On multi-orthogonal bases in finite-dimensional non-Archimedean normed spaces

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ABSTRACT

The paper deals with the problem of the existence multi-orthogonal bases in finite-dimensional normed spaces over K, where K is a non-Archimedean complete valued field.

1. INTRODUCTION

Throughout this paper K will denote a non-Archimedean, non-trivially valued field which is complete under the metric induced by the valuation $|.| : K \to [0, \infty)$ and E will denote a finite-dimensional linear space over K. Every considered norm, defined on E, will be *non-Archimedean* (i.e. it satisfies 'the strong triangle inequality': $\|x + y\| \leqslant \max\{\|x\|, \|y\|\}$ for all $x, y \in E$). Recall that for a given norm $\|.\|$, defined on E, a sequence $(x_i)_{i=1}^n \subset E$ $(n \in N)$ is called *orthogonal* if $\|\lambda_1 x_1 + \cdots + \lambda_n x_n\| = \max_{i=1,\ldots,n} \|\lambda_i x_i\|$ for any $\lambda_1, \ldots, \lambda_n \in K$. Additionally, we say that an orthogonal sequence $(x_i)_{i=1}^n \subset E$ is a *base* of E if $[x_1, \ldots, x_n] = E$. Then, for every $x \in E$ there is a unique $(\lambda_i)_{i=1}^n \in K^n$ such that $x = \sum_{i=1}^n \lambda_i x_i$. A linear subspace D of E is said to be *orthocomplemented in* E if there is a linear subspace D_0 of E such that $D + D_0 = E$ and $D \perp D_0$ (i.e. $\|x + y\| = \max\{\|x\|, \|y\|\}$ for all $x \in D$, $y \in D_0$). Let $\|.\|_1, \ldots, \|.\|_k$ be norms defined on E. We say that a sequence $(x_i)_{i=1}^n \subset E$ $(n \in N)$ is *multi-orthogonal* in E if it is orthogonal with respect to all $\|.\|_1, \ldots, \|.\|_k$ and we say that a linear subspace D of E is

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multi-orthocomplemented if there exists a linear subspace D_0 of E which satisfies $D + D_0 = E$ and $D \perp D_0$ with respect to all $\|.\|_1, \ldots, \|.\|_k$.

The problem of the existence of multi-orthogonal bases in finite-dimensional normed spaces was presented by A. van Rooij and W. Schikhof in 1992 (see Problem 3 of [3]). They noted that if $\|.\|_1$, $\|.\|_2$ are norms, defined on E, such that $(E, \|.\|_1)$ and $(E, \|.\|_2)$ both have orthogonal bases, then there exists a base of E, so called a *multi-orthogonal base*, which is orthogonal to both $\|.\|_1$ and $\|.\|_2$. In [3] A. van Rooij and W. Schikhof ask if the similar result is true for three or finitely many norms.

In this paper we solve this problem. In Theorem 5 we give a negative answer for this question, proving that there exist three norms defined on two-dimensional linear space for which there is no multi-orthogonal base, although for every defined norm there exists an orthogonal base. In Theorem 9 we present some equivalent conditions for existence of multi-orthogonal bases in finite-dimensional normed space. Example 6 contains the construction of the linear space, where three norms are defined, with a multi-orthogonal base and a linear subspace without such base.

For more background of the theory of non-Archimedean normed spaces we refer the reader to [1] and [2].

2. RESULTS

To obtain the main result (Theorem 5), the construction of three norms defined on two-dimensional E in such way that there is no base which is orthogonal with respect to all three norms, although E possesses an orthogonal base for every defined norm, we need some preparation.

Lemma 1. *Let* $\dim E = 2$ *and let* $\|.\|$ *be a norm on E with orthogonal base* $\{e_1, e_2\}$, *where* $\|e_1\| > \|e_2\|$. *Take nonzero* $u = c_1e_1 + c_2e_2 \in E$ $(c_1, c_2 \in K)$ *such that* $u \perp (e_1 + e_2)$. *Then,* $|c_1| < |c_2|$.

Proof. Assume that $|c_1| \geqslant |c_2|$, then $\|u\| = \max\{\|c_1e_1\|, \|c_2e_2\|\} = \|c_1e_1\|$. But, we obtain

$$\begin{aligned}\|u - c_1(e_1 + e_2)\| &= \|(c_1e_1 + c_2e_2) - (c_1e_1 + c_1e_2)\| = \|c_2e_2 - c_1e_2\| \\ &\leqslant \max\{\|c_2e_2\|, \|c_1e_2\|\} = \max\{|c_2|, |c_1|\} \cdot \|e_2\| \\ &= \|c_1e_2\| < \|c_1e_1\| = \|u\|,\end{aligned}$$

a contradiction with $u \perp (e_1 + e_2)$. □

Lemma 2. *Let* $\dim E = 2$ *and* $\|.\|_1, \|.\|_2$ *be norms defined on E. Assume that* $\{e_1, e_2\}$ *is a multi-orthogonal base (i.e. orthogonal with respect to* $\|.\|_1$ *and* $\|.\|_2$*) on E such that*

$$(1) \qquad \|e_1\|_1 > \|e_2\|_1 \quad \text{and} \quad \|e_1\|_2 < \|e_2\|_2.$$

Then, $z := e_1 + e_2 \in E$ *possesses no nonzero multi-orthogonal element in E.*

Proof. Assume that there exists $u = c_1 e_1 + c_2 e_2$ ($c_1, c_2 \in K$), an element of E which is multi-orthogonal to z. Then, applying Lemma 1 to $\|.\|_1$ and the base $\{e_1, e_2\}$, we imply $|c_1| < |c_2|$. On the other hand, using Lemma 1 to $\|.\|_2$ and the base $\{e_2, e_1\}$, we obtain $|c_1| > |c_2|$, a contradiction. $\square$

We note that norms defined on two-dimensional E, which satisfies the condition (1), really exist.

Example 3. Let $E = K^2$ and let $\lambda \in K$, $|\lambda| < 1$. Define two norms on E by

$$\|(x_1, x_2)\|_1 := \max\{|x_1|, |\lambda x_2|\},$$
$$\|(x_1, x_2)\|_2 := \max\{|\lambda x_1|, |x_2|\}.$$

Then, it is easy to check that $\{e_1, e_2\}$ (the standard base of K^2) is an orthogonal base of E with respect to $\|.\|_1$ and $\|.\|_2$, which satisfies the condition (1).

Lemma 4. *Let* $\dim E = 2$ *and let* e_1, e_2 *be nonzero, linearly independent elements of* E. *Take* $\lambda \in K$, *such that* $|\lambda| > 1$. *Then,*

$$\|c_1 e_1 + c_2 e_2\|_3 := \max\left\{\left|\left(1 + \frac{1}{\lambda^2}\right)c_1 - c_2\right|, \left|c_1 - \left(1 + \frac{1}{\lambda}\right)c_2\right|\right\} \quad (c_1, c_2 \in K) \tag{2}$$

is a norm on E *for which* $\|e_1\|_3 = \|e_2\|_3 = 1$, $\|e_1 + e_2\|_3 = \frac{1}{\lambda} < 1$ *and* $\{e_1 + e_2, e_2\}$ *is an orthogonal base of* $(E, \|.\|_3)$.

Proof. It is easy to verify that $\|.\|_3$ is a norm and $\|e_1\|_3 = \|e_2\|_3 = 1$, $\|e_1 + e_2\|_3 = \frac{1}{\lambda} < 1$. Now, we prove that $\{e_1 + e_2, e_2\}$ is an orthogonal base of $(E, \|.\|_3)$. Taking $a \in K$, we get

$$\begin{aligned}
\|e_1 + e_2 + a e_2\|_3 &= \max\left\{\left|\left(1 + \frac{1}{\lambda^2}\right) - (1 + a)\right|, \left|1 - \left(1 + \frac{1}{\lambda}\right)(1 + a)\right|\right\} \\
&= \max\left\{\left|\frac{1}{\lambda^2} - a\right|, \left|\frac{1}{\lambda} + a + \frac{a}{\lambda}\right|\right\} = \max\left\{\left|\frac{1}{\lambda}\right|, |a|\right\} \\
&= \max\{\|e_1 + e_2\|_3, \|a e_2\|_3\}. \quad \square
\end{aligned}$$

Now, we are ready to prove

Theorem 5. *Let* $\dim E = 2$. *Then, there exist* $\|.\|_1, \|.\|_2, \|.\|_3$, *three norms defined on* E, *such that* $(E, \|.\|_i)$ *has an orthogonal base for every* $i \in \{1, 2, 3\}$, *but there is no base on* E *which is orthogonal with respect to all* $\|.\|_1, \|.\|_2, \|.\|_3$.

Proof. First, we observe that using Example 3 we can define $\|.\|_1, \|.\|_2$ on E in such a way that there exists $\{e_1, e_2\}$, a multi-orthogonal base (i.e. orthogonal with respect to $\|.\|_1$ and $\|.\|_2$) on E, which satisfies condition (1). Next, we define $\|.\|_3$, as the norm introduced in (2), applying Lemma 4 to the base $\{e_1, e_2\}$, mentioned above. In this way, we equip E in three norms $\|.\|_1$, $\|.\|_2$ and $\|.\|_3$, such that for every one there exists an orthogonal base.

Now, suppose that there exists $\{u, w\}$, a base of E which is orthogonal with respect to all three norms. Let $u := a_1e_1 + a_2e_2$, $w := b_1e_1 + b_2e_2$. We may assume that $a_2 = 1$ (by linear independence either $a_2 \neq 0$ or $b_2 \neq 0$; by symmetry we may suppose that $a_2 \neq 0$). Since by assumption, $u = a_1e_1 + e_2$ possesses an orthogonal element in E with respect to $\|.\|_1$ and $\|.\|_2$; hence, applying Lemma 2 to the base $\{a_1e_1, e_2\}$, we conclude that $|a_1\| \leqslant \frac{\|e_2\|_1}{\|e_1\|_1} < 1$ or $|a_1| \geqslant \frac{\|e_2\|_2}{\|e_1\|_2} > 1$.

Consider the case where $b_2 = 0$ (then, obviously $b_1 \neq 0$). If $|a_1| \geqslant \frac{\|e_2\|_2}{\|e_1\|_2}$ then

$$\left\|u - \frac{a_1}{b_1}w\right\|_3 = \left\|a_1e_1 + e_2 - \frac{a_1}{b_1}b_1e_1\right\|_3 = \|e_2\|_3 = 1.$$

But $\|u\|_3 = \|a_1e_1 + e_2\|_3 = |a_1| > 1$, by assumption; hence, $\|u - \frac{a_1}{b_1}w\|_3 < \|u\|_3$, a contradiction. If $|a_1| \leqslant \frac{\|e_2\|_1}{\|e_1\|_1} < 1$, then

$$\|u\|_3 = \max\left\{\left|\left(1 + \frac{1}{\lambda^2}\right)a_1 - 1\right|, \left|a_1 - \left(1 + \frac{1}{\lambda}\right)\right|\right\} = 1$$

and

$$\begin{aligned}\left\|u + \frac{1}{b_1}w\right\|_3 &= \|a_1e_1 + e_2 + e_1\|_3 \\ &= \max\left\{\left|\left(1 + \frac{1}{\lambda^2}\right)(a_1 + 1) - 1\right|, \left|a_1 + 1 - \left(1 + \frac{1}{\lambda}\right)\right|\right\} \\ &= \max\left\{\left|a_1 + \frac{1}{\lambda^2} + \frac{a_1}{\lambda^2}\right|, \left|a_1 - \frac{1}{\lambda}\right|\right\} < 1 = \|u\|_3.\end{aligned}$$

This contradicts to $u \perp w$ with respect to $\|.\|_3$.

Let $b_2 \neq 0$. Without loss of generality we can assume that $b_2 = 1$. Since, we suppose that $w = b_1e_1 + e_2$ possesses an orthogonal element in E with respect to $\|.\|_1$ and $\|.\|_2$ and $\{e_1, e_2\}$ satisfies condition (1), we imply that $|b_1\| \neq 1$.

Suppose that $|a_1| \geqslant \frac{\|e_2\|_2}{\|e_1\|_2} > 1$. Then $\|u\|_3 = \|a_1e_1 + e_2\|_3 = |a_1|$. Let $|b_1| > 1$. We obtain

$$\begin{aligned}\left\|u - \frac{a_1}{b_1}w\right\|_3 &= \left\|a_1e_1 + e_2 - a_1e_1 - \frac{a_1}{b_1}e_2\right\|_3 \\ &= \left|1 - \frac{a_1}{b_1}\right|\|e_2\|_3 < |a_1| = \|u\|_3,\end{aligned}$$

a contradiction. Taking $|b_1| < 1$, we get

$$\begin{aligned}\|u + a_1w\|_3 &= \|a_1e_1 + e_2 + a_1b_1e_1 + a_1e_2\|_3 \\ &\leqslant \max\{\|a_1(e_1 + e_2)\|_3, \|e_2 + a_1b_1e_1\|_3\} < |a_1| = \|u\|_3,\end{aligned}$$

since $\|e_1 + e_2\|_3 < 1$, a contradiction.

Let $|a_1| \leqslant \frac{\|e_2\|_1}{\|e_1\|_1} < 1$. If $|b_1| < 1$, we obtain

$$\|u - w\|_3 = \|a_1 e_1 + e_2 - b_1 e_1 - e_2\|_3 = |a_1 - b_1| \cdot \|e_1\|_3 < \|e_1\|_3 = 1.$$

For $|b_1| > 1$ we get

$$\begin{aligned}\left\|u + \frac{1}{b_1} w\right\|_3 &= \left\|a_1 e_1 + e_2 + \frac{1}{b_1}(b_1 e_1 + e_2)\right\|_3 \\ &= \left\|(a_1 + 1)e_1 + \left(1 + \frac{1}{b_1}\right)e_2\right\|_3 \\ &= \max\left\{\left|\left(1 + \frac{1}{\lambda^2}\right)(a_1 + 1) - \left(1 + \frac{1}{b_1}\right)\right|,\right. \\ &\qquad \left.\left|a_1 + 1 - \left(1 + \frac{1}{\lambda}\right)\left(1 + \frac{1}{b_1}\right)\right|\right\} < 1\end{aligned}$$

but $\|u\|_3 = 1$, a contradiction. □

Gruson's theorem (Theorem 5.9 of [2]) says that every closed linear subspace of a Banach space with an orthogonal base has an orthogonal base, either. The following example shows that the counterpart for multi-orthogonal bases is not true.

Example 6. Let $E := K^3$ and let $\lambda \in K$, $|\lambda| > 1$. We define three norms on E by

$$\begin{aligned}\|(x_1, x_2, x_3)\|_1 &:= \max\{x_1|, |x_2|, |x_3|\}, \\ \|(x_1, x_2, x_3)\|_2 &:= \max\left\{|\lambda x_1|, \left|\frac{x_2}{\lambda^2}\right|, |\lambda x_3|\right\}, \\ \|(x_1, x_2, x_3)\|_3 &:= \max\left\{\left|\frac{x_1}{\lambda^3}\right|, \left|\frac{x_2}{\lambda^3}\right|, |x_3|\right\}.\end{aligned}$$

Then, E has a multi-orthogonal base (i.e. orthogonal with respect to all three norms $\|.\|_1, \|.\|_2, \|.\|_3$), but the subspace $[u, w]$, where $u := \lambda e_1 + e_2 + e_3$, $w := e_1 + \lambda^2 e_2 + e_3$, has no multi-orthogonal base ($\{e_1, e_2, e_3\}$ denotes a standard base of K^3).

Proof. It is easy to check that the standard base $\{e_1, e_2, e_3\}$ is orthogonal with respect to all three norms $\|.\|_1, \|.\|_2, \|.\|_3$. First, we prove that u, w are orthogonal with respect to $\|.\|_1$ and $\|.\|_2$. Let $\lambda_0 \in K$. Then

$$\|u + \lambda_0 w\|_1 = \max\{|\lambda + \lambda_0|, |1 + \lambda_0 \lambda^2|, |1 + \lambda_0|\} \geqslant |\lambda| = \|u\|_1$$

and

$$\|u + \lambda_0 w\|_2 = \max\left\{|\lambda| \cdot |\lambda + \lambda_0|, \left|\frac{1}{\lambda^2} + \lambda_0\right|, |\lambda| \cdot |1 + \lambda_0|\right\} \geqslant |\lambda^2| = \|u\|_2.$$

Observe that u, w are not orthogonal with respect to $\|.\|_3$:

$$\|u - w\|_3 = \max\left\{\left|\frac{\lambda - 1}{\lambda^3}\right|, \left|\frac{1 - \lambda^2}{\lambda^3}\right|, |1 - 1|\right\} < 1 = \|u\|_3.$$

Take $\mu_1 u + \mu_2 w \in [u, w]$ $(\mu_1, \mu_2 \in K)$ and assume that $\mu_1 u + \mu_2 w$ has a multi-orthogonal element in $[u, w]$.

Consider the case where $\mu_1 \neq 0$; without loss of generality we can assume that $\mu_1 = 1$. It follows from Lemma 2 that $u + \mu_2 w$ has an orthogonal element in $[u, w]$ with respect to $\|.\|_1, \|.\|_2$ only if

$$(3)\qquad \begin{aligned} \|\mu_2 w\|_1 \leqslant \|u\|_1 \quad &\Longrightarrow \quad |\mu_2| \leqslant \frac{\|u\|_1}{\|w\|_1} = \frac{1}{|\lambda|}, \quad \text{or} \\ \|\mu_2 w\|_2 \geqslant \|u\|_2 \quad &\Longrightarrow \quad |\mu_2| \geqslant \frac{\|u\|_2}{\|w\|_2} = |\lambda|. \end{aligned}$$

Now, assume that $\beta_1 u + \beta_2 w$ $(\beta_1, \beta_2 \in K)$ is a multi-orthogonal element to $u + \mu_2 w$ in $[u, w]$. Note that by Lemma 2, since the base $\{w, u\}$ satisfies condition (1), it is necessary to have $|\beta_1| \neq |\beta_2|$. Take $\lambda_0 = -1/(\beta_1 + \beta_2)$. If $|\mu_2| \leqslant \frac{1}{|\lambda|} < 1$, we obtain

$$\begin{aligned} &\|u + \mu_2 w + \lambda_0(\beta_1 u + \beta_2 w)\|_3 \\ &\quad = \max\left\{\left|\frac{\lambda + \mu_2 + \lambda_0(\beta_1\lambda + \beta_2)}{\lambda^3}\right|, \left|\frac{1 + \mu_2\lambda^2 + \lambda_0(\beta_1 + \beta_2\lambda^2)}{\lambda^3}\right|,\right. \\ &\qquad\qquad \left. |1 + \mu_2 + \lambda_0(\beta_1 + \beta_2)|\right\} \\ &\quad \leqslant \max\left\{\frac{1}{|\lambda^2|}, \frac{1}{|\lambda|}, |\mu_2|\right\} < 1, \end{aligned}$$

a contradiction to the assumption, since

$$\|u + \mu_2 w\|_3 = \max\left\{\left|\frac{\lambda + \mu_2}{\lambda^3}\right|, \left|\frac{1 + \mu_2\lambda^2}{\lambda^3}\right|, |1 + \mu_2|\right\} = 1.$$

If $|\mu_2| \geqslant |\lambda|$ then, taking $\lambda_0 = -\mu_2/(\beta_1 + \beta_2)$, we get

$$\begin{aligned} &\|u + \mu_2 w + \lambda_0(\beta_1 u + \beta_2 w)\|_3 \\ &\quad = \max\left\{\left|\frac{\lambda + \mu_2 + \lambda_0(\beta_1\lambda + \beta_2)}{\lambda^3}\right|, \left|\frac{1 + \mu_2\lambda^2 + \lambda_0(\beta_1 + \beta_2\lambda^2)}{\lambda^3}\right|,\right. \\ &\qquad\qquad \left. |1 + \mu_2 + \lambda_0(\beta_1 + \beta_2)|\right\} \\ &\quad \leqslant \max\left\{\frac{|\mu_2|}{|\lambda^3|}, \frac{|\mu_2|}{|\lambda|}, 1\right\} < |\mu_2|, \end{aligned}$$

but

$$\|u + \mu_2 w\|_3 = \max\left\{\left|\frac{\lambda + \mu_2}{\lambda^3}\right|, \left|\frac{1 + \mu_2\lambda^2}{\lambda^3}\right|, |1 + \mu_2|\right\} = |\mu_2|,$$

a contradiction.

Now, let $\mu_1 = 0$. We can assume that $\mu_2 = 1$ and that there exists $\beta_1 u + \beta_2 w \in [u, w]$ $(\beta_1, \beta_2 \in K)$, a multi-orthogonal element to u. Since u and w are not orthogonal with respect to $\|.\|_3$; hence $\beta_1 \neq 0$, without loss of generality we can assume that $\beta_1 = 1$. Using Lemma 2, we imply that there exists an orthogonal element to $u+\beta_2 w$ with respect to $\|.\|_1, \|.\|_2$ only if $|\beta_2| \leqslant \frac{1}{|\lambda|}$ or $|\beta_2| \geqslant |\lambda|$. Suppose that $|\beta_2| \geqslant |\lambda|$. Then, taking $\lambda_0 = -1/\beta_2$ we get

$$\|u + \lambda_0(u + \beta_2 w)\|_3 = \left\| u - \frac{u}{\beta_2} - w \right\|_3 \leqslant \max\left\{ \|u - w\|_3, \left\| \frac{u}{\beta_2} \right\|_3 \right\} < \|u\|_3,$$

a contradiction. Considering the case where $|\beta_2| \leqslant \frac{1}{|\lambda|}$, choosing $\lambda_0 = -1$, we obtain

$$\|u + \lambda_0(u + \beta_2 w)\|_3 = \|\beta_2 w\|_3 \leqslant \frac{1}{|\lambda|}\|w\|_3 < \|u\|_3$$

and finishing the proof. □

Observe the following fact:

Proposition 7. *Let $\|.\|_1, \|.\|_2$ be norms on E and let $x_0 \in E$ ($x_0 \neq 0$) be such that $[x_0]$ is multi-orthocomplemented in E. If $D \subset E$ is a multi-orthocomplement of $[x_0]$ in E then there exists $\lambda_0 \in K$ or there exists $x_d \in D$ such that*

$$\max_{x\in E, x\neq 0} \frac{\|x\|_2}{\|x\|_1} = \frac{\|x_d\|_2}{\|x_d\|_1} \quad or \quad \max_{x\in E, x\neq 0} \frac{\|x\|_2}{\|x\|_1} = \frac{\|x_0\|_2}{\|x_0\|_1}.$$

Proof. Take $y \in E$, $y \neq 0$, where $y = \lambda x_0 + d$, $\lambda \in K$, $d \in D$, such that $\frac{\|y\|_2}{\|y\|_1} = \max_{x\in E, x\neq 0} \frac{\|x\|_2}{\|x\|_1}$. Then

$$\frac{\|y\|_2}{\|y\|_1} = \frac{\|\lambda x_0 + d\|_2}{\|\lambda x_0 + d\|_1} = \frac{\max\{\|\lambda x_0\|_2, \|d\|_2\}}{\max\{\|\lambda x_0\|_1, \|d\|_1\}}.$$

If $\|\lambda x_0\|_2 \geqslant \|d\|_2$, then

$$\frac{\|y\|_2}{\|y\|_1} = \frac{\|\lambda x_0 + d\|_2}{\|\lambda x_0 + d\|_1} = \frac{\|\lambda x_0\|_2}{\max\{\|\lambda x_0\|_1, \|d\|_1\}} \leqslant \frac{\|\lambda x_0\|_2}{\|\lambda x_0\|_1} = \frac{\|x_0\|_2}{\|x_0\|_1}.$$

On the other hand, if $\|\lambda x_0\|_2 < \|d\|_2$ then

$$\frac{\|y\|_2}{\|y\|_1} = \frac{\|\lambda x_0 + d\|_2}{\|\lambda x_0 + d\|_1} = \frac{\|d\|_2}{\max\{\|\lambda x_0\|_1, \|d\|_1\}} \leqslant \frac{\|d\|_2}{\|d\|_1} \leqslant \max_{x\in D, x\neq 0} \frac{\|x\|_2}{\|x\|_1}. \quad \square$$

We can easily conclude the corollary:

Corollary 8. *Let $\|.\|_1, \|.\|_2$ be norms on E and let $\{e_1, \dots, e_n\}$ be a multi-orthogonal base in E. Then, there exist indices $i, j \in \{1, \dots, n\}$ such that*

$$\max_{x\in E, x\neq 0} \frac{\|x\|_2}{\|x\|_1} = \frac{\|e_i\|_2}{\|e_i\|_1} \quad and \quad \max_{x\in E, x\neq 0} \frac{\|x\|_1}{\|x\|_2} = \frac{\|e_j\|_1}{\|e_j\|_2}.$$

Next theorem gives some conditions of the existence of a multi-orthogonal base in finite-dimensional normed space for given three norms.

Theorem 9. *Let $\|.\|_1$, $\|.\|_2$ be norms on E and $\{e_1, \dots, e_n\}$ be a multi-orthogonal base (orthogonal with respect to $\|.\|_1$ and $\|.\|_2$) on E. Then, the following conditions are equivalent:*

(1) $\frac{\|e_1\|_1}{\|e_1\|_2} = \dots = \frac{\|e_n\|_1}{\|e_n\|_2} = p$;

(2) $\frac{\|x\|_1}{\|x\|_2} = p$ *for every nonzero* $x \in E$;

(3) *for every norm* $\|.\|_3$ *on* E, *every two-dimensional linear subspace of* E *possesses a base, orthogonal with respect to* $\|.\|_1$, $\|.\|_2$ *and* $\|.\|_3$;

(4) *for every norm* $\|.\|_3$ *on* E, E *possesses a base, orthogonal with respect to* $\|.\|_1$, $\|.\|_2$ *and* $\|.\|_3$.

Proof. (1) $\Rightarrow$ (2). Assume that $\|e_i\|_1 = p \cdot \|e_i\|_2$ for each $i \in \{1, \dots, n\}$. Let $x \in E$ $(x \neq 0)$. Then, there exist $a_1, \dots, a_n \in K$ such that $x = \sum_{i=1}^n a_i e_i$ and we get

$$\|x\|_1 = \left\| \sum_{i=1}^n a_i e_i \right\|_1 = \max_{i=1,\dots,n} \{|a_i| \cdot \|e_i\|_1\} = \max_{i=1,\dots,n} \{|a_i| \cdot p \cdot \|e_i\|_2\}$$
$$= p \cdot \max_{i=1,\dots,n} \{\|a_i e_i\|_2\} = p \cdot \left\| \sum_{i=1}^n a_i e_i \right\|_2 = p \cdot \|x\|_2.$$

The implication (2) $\Rightarrow$ (1) is obvious.

(2) $\Rightarrow$ (3). Let $\|.\|_3$ be a norm defined on E and let F be a two-dimensional linear subspace of E. It follows from the assumption that every orthogonal base of F with respect to $\|.\|_1$ is also orthogonal with respect to $\|.\|_2$. Since, by Theorem 1.11 of [1], there exists a base $\{z_1, z_2\}$ in F which is orthogonal with respect to $\|.\|_1$ and $\|.\|_3$, $\{z_1, z_2\}$ is also orthogonal with respect to $\|.\|_2$.

(3) $\Rightarrow$ (2). Assume the contrary and suppose that there exist nonzero $z_1, z_2 \in E$ such that $\frac{\|z_1\|_1}{\|z_1\|_2} > \frac{\|z_2\|_1}{\|z_2\|_2}$. From Corollary 8, we get that there exists a base $\{x_1, x_2\}$ of $[z_1, z_2]$, which is orthogonal with respect to $\|.\|_1$ and $\|.\|_2$, such that $\frac{\|x_1\|_1}{\|x_1\|_2} > \frac{\|x_2\|_1}{\|x_2\|_2}$. Then $\frac{\|x_1\|_1}{\|x_2\|_1} > \frac{\|x_1\|_2}{\|x_2\|_2}$. Define a norm $\|.\|_3$ on E which satisfies the following properties:

- $\|x_1 - [x_2]\|_3 < \|x_1\|_3$,
- $\frac{\|x_1\|_1}{\|x_2\|_1} > \frac{\|x_1\|_3}{\|x_2\|_3} > \frac{\|x_1\|_2}{\|x_2\|_2}$.

Now, assume that there exist $a, b \in K$ such that $ax_1 + bx_2$ has an element in $[z_1, z_2]$, say $c_1x_1 + c_2x_2$ $(c_1, c_2 \in K)$, orthogonal with respect to $\|.\|_1$, $\|.\|_2$ and $\|.\|_3$.

Assuming that $\|ax_1\|_3 = \|bx_2\|_3$ (then $a \neq 0$ and $b \neq 0$) we get $\frac{\|x_1\|_3}{\|x_2\|_3} = \frac{|b|}{|a|}$ and $\frac{\|x_1\|_1}{\|x_2\|_1} > \frac{|b|}{|a|} > \frac{\|x_1\|_2}{\|x_2\|_2}$. Next, we get $\|\frac{a}{b}x_1\|_1 > \|x_2\|_1$ and $\|\frac{a}{b}x_1\|_2 < \|x_2\|_2$. Using Lemma 2, we conclude that $ax_1 + bx_2$ has no orthogonal element in $[z_1, z_2]$ with respect to $\|.\|_1$, $\|.\|_2$.

Hence, $\|ax_1\|_3 \neq \|bx_2\|_3$ (and $\|c_1x_1\|_3 \neq \|c_2x_2\|_3$). Suppose that $\|ax_1\|_3 > \|bx_2\|_3$. If $\|c_1x_1\|_3 > \|c_2x_2\|_3$, then $\|x_1\|_3 > \|\frac{c_2}{c_1}x_2\|_3$ and taking $\lambda := -\frac{a}{c_1}$ we get

$$\begin{aligned}\|ax_1 + bx_2 + \lambda(c_1x_1 + c_2x_2)\|_3 &= \left\|bx_2 - a\frac{c_2}{c_1}x_2\right\|_3 \\ &< \|ax_1\|_3 = \|ax_1 + bx_2\|_3,\end{aligned}$$

a contradiction. If $\|c_1x_1\|_3 < \|c_2x_2\|_3$, since by assumption there exists $\mu \in K$ with $\|x_1 - \mu x_2\|_3 < \|x_1\|_3$, taking $\lambda := -\frac{a\mu}{c_2}$ we get

$$\begin{aligned}\|ax_1 + bx_2 + \lambda(c_1x_1 + c_2x_2)\|_3 &= \left\|a(x_1 - \mu x_2) + bx_2 - \frac{a\mu}{c_2}c_1x_1\right\|_3 \\ &< \|ax_1\|_3 = \|ax_1 + bx_2\|_3,\end{aligned}$$

since

$$\left\|\frac{a\mu}{c_2}c_1x_1\right\|_3 < \left\|\frac{a\mu}{c_2}c_2x_2\right\|_3 = \|a\mu x_2\|_3 = \|ax_1\|_3.$$

For the case $\|ax_1\|_3 < \|bx_2\|_3$, by symmetry, we obtain the same conclusion.

Finally, assume that $a = 0$ or $b = 0$. Then, it is easy to verify, that in this case we have $\|c_1x_1\|_3 = \|c_2x_2\|_3$. But, as we observe above, such element $c_1x_1 + c_2x_2$ has no orthogonal element with respect to $\|.\|_1$, $\|.\|_2$ in $[z_1, z_2]$, a contradiction.

(2) $\Rightarrow$ (4). Let $\|.\|_3$ be a norm defined on E. It follows from the assumption that every orthogonal base with respect to $\|.\|_1$ is also orthogonal with respect to $\|.\|_2$. Applying the same argumentation as in (2) $\Rightarrow$ (3), by Theorem 1.11 of [1], there exists a base $\{z_1, \dots, z_n\}$ in E which is orthogonal with respect to $\|.\|_1$ and $\|.\|_3$; thus, orthogonal with respect to $\|.\|_2$.

(4) $\Rightarrow$ (1). Assume the contrary and suppose that for every $\|.\|_3$, a norm defined on E, E possesses a base orthogonal with respect to $\|.\|_1$, $\|.\|_2$ and $\|.\|_3$. Suppose that

$$\frac{\|e_1\|_1}{\|e_1\|_2} \geqslant \frac{\|e_2\|_1}{\|e_2\|_2} \geqslant \cdots \geqslant \frac{\|e_n\|_1}{\|e_n\|_2} \quad \text{and} \quad \frac{\|e_1\|_1}{\|e_1\|_2} > \frac{\|e_n\|_1}{\|e_n\|_2}. \tag{4}$$

Choose $\mu \in K$ such that

$$\frac{\|e_n\|_2}{\|e_1\|_2} > \|\mu\| > \frac{\|e_n\|_1}{\|e_1\|_1} \tag{5}$$

and $p \in K$, $|p| < 1$. Next, we define the norm on E by

$$\begin{aligned}\|x\|_3 := \max\{&\left|(1+p^2)x_1 - \mu x_n\right| \cdot \|e_1\|_2, |x_1 - \mu(1+p)x_n| \cdot \|e_1\|_2, \\ &\|x_2e_2\|_2, \dots, \|x_{n-1}e_{n-1}\|_2\},\end{aligned}$$

where $x \in E$ is given by $x = \sum_{i=1}^{n} x_ie_i$.

First, we prove that $\max_{x\in E} \frac{\|x\|_1}{\|x\|_3}$ is attained for

$$(6)\qquad u_0 = \lambda\left(e_1 + \frac{1}{\mu}\lambda_2 e_n + a_2e_2 + \cdots + a_{n-1}e_{n-1}\right)$$

if $\lambda_2 = 1+\varepsilon$ ($\varepsilon, \lambda \in K$, $\|\varepsilon\| \leqslant |p|$, $\lambda \neq 0$) and $\max_{i=2,\dots,n-1} \|a_ie_i\|_2 \leqslant |p|\cdot\|e_1\|_2$. Indeed, then

$$\begin{aligned}&\max\left\{\left|(1+p^2) - \mu\frac{\lambda_2}{\mu}\right|, \left|1-\mu(1+p)\frac{1}{\mu}\lambda_2\right|\right\}\\ &\quad = \max\{|p^2-\varepsilon|, |p+\varepsilon+p\varepsilon|\} = |p|,\\ &\|a_2e_2+\cdots+a_{n-1}e_{n-1}\|_2 = \max_{i=2,\dots,n-1} \|a_ie_i\|_2 \leqslant |p|\cdot\|e_1\|_2\end{aligned}$$

and

$$\begin{aligned}\frac{\|u_0\|_1}{\|u_0\|_3} &= \frac{\max\{\|e_1\|_1, \|\frac{1}{\mu}(1+\varepsilon)e_n\|_1, \|a_2e_2\|_1, \dots, \|a_{n-1}e_{n-1}\|_1\}}{|p|\cdot\|e_1\|_2}\\ &= \frac{\|e_1\|_1}{|p|\cdot\|e_1\|_2},\end{aligned}$$

since $\|\frac{1}{\mu}(1+\varepsilon)e_n\|_1 = \|\frac{1}{\mu}e_n\|_1 < \|e_1\|_1$ by (5) and

$$(7)\qquad \frac{\|e_1\|_2}{\|e_1\|_1} \leqslant \frac{\|a_ie_i\|_2}{\|a_ie_i\|_1} \leqslant \frac{|p|\cdot\|e_1\|_2}{\|a_ie_i\|_1} \quad\Longrightarrow\quad \|a_ie_i\|_1 \leqslant |p|\cdot\|e_1\|_1$$

if $i \in \{2,\dots,n-1\}$ and $a_i \neq 0$.

Next, we prove that for nonzero (assuming that $a_1 = 1$ or $a_1 = 0$, $a_1 \in K$)

$$u = \lambda\left(a_1e_1 + \frac{1}{\mu}\lambda_2e_n + a_2e_2 + \cdots + a_{n-1}e_{n-1}\right) \quad (\lambda \in K),$$

where there exists $j \in \{2,\dots,n-1\}$ such that $\|a_je_j\|_2 = \max_{i=2,\dots,n-1}\|a_ie_i\|_2 > |p|\cdot\|e_1\|_2$ or $\lambda_2 = 1+\varepsilon$ for $|\varepsilon| > |p|$ ($\varepsilon \in K$), we obtain $\frac{\|u\|_1}{\|u\|_3} < \frac{\|e_1\|_1}{|p|\cdot\|e_1\|_2}$.

Let $j \in \{2,\dots,n-1\}$ with $\|a_je_j\|_2 = \max_{i=2,\dots,n-1}\|a_ie_i\|_2 > |p|\cdot\|e_1\|_2$ and assume that $|\varepsilon| \leqslant |p|$. For $a_1 = 1$, applying (5), we get

$$\begin{aligned}\frac{\|u\|_1}{\|u\|_3} &= \frac{\max\{\|e_1\|_1, \|\frac{1}{\mu}(1+\varepsilon)e_n\|_1, \|a_2e_2\|_1, \dots, \|a_{n-1}e_{n-1}\|_1\}}{\max\{|(1+p^2)-(1+\varepsilon)|\cdot\|e_1\|_2, |1-(1+p)(1+\varepsilon)|\cdot\|e_1\|_2, \|a_je_j\|_2\}}\\ &\leqslant \frac{\max\{\|e_1\|_1, \|(1+\varepsilon)e_1\|_1, \|a_2e_2\|_1, \dots, \|a_{n-1}e_{n-1}\|_1\}}{\max\{|p^2-\varepsilon|\cdot\|e_1\|_2, |p+\varepsilon+p\varepsilon|\cdot\|e_1\|_2, \|a_je_j\|_2\}}\\ &= \frac{\max\{\|e_1\|_1, \|a_2e_2\|_1, \dots, \|a_{n-1}e_{n-1}\|_1\}}{\|a_je_j\|_2}.\end{aligned}$$

Then, using (4), we get

$$\frac{\|u\|_1}{\|u\|_3} \leqslant \max\left\{\frac{\|e_1\|_1}{\|a_je_j\|_2}, \frac{\|a_ke_k\|_1}{\|a_ke_k\|_2}\right\} < \frac{\|e_1\|_1}{|p|\cdot\|e_1\|_2},$$

where $k \in \{2, \dots, n-1\}$ and $\|a_k e_k\|_1 = \max_{i=2,\dots,n-1} \|a_i e_i\|_1$ (without loss of generality we can assume that $a_k \neq 0$).

Now, suppose only that $|\varepsilon| > |p|$. Then, assuming that $a_k \neq 0$ (if not, with slight modifications we can also get the same final evaluation) we obtain

$$\frac{\|u\|_1}{\|u\|_3} \leqslant \frac{\max\{\|e_1\|_1, \|(1+\varepsilon)e_1\|_1, \|a_2e_2\|_1, \dots, \|a_{n-1}e_{n-1}\|_1\}}{\max\{|\varepsilon| \cdot \|e_1\|_2, \|a_2e_2\|_2, \dots, \|a_{n-1}e_{n-1}\|_2\}}$$
$$\leqslant \max\left\{\frac{\|e_1\|_1}{|\varepsilon| \cdot \|e_1\|_2}, \frac{\|e_1\|_1}{\|e_1\|_2}, \frac{\|a_ke_k\|_1}{\|a_ke_k\|_2}\right\} < \frac{\|e_1\|_1}{|p| \cdot \|e_1\|_2}.$$

Considering the case if $a_1 = 0$, we see that

$$\frac{\|u\|_1}{\|u\|_3} = \frac{\max\{\|\frac{1}{\mu}(1+\varepsilon)e_n\|_1, \|a_2e_2\|_1, \dots, \|a_{n-1}e_{n-1}\|_1\}}{\max\{|(1+\varepsilon)| \cdot \|e_1\|_2, |(1+p)(1+\varepsilon)| \cdot \|e_1\|_2, \|a_je_j\|_2\}}$$
$$\leqslant \frac{\max\{\|(1+\varepsilon)e_1\|_1, \|a_2e_2\|_1, \dots, \|a_{n-1}e_{n-1}\|_1\}}{\max\{|(1+\varepsilon)| \cdot \|e_1\|_2, |(1+p)(1+\varepsilon)| \cdot \|e_1\|_2, \|a_je_j\|_2\}}$$
$$\leqslant \max\left\{\frac{\|e_1\|_1}{\|e_1\|_2}, \frac{\|a_ke_k\|_1}{\|a_ke_k\|_2}\right\} < \frac{\|e_1\|_1}{|p| \cdot \|e_1\|_2}.$$

By Corollary 8, in every base of E, orthogonal with respect to $\|.\|_1$, $\|.\|_2$ and $\|.\|_3$, there is an element u_0 given by (6). Without loss of generality, we can assume that $\lambda = 1$. Hence, for such u_0 we can find an $(n-1)$-dimensional linear subspace D of E such that $E = [u_0] \oplus D$ ($[u]$ and D are orthogonal with respect to $\|.\|_1$, $\|.\|_2$ and $\|.\|_3$). Now, we can write $e_n = cu_0 + d_0$ for some $c \in K, d_0 \in D$. Note that

$$\|u_0\|_1 = \max\left\{\|e_1\|_1, \left\|\frac{1}{\mu}(1+\varepsilon)e_n\right\|_1, \|a_2e_2\|_1, \dots, \|a_{n-1}e_{n-1}\|_1\right\} = \|e_1\|_1$$

and

$$\|u_0\|_2 = \max\left\{\|e_1\|_2, \left\|\frac{1}{\mu}e_n\right\|_2, \max_{i=2,\dots,n-1} \|a_ie_i\|_2\right\} = \left\|\frac{1}{\mu}e_n\right\|_2,$$

since

$$\left\|\frac{1}{\mu}(1+\varepsilon)e_n\right\|_1 < \|e_1\|_1, \|a_2e_2 + \dots + a_{n-1}e_{n-1}\|_2 \leqslant |p| \cdot \|e_1\|_2$$

and $\|e_1\|_2 < \frac{1}{|\mu|}\|e_n\|_2$ by (5), $\|a_ie_i\|_1 \leqslant |p| \cdot \|e_1\|_1$ by (7).

Applying multi-orthogonality u_0 and d_0 we get $\|e_n\|_1 = \max\{\|cu_0\|_1, \|d_0\|_1\}$; hence, $\|cu_0\|_1 = \|ce_1\|_1 \leqslant \|e_n\|_1$. Using (5) again, we imply

$$|c| \leqslant \frac{\|e_n\|_1}{\|e_1\|_1} < |\mu|.$$

Then $\|cu_0\|_2 = \frac{|c|}{|\mu|}\|e_n\|_2 < \|e_n\|_2$. Taking $d_0 = e_n - cu_0$, noting that $\|d_0\|_2 = \max\{\|cu_0\|_2, \|e_n\|_2\} = \|e_n\|_2$, we obtain

$$\left\|\frac{\mu}{\lambda_2}u_0 - d_0\right\|_2 = \left\|\frac{\mu}{\lambda_2}e_1 + e_n + \frac{\mu}{\lambda_2}(a_2e_2 + \cdots + a_{n-1}e_{n-1}) - e_n + cu_0\right\|_2$$
$$\leqslant \max\{\|\mu e_1\|_2, \|\mu(a_2e_2 + \cdots + a_{n-1}e_{n-1})\|_2, \|cu_0\|_2\}$$
$$< \|e_n\|_2,$$

a contradiction with orthogonality $[u_0]$ and D with respect to $\|.\|_2$. □

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