[
\begin{align*}
\sin \varphi_i &= \frac{\hat{h}_i}{\mu v_i} \cdot r_0 \cdot r_i \\
\cos \varphi_i &= \frac{r_0 \cdot \hat{e}_i}{r_0 \cdot \hat{e}_i}
\end{align*}
\]

(13)

With the above expressions, all six classical orbital elements of the final orbit are obtained.

3. Analysis of the problem

The eccentricity \( e_t \) of the transfer orbit is obtained by Eq. (9). However, a simple explicit expression for \( e_t \) only about the initial true anomaly will be derived in the following paragraphs. Note that the flight-direction angles at the impulse point \( P_1 \) for initial and final orbits are the same. From Eq. (12) it is known

\[
\frac{\mu^2}{\alpha^2} \sin^2 \gamma_0 = \frac{h_i^2}{\mu} = p_t
\]

(14)

Since the flight-direction angle is

\[
\cos \gamma_0 = \frac{e_0 \sin \varphi_0}{1 + e_0 \cos \varphi_0}
\]

(15)

we have

\[
\sin^2 \gamma_0 = \frac{1}{1 + \cos^2 \gamma_0} = \frac{(1 + e_0 \cos \varphi_0)^2}{1 + e_0^2 + 2e_0 \cos \varphi_0}
\]

(16)

Substituting Eq. (16) into Eq. (14) yields

\[
\frac{p_t^2}{\mu} = \frac{1}{1 + e_0^2 + 2e_0 \cos \varphi_0} \left[ (v_\infty^2 - \frac{2p_t}{\mu}) (1 + e_0 \cos \varphi_0) \right]
\]

(17)

By using Eq. (4) the eccentricity can be obtained as

\[
e_t = \frac{1}{\sqrt{1 + e_0^2 + 2e_0 \cos \varphi_0}} \cdot \left( \frac{e_0 \sin \varphi_0}{\mu} + \left( v_\infty^2 \frac{p_t}{\mu} + (1 + e_0 \cos \varphi_0) \right)^2 \right)^{1/2}
\]

(18)

which is a function only of the initial true anomaly \( \varphi_0 \).

Define an \( XYZ \) frame, which is obtained by rotating the ECI frame with 3–1 sequence for angles \( \Omega \) and \( i \), respectively. The origin of the \( XYZ \) frame is also the center of the Earth. Then the \( X \) and \( Y \) axes are in the orbit plane, and \( Z \) axis is aligned with the angular momentum vector of the initial (or final) orbit.

The eccentricity vector Eq. (9) in the ECI frame can be transformed into that in the \( XYZ \) frame. Note that the flight-direction angle \( \gamma_0 \) the angle from the position vector \( r_0 \) to the velocity vector \( v_0 \), then from Eq. (9) the component of the eccentricity vector in the \( Y \) axis is

\[
e_{0Y} = \frac{p_0}{1 + e_0 \cos \theta_0} \left( v_\infty^2 - \frac{2p_t}{\mu} \right) \left( 1 + e_0 \cos \varphi_0 \right)
\]

(19)

which includes two variables, \( \varphi_0 \) and \( \gamma_0 \). Since the flight-direction angle is a function only of the initial true anomaly

\[
\gamma_0 = \frac{\pi}{2} - \arctan \left( \frac{e_0 \sin \varphi_0}{1 + e_0 \cos \varphi_0} \right)
\]

(20)

it is known that

\[
\sin \gamma_0 = \frac{e_0 \sin \varphi_0}{\sqrt{1 + e_0^2 + 2e_0 \cos \varphi_0}}
\]

(21)

Then, Eq. (19) can be rewritten as a function only of \( \varphi_0 \),

\[
e_{0Y} = \frac{p_0}{1 + e_0 \cos \theta_0} \left( v_\infty^2 - \frac{2p_t}{\mu} \right) \left( 1 + e_0 \cos \varphi_0 \right)
= \frac{e_0 \sin \varphi_0}{\sqrt{1 + e_0^2 + 2e_0 \cos \varphi_0}}
\]

(22)

Similarly, the component in the \( X \) axis is

\[
e_{0X} = \frac{p_0}{1 + e_0 \cos \theta_0} \left( v_\infty^2 - \frac{2p_t}{\mu} \right) \left( 1 + e_0 \cos \varphi_0 \right)
= \frac{e_0 \sin \varphi_0}{\sqrt{1 + e_0^2 + 2e_0 \cos \varphi_0}}
\]

(23)

For the eccentricity vector, there is no component in the \( Z \) axis, thus the eccentricity can also be obtained as

\[
e_t = \sqrt{e_{0X}^2 + e_{0Y}^2}
\]

(24)

where \( e_{0X} \) and \( e_{0Y} \) are the components in the \( X \) and \( Y \) axes, respectively. The components in the \( Z \) axis is \( e_{0Z} = 0 \).

The true anomaly \( \varphi_\infty \) associated with \( \gamma_\infty \) on the final orbit is then

\[
\varphi_\infty = \arccos \left( \frac{1}{e_t} \right)
\]

(25)
For the coplanar orbit escape problem with the tangent impulse, we have

\[ F \triangleq \text{atan2}(e_f, e_{i_f}) + \phi_\infty^* - \text{atan2}(\hat{v}_{\infty x}, \hat{v}_{\infty y}) = 0 \] (26)

where \( \text{atan2}(e_y, e_x) \) denotes the argument of perigee for the final orbit, \( \phi_\infty^* \) the true anomaly associated with \( v_\infty^* \) in the final orbit, and \( \text{atan2}(\hat{v}_{\infty x}, \hat{v}_{\infty y}) \) the angle determined by the excess velocity components in the X and Y axes (see Fig. 1). Note that \( F \) is a function only of the initial true anomaly \( \phi_0 \). In Section 4, Eq. (26) will be solved.

4. Solution of the problem

For a given \( v_\infty^* \), the value of \( \text{atan2}(\hat{v}_{\infty x}, \hat{v}_{\infty y}) \) in Eq. (26) is invariable. In addition, \( e_f > 1 \) is satisfied for the final hyperbolic orbit, then from Eq. (25) it is known that \( \phi_\infty^* \in (\pi/2, \pi) \). In the following subsections, two methods are proposed to obtain the solutions for Eq. (26).

4.1. Discontinuous point \( \phi_k \)

In Eq. (26), there is a discontinuous point for \( \text{atan2}(e_y, e_x) \) at \( \phi_k \) which satisfies \( e_f(\phi) = 0 \) and \( e_f(\phi) - \delta > 0 \) (or \( e_f(\phi) + \delta < 0 \)), where \( \delta \) is a constant small enough.

Multiplying both sides of the equation \( e_f = 0 \) by \( \mu(1 + e_0 \cos \phi_0)(1 + e_0^2 + 2e_0 \cos \phi_0) \) gives

\[
(1 + e_0^2 + 2e_0 \cos \phi_0) \left[ p_0 \left( v_\infty^* \right)^2 + \mu(1 + e_0 \cos \phi_0) \right] \sin(e_0 + \phi_0) \\
- e_0 \sin \phi_0 \left[ p_0 \left( v_\infty^* \right)^2 + 2\mu(1 + e_0 \cos \phi_0) \right] \\
[0, \cos \phi_0 + \cos(\phi_0 + \phi_0)] = 0
\] (27)

Simplifying it yields

\[
g(\phi_0) \triangleq \left\{ (1 + e_0^2) \left[ p_0 \left( v_\infty^* \right)^2 + \mu \right] + \left[ (1 + e_0^2) \mu e_0 + e_0 p_0 \left( v_\infty^* \right)^2 \right] \cos \phi_0 \right\} \times \sin(e_0 + \phi_0) + e_0 \left[ p_0 \left( v_\infty^* \right)^2 + 2\mu \right] \sin \phi_0 + 2e_0^2 \sin \phi_0 \cos \phi_0 - e_0^2 \left[ p_0 \left( v_\infty^* \right)^2 + 2\mu \right] \cos e_0 \sin \phi_0 - 2e_0^2 \cos \phi_0 \sin(2\phi_0) = 0
\] (28)

The derivative of \( g(\phi_0) \) with respect to \( \phi_0 \) is

\[
g'(\phi_0) \triangleq \frac{dg(\phi_0)}{d\phi_0} = (1 + e_0^2) \left[ p_0 \left( v_\infty^* \right)^2 + \mu \right] \cos(e_0 + \phi_0) \\
+ \left[ (1 + e_0^2) \mu e_0 + e_0 p_0 \left( v_\infty^* \right)^2 \right] \cos(2\phi_0 + e_0) \\
- 2e_0^2 \sin \phi_0 \cos \phi_0 - e_0^2 \left[ p_0 \left( v_\infty^* \right)^2 + 2\mu \right] \cos e_0 \cos \phi_0 \\
- 2e_0^2 \cos \phi_0 \cos(2\phi_0)
\] (29)

Then, there are two extreme points for \( g(\phi_0) \) in the range \([0, 2\pi]\):

1. If \( g'(0) > 0 \), the maximum point \( \phi_{\max} \) and the minimum point \( \phi_{\min} \) satisfy \( \phi_{\max} < \phi_{\min} \), then \( \phi_k \epsilon (\phi_{\max}, \phi_{\min}) \), thus \( \phi_k \) can be obtained by the golden search with the initial guesses \( \phi_{\max} \) and \( \phi_{\min} \).

2. If \( g'(0) < 0 \), the maximum point \( \phi_{\max} \) and the minimum point \( \phi_{\min} \) satisfy \( \phi_{\max} > \phi_{\min} \). It is known that \( \epsilon(0, \phi_{\min}) \) if \( g(0) > 0 \) and \( \epsilon(\phi_{\min}, 2\pi) \) if \( g(0) < 0 \). Note that \( g(0) = (1 + e_0^2) \left[ p_0 \left( v_\infty^* \right)^2 + \mu(1 + e_0) \right] \sin e_0 \), then \( g(0) > 0 \) when \( \epsilon \in [0, \pi] \) and \( g(0) < 0 \) when \( \epsilon \in (\pi, 2\pi) \). Finally, the numerical value of \( \phi_k \) can be obtained by the golden search with the boundary values as the initial guesses.

4.2. Solution for initial circular orbit

For the initial circular orbit, \( e_0 = 0 \) is satisfied. From Eq. (9) it is known that the eccentricity vector of the final orbit is

\[ e_f = \frac{1}{\mu} \left( \frac{v_1^2 - \mu}{r_0} \right) r_0 = \frac{1}{\mu} \left( \frac{v_\infty^2}{r_0} + \frac{\mu}{r_0} \right) r_0 \] (30)

whose magnitude is

\[ e_f = \frac{1}{\mu} \left( \frac{v_\infty^2}{r_0} + \frac{\mu}{r_0} \right) r_0 = 1 + \frac{\rho_0}{\mu} \left( \frac{v_\infty^2}{r_0} \right)^2 \] (31)

For the initial circular orbit with a tangent impulse, the eccentricity vector of the final orbit is aligned with the radius vector of the impulse point. Then the argument of the impulse point is

\[ \phi_0 = \text{atan2}(\hat{v}_{\infty x}, \hat{v}_{\infty y}) - \phi_\infty^* \]

\[ = \text{atan2}(\hat{v}_{\infty x}, \hat{v}_{\infty y}) - \arccos \left( -\frac{1}{e_f} \right) \] (32)

which is a closed-form solution.

4.3. Solution for initial elliptic orbit

Once the discontinuous point of \( F \) is obtained, the equation \( F = 0 \) can be solved by the secant method. In the whole range \([0, 2\pi]\), there are two piecewise ranges, e.g., \([0, \phi_k]\) and \([\phi_k, 2\pi]\). In addition, it is known that \( F(0) = F(2\pi) = F(\phi_k) + 2\pi \). Extensive numerical examples show that \( F \) changes monotonically. Then there is only one solution in the range \([0, 2\pi]\). If \( F(0) F(\phi_k) < 0 \), the solution is in \([0, \phi_k]\) and it can be obtained by the secant method,

\[ \phi_{0,n+1} = \phi_{0,n} - \frac{F(\phi_{0,n})}{F(\phi_{0,n}) - F(\phi_{0,n-1})} \times (n = 1, 2, 3, \ldots) \] (33)

where the initial guesses are \( \phi_{0,0} = 0 \), and \( \phi_{0,1} = \phi_k - \delta \). Otherwise, the solution is in \([\phi_k, 2\pi]\) and it can be obtained by Eq. (33) with the initial guesses \( \phi_{0,0} = \phi_k \) and \( \phi_{0,1} = 2\pi - \delta \).

Next, some numerical examples are given to show the monotonicity of function \( F \). The perigee height of the initial orbit is \( h_p = 1000 \text{ km} \), then the semimajor axis is \( a_0 = (R_E + h_p)/(1 - e_0) \), where \( R_E = 6378.13 \text{ km} \) is the Earth radius, the initial orbital elements are \( I_0 = 40^\circ \), \( \Omega_0 = 30^\circ \), and \( f_0 = 50^\circ \). If the excess velocity vector is \( v_\infty^* = [-2.4888, 0.1302, 1.6700]^T \text{ km/s} \) in the ECI frame, then the normalized excess vector in the \( XYZ \) frame is \([-0.5000, 0.8660, 0]^T \text{ km/s} \). For different eccentricities \( e_0 \) with 0.1, 0.3, 0.5, 0.7, 0.9, the plots of \( F \) are given in Fig. 2, which shows that \( F \) monotonously increases in both piecewise ranges, and the curve of \( F \) is more linear for a smaller eccentricity.

In addition, for the eccentricity \( e_0 = 0.5 \) and different parameters \( k_\star \), the final excess velocity vector is
\[ v_0 + 1 = k \frac{v}{C_0^2}. \]

The plots of \( F \) are given in Fig. 3, which shows that \( F \) monotonously increases in both piecewise ranges. Therefore, for different eccentricities and excess velocity vectors, \( F \) always changes monotonously in both piecewise ranges.

4.4. Alternative method for avoiding the singularity

Although Section 4.3 gives a method to solve the one-tangent-impulse transfer problem from an elliptic parking orbit to a given excess velocity vector, there is a discontinuous point when solving the equation \( F = 0 \). This subsection provides an alternative method to avoid the singularity.

Eq. (26) can be rewritten as

\[ F_s \triangleq \frac{k}{C_0} v_0 + \sin \left( \varphi_\infty^+ - \text{atan} \left( v_\infty^+ , \hat{v}_\infty^+ \right) \right) = 0 \]

and

\[ F_c \triangleq \frac{k}{C_0} v_0 - \cos \left( \varphi_\infty^+ - \text{atan} \left( v_\infty^+ , \hat{v}_\infty^+ \right) \right) = 0 \]

Then solving Eq. (26) can be transformed into solving Eqs. (34) and (35).

When Eq. (34) is considered, for the example in Section 4.3 of different eccentricities \( e_0 \) with 0.1, 0.3, 0.5, 0.7, 0.9, the plots of \( F_s \) and \( F_c \) are given in Figs. 4 and 5, respectively. The figures show that there are no discontinuous points for \( F_s \) and \( F_c \). However, we need to solve two related functions, i.e., Eqs. (34) and (35); or we need to obtain all solutions for either of Eqs. (34) and (35) and delete the ones not satisfying the other. In addition, \( F_s \) and \( F_c \) do not monotonously change in the whole range of \( \varphi_0 \) (but function \( F \) monotonously increases in both piecewise ranges). The same result can be obtained for the cases with the same eccentricity and different \( v_0^+ \). Thus, the extreme points need to be solved and then the solutions for Eqs. (34) and (35) are obtained. As a result, although the singularity problem is avoided, solving Eqs. (34) and (35) is not much simpler than the proposed method with a discontinuous point.

5. The minimum-fuel multi-tangent-impulse transfer

The previous section solves the one-tangent-impulse transfer problem from a coplanar elliptic orbit to an excess velocity vector. However, if multiple tangent impulses are used to fulfill the mission, there are infinite solutions, and the minimum-fuel solution is the most interesting one. Assume that \( N \) tangent impulses are adopted, the orbital elements for each orbit arc are \( E_i = [a_i, e_i, I_i, \omega_i, \Omega_i, \varphi_i] \), then the initial orbital elements are \( E_0 \), the final orbital elements are \( E_N = E_f \). The position vector \( r_i \) and the velocity vector \( v_i \) for the orbital elements \( E_i \) are obtained by the transformation from orbital elements to position and velocity vectors in the ECI frame. For the \((i + 1)\)th tangent impulse, the impulse position is \( \varphi_i \) and the impulse magnitude is \( k_i \), then the velocity vector at \( \varphi_i \) reaches

\[ v_{i+1} = (1 + \lambda_i) \frac{V_r}{|V_r|}. \]

With \( r_i \) and \( v_{i+1} \), the orbital element \( E_{i+1} \) is obtained by the transformation from position and velocity vectors in the ECI frame to orbital elements, or directly by the proposed method.
in Section 2. Then, our purpose is to solve the true anomaly $\phi_i$ of the impulse position and the corresponding impulse magnitude $\lambda_i$ ($i = 1, 2, \ldots, N$) such that the total cost

$$\Delta V_{\text{total}} = \sum_{i=1}^{N} \lambda_i$$

(37)

is minimized.

When $\phi_i$ and $\lambda_i$ of the tangent impulse for $i = 1, 2, \ldots, N-1$ are fixed, the values of $\phi_N$ and $\lambda_N$ for the final impulse can be obtained by using the proposed numerical method in Section 4. Thus, for the $N$ tangent-impulse case, only 2($N-1$) variables are needed in the numerical optimization approach. This paper will use the genetic method to obtain a numerical solution to the minimum-fuel multi-tangent-impulse transfer problem.

6. Numerical simulations

Assume that a spacecraft moves in an initial elliptic orbit, where the perigee height is $h_0 = 1000$ km, the semimajor axis is $a_0 = (R_E + h_0)/(1 - e_0)$, where $R_E = 6378.13$ km is the Earth radius, the orbit inclination is $i_0 = 45^\circ$, the argument of perigee is $\omega_0 = 30^\circ$, and the right ascension of ascending node is $\Omega_0 = 50^\circ$.

6.1. One tangent impulse

If the excess velocity vector is reached by a single tangent impulse, the solution is obtained by the secant method in Section 4. If the excess velocity vector is $v^* = [-2.4888, 0.1302, 1.6700]^T$ km/s in the ECI frame, which is three times the velocity vector at $\phi_0 = 0^\circ$ with $e_0 = 0.5$, then the normalized excess vector in the $XYZ$ frame is $[-0.5000, 0.8660, 0]^T$ km/s; for different eccentricities $e_0$ with 0.1, 0.3, 0.5, 0.7, 0.9, the plots of $F$ are given in Fig. 2, and the plots of $P$ and $F$ are given in Figs. 4 and 5, respectively. The solutions are listed in Table 1.

If the eccentricity $e_0 = 0.5$, for different $k$, the final excess velocity vector is $v^* = k[-2.4888, 0.1302, 1.6700]^T$ km/s in the ECI frame, the plots of $F$ are given in Fig. 3, and the solutions are listed in Table 2, which shows that more energy cost is required for a larger final excess velocity.

When $e_0 = 0.5$ and $v^* = [-2.4888, 0.1302, 1.6700]^T$ km/s, the solution is $\phi_0 = 252.8762^\circ$ and the corresponding cost is $\lambda = 2.5256$ km/s. The trajectories of the initial and final excess orbits in the orbit plane (or the $XYZ$ frame) are plotted in Fig. 1, where $F_O$ is the direction of the final eccentricity vector in the $X$–$Y$ plane. The initial position vector at $P_1$ is $r_0 = [9283.1, -4014.2, -8132.2]^T$ km, and the velocity vector of the initial orbit at $P_1$ is $v_0 = [0.4864, 5.2705, 2.5300]^T$ km/s in the ECI frame. With a tangent impulse whose magnitude is $\lambda = 2.5256$ km/s, the orbit elements of the final hyperbolic orbit at $P_1$ are $E_f = [-44288.9, 1.2294, 40^\circ, 335.5685^\circ, 50^\circ, 307.3077^\circ]$. With these hyperbolic orbital parameters, after the time-of-flight $1.0 \times 10^5$ s for the two-body motion, the error of the velocity vector and the required excess velocity vector is $1.861 \times 10^6$ km/s.

When $e_0 = 0.5$ and $v^* = 7 \times [-2.4888, 0.1302, 1.6700]^T$ km/s, the solution is $\phi_0 = 302.2874^\circ$ and the corresponding cost is $\lambda = 3.8274$ km/s. The trajectories of the initial and final excess orbits in the orbit plane are plotted in Fig. 6. The initial position vector at $P_1$ is $r_0 = [7353.9, 3923.4, -2610.9]^T$ km, and the velocity vector of the initial orbit at $P_1$ is $v_0 = [-3.8132, 4.8843, 5.0855]^T$ km/s in the ECI frame. With a tangent impulse whose magnitude is $\lambda = 3.8274$ km/s, the orbit elements of the final hyperbolic orbit at $P_1$ are $E_f = [-8134.7, 1.9924, 40^\circ, 359.8744^\circ, 50^\circ, 332.4130^\circ]$. With these hyperbolic orbital parameters, after the time-of-flight $1.0 \times 10^5$ s for the two-body motion, the error of the velocity vector and the required excess velocity vector is $8.1861 \times 10^6$ km/s.

6.2. Multiple tangent impulse

If multiple tangent impulses are used for the orbit transfer to the excess velocity vector problem, the minimum-energy solution of $[\phi_i, \lambda_i], i = 1, 2, \ldots, N-1$ can be obtained by numerical optimization methods; the solution of $[\phi_N, \lambda_N]$ for the final impulse is obtained by the method in Section 4. Herein, the genetic algorithm ("ga" in MATLAB) is used for this optimization problem. The orbital elements of the initial orbit are the same as that in Section 6.1 with $e_0 = 0.5$ and $v^* = [-2.4888, 0.1302, 1.6700]^T$ km/s. Then the two, three,

<table>
<thead>
<tr>
<th>$e_0$</th>
<th>Solution of $\phi_0$ (°)</th>
<th>Impulse magnitude $\lambda$ (km/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>292.2646</td>
<td>3.2307</td>
</tr>
<tr>
<td>0.3</td>
<td>273.3140</td>
<td>2.8359</td>
</tr>
<tr>
<td>0.5</td>
<td>252.8762</td>
<td>2.5256</td>
</tr>
<tr>
<td>0.7</td>
<td>231.7169</td>
<td>2.2595</td>
</tr>
<tr>
<td>0.9</td>
<td>210.2594</td>
<td>1.9600</td>
</tr>
</tbody>
</table>

Table 1. Solutions of $\phi_0$ for the same eccentricity and different excess velocity vectors.

<table>
<thead>
<tr>
<th>$k_1$</th>
<th>Solution of $\phi_1$ (°)</th>
<th>Impulse magnitude $\lambda$ (km/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>227.2309</td>
<td>2.4349</td>
</tr>
<tr>
<td>3</td>
<td>252.8762</td>
<td>2.5256</td>
</tr>
<tr>
<td>5</td>
<td>279.9323</td>
<td>3.0033</td>
</tr>
<tr>
<td>7</td>
<td>302.2874</td>
<td>3.8274</td>
</tr>
<tr>
<td>9</td>
<td>318.2679</td>
<td>4.9338</td>
</tr>
</tbody>
</table>

Table 2. Solutions of $\phi_0$ for the same eccentricity and different excess velocity vectors.
and four tangent impulses are used, respectively. The solutions are obtained as listed in Table 3, which shows that the costs of optimal one-, two-, three- and four-impulse transfer are almost the same. For the one-tangent-impulse case, the solution is obtained directly by the method in Section 4, thus this optimization method is not needed. It should be notified that the orbital elements $E_i$ will not change when $\lambda = 0$ even though the true anomalies are different.

As a result, the optimal multi-tangent-impulse transfer from coplanar elliptic orbit to an excess velocity vector is the same as that with one tangent impulse. Because this result is only obtained by numerical methods, it is not easy to give a stick explanation. However, for different eccentricities and different final excess velocity vectors, the result still holds by using numerical optimization algorithms.

7. Conclusions

In this paper the coplanar orbit escape problem from an elliptic orbit to an excess velocity vector is studied with the condition that only tangent impulses are used. Since the impulse direction is aligned with the velocity vector, the magnitude and the true anomaly of the position point of the tangent impulse are solved. For initial circular orbit, the solution can be obtained in closed-form, whereas for initial elliptic orbit, the solution is only obtained by a numerical iterative algorithm. The one-tangent-impulse solution is the optimal one even though the multi-tangent-impulse can be used. The proposed method for the coplanar tangent-orbit design of escape trajectories can be used in space exploration, such as human Lunar and Mars exploration missions. The tangent escape trajectories provide a new approach for orbit escape transfer with only speed changes.

Acknowledgements

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Table 3 Solutions of $[\varphi_i, \lambda_i], i = 1, 2, ..., N - 1$ and the total costs for different numbers of tangent impulses.

<table>
<thead>
<tr>
<th>Impulse number $N$</th>
<th>Solution of $[\varphi_i, \lambda_i]$ ($^\circ$), km/s</th>
<th>Solution of $[\varphi_N, \lambda_N]$ ($^\circ$), km/s</th>
<th>Total cost $\Delta v_{\text{total}}$ (km/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$[0.1561, 4.3185 \times 10^{-4}]$</td>
<td>$[252.8762, 2.525643]$</td>
<td>2.525643</td>
</tr>
<tr>
<td>2</td>
<td>$[34.4573, 0], [29.0005, 0]$</td>
<td>$[252.8762, 2.525643]$</td>
<td>2.525667</td>
</tr>
<tr>
<td>3</td>
<td>$[5.9257, 0], [84.6521, 0], [12.9435, 4.8346 \times 10^{-4}]$</td>
<td>$[252.8742, 2.525225]$</td>
<td>2.525674</td>
</tr>
<tr>
<td>4</td>
<td>$[252.8742, 2.525225]$</td>
<td>$[252.8742, 2.525225]$</td>
<td>2.525674</td>
</tr>
</tbody>
</table>

References