Dual greedy polyhedra, choice functions, and abstract convex geometries

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Abstract

We consider a system of linear inequalities and its associated polyhedron for which we can maximize any linear objective function by finding tight inequalities at an optimal solution in a greedy way. We call such a system of inequalities a dual greedy system and its associated polyhedron a dual greedy polyhedron. Such dual greedy systems have been considered by Faigle and Kern, and Krüger for antichains of partially ordered sets, and by Kashiwabara and Okamoto for extreme points of abstract convex geometries. Faigle and Kern also considered dual greedy systems in a more general framework than antichains. A related dual greedy algorithm was proposed by Frank for a class of lattice polyhedra. In the present paper we show relationships among dual greedy systems, substitutable choice functions, and abstract convex geometries. We also examine the submodularity and facial structures of the dual greedy polyhedra determined by dual greedy systems. Furthermore, we consider an extension of the class of dual greedy polyhedra.

Keywords: Dual greedy algorithm; Choice function; Convex geometry; Submodularity

1. Introduction

We consider a system of linear inequalities and its associated polyhedron for which we can maximize any linear objective function by finding tight inequalities at an optimal solution in a greedy way. We call such a system of inequalities a dual greedy system and its associated polyhedron a dual greedy polyhedron. A polymatroid [3] is a typical classic example of such a dual greedy polyhedron. Furthermore, dual greedy systems have recently been considered by Faigle and Kern [4,5], and Krüger [14] for antichains of partially ordered sets (also see [17]), and by Kashiwabara and Okamoto [11] for extreme points of abstract convex geometries [2]. Faigle and Kern [6] also considered dual greedy systems in a more general framework than antichains. A related dual greedy algorithm was proposed by Frank [7] for a class of lattice polyhedra [10].

In the present paper we show relationships among dual greedy systems, substitutable choice functions, and abstract convex geometries. We also examine the submodularity and facial structures of the dual greedy polyhedra determined by dual greedy systems. Furthermore, we consider an extension of the class of dual greedy polyhedra.

2. Dual greedy polyhedra

The dual greedy systems considered in [3–6,11,14,17] have the following common features.
Let $E$ be a finite nonempty set with $n = |E|$. Consider

(i) a nonempty family $\mathcal{A} \subseteq 2^E$,
(ii) a function $f : \mathcal{A} \rightarrow \mathbb{R}$,
(iii) a system of linear inequalities

$$x(X) \leq f(X) \quad (X \in \mathcal{A}), \quad \text{(2.1)}$$

where $x$ is a variable vector in $\mathbb{R}^E$ and for any $X \in \mathcal{A}$ we define $x(X) = \sum_{e \in X} x(e)$.

Note that (2.1) has only $\{0, 1\}$-coefficients in the left-hand side. Define the polyhedron

$$P(f) = \{ x \in \mathbb{R}^E \mid \forall X \in \mathcal{A} : x(X) \leq f(X) \} \quad \text{(2.2)}$$

determined by (2.1).

For any nonnegative vector $w \in \mathbb{R}_+^E$ consider a linear programming problem:

$$(P_w) \quad \text{Maximize} \quad \sum_{e \in E} w(e) x(e)$$

subject to $x \in P(f)$ \quad \text{(2.3)}

and its dual linear programing problem:

$$(P^*_w) \quad \text{Minimize} \quad \sum_{X \in \mathcal{A}} f(X) \lambda_X$$

subject to $\sum_{X \in X \in \mathcal{A}} \lambda_X = w(e) \quad (e \in E)$, \quad \text{(2.4)}$

$$\lambda_X \geq 0 \quad (X \in \mathcal{A}).$$

Now, suppose that we are given a function $C : 2^E \rightarrow \mathcal{A}$ such that for any $X \subseteq E$ we have (i) $C(X) \subseteq X$ and (ii) $C(X) \neq \emptyset$ if $X \neq \emptyset$. Such a function $C$ is called a choice function in the literature (see, e.g., [15]). We assume that $\mathcal{A}$ is the image of $C$, i.e., $\mathcal{A} = \{ C(X) \mid X \in 2^E \}$.

Then, consider Procedure Dual_GreedyAlgorithm described as follows.

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**Dual_GreedyAlgorithm**

Put $w' \leftarrow w$ and $X \leftarrow E$.

For each $i = 1, 2, \ldots, n$ do the following:

- Put $X_i \leftarrow C(X_i)$.
- Find $e_i \in X_i$ such that $w'(e_i) = \min \{ w'(e) \mid e \in X_i \}$.
- Put $\lambda_{X_i} \leftarrow w'(e_i)$, $X \leftarrow X \setminus \{ e_i \}$, and $w'(e) \leftarrow w'(e) - \lambda_{X_i}$ for each $e \in X_i$.

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Through Dual_GreedyAlgorithm we get $X_i \in \mathcal{A}$, $\lambda_X \geq 0 \quad (i = 1, 2, \ldots, n)$ such that

$$w = \sum_{i=1}^n \lambda_{X_i} X_i. \quad \text{(2.5)}$$

Note that we get a dual feasible solution formed by $\lambda_{X_i} \quad (i = 1, 2, \ldots, n)$ together with $\lambda_X = 0$ for any other $X \in \mathcal{A}$.

We assume

(A0) Each $X \in \mathcal{A}$ arises as an $X_i$ by Dual_GreedyAlgorithm for some $w \in \mathbb{R}_+^E$.

(A1) The expression of $w$ in (2.5) is unique up to terms of zero coefficients, independently of the choice of $e_i$’s in Dual_GreedyAlgorithm.

The sequence of $X_i \quad (i = 1, 2, \ldots, n)$ obtained by Procedure Dual_GreedyAlgorithm defines a system of equations

$$x(X_i) = f(X_i) \quad (i = 1, 2, \ldots, n). \quad \text{(2.6)}$$

We call the coefficient matrix of (2.6) a dual greedy basis matrix and $(X_i \mid i = 1, 2, \ldots, n)$ a dual greedy basis.
Remark 1. From Dual_Greedy_Algorithm we can easily see the following:

After appropriately rearranging the columns of the dual greedy basis matrix \( A = [a_{ij}] \) of (2.6), \( A \) satisfies the following properties:

(a) \( A \) is an upper triangular matrix,
(b) \( a_{ii} = 1 \) for all \( i = 1, 2, \ldots, n \).

The dual greedy basis determines a primal solution \( x \), which we call a dual greedy solution. If a dual greedy solution is primal feasible for every \( w \in \mathbb{R}_+^k \), we say that the dual greedy algorithm works. We then also call system (2.1) of inequalities a dual greedy system and the polyhedron \( P(f) \) a dual greedy polyhedron associated with it.

Remark 2. When the dual greedy algorithm works, the optimal objective function value \( \mu(w) \) of the dual problems \((P_n^*)\) and \((P_{n+}^*)\) is given by

\[
\mu(w) = \sum_{i=1}^n \lambda_i f(X_i) \tag{2.7}
\]

according to (2.5). The function \( \mu : \mathbb{R}_+^k \rightarrow \mathbb{R} \) is what is called the support function of \( P(f) \), which is convex.

Conversely, without assuming that the dual greedy algorithm works, we can define a function \( \tilde{\mu} : \mathbb{R}_+^k \rightarrow \mathbb{R} \) by (2.7) according to (2.5), where note that the expression (2.5) is unique up to terms with zero coefficients. Also, we put \( \tilde{\mu}(w) = +\infty \) for \( w \in \mathbb{R}_+^k \setminus \mathbb{R}_+^k \). The function \( \tilde{\mu} \) thus defined is convex only if the dual greedy algorithm works, as shown below.

Theorem 2.1. Under Assumptions (A0) and (A1) the function \( \tilde{\mu} : \mathbb{R}_+^k \rightarrow \mathbb{R} \cup \{+\infty\} \) is convex if and only if the dual greedy algorithm works.

Proof. If the dual greedy algorithm works, then we have \( \tilde{\mu} = \mu \) (the support function of \( P(f) \)) and hence \( \tilde{\mu} \) is convex.

Conversely, suppose that \( \tilde{\mu} \) is convex. Note that \( \tilde{\mu} \) is positively homogeneous by definition and is continuous on \( \mathbb{R}_+^k \) by Assumption (A1). Hence, it is a support function of a convex set \( P \subseteq \mathbb{R}_+^k \) defined by

\[
P = \left\{ x \mid x \in \mathbb{R}_+^k, \, \forall w \in \mathbb{R}_+^k : \sum_{e \in E} w(e)x(e) \leq \tilde{\mu}(w) \right\} \tag{2.8}
\]

(see [18, Corollary 13.2.1]). It follows from the definition of \( \tilde{\mu} \) by (2.5) and (2.7) and Assumption (A0) that we have \( P = P(f) \). Now, for any \( w \in \mathbb{R}_+^k \) let \( (X_i \mid i = 1, 2, \ldots, n) \) be the corresponding dual greedy basis determined by Dual_Greedy_Algorithm. We shall show that the system of equations \( x(X_i) = f(X_i) \left( i = 1, 2, \ldots, n \right) \) determines a primal feasible solution \( \tilde{x} \in P(f) \).

For any \( \tilde{\lambda}_i > 0 \left( i = 1, 2, \ldots, n \right) \) define \( \tilde{w} \in \mathbb{R}_+^k \) by (2.5) with \( \lambda_i \) being replaced by \( \tilde{\lambda}_i \). Then we have

\[
\tilde{\mu}(\tilde{w}) = \sum_{i=1}^n \tilde{\lambda}_i f(X_i). \tag{2.9}
\]

Let \( \hat{x} \) be a vector in \( P(=P(f)) \) such that

\[
\sum_{e \in E} \tilde{w}(e)\hat{x}(e) = \tilde{\mu}(\tilde{w}). \tag{2.10}
\]

Here, recall that \( \tilde{\mu} \) is the support function of \( P(=P(f)) \), so that such a vector \( \hat{x} \) exists. Since \( \hat{x}(X_i) \leq f(X_i) \) and \( \tilde{\lambda}_i > 0 \) \( (i = 1, 2, \ldots, n) \), it follows from (2.5), (2.9), and (2.10) that we have \( \hat{x}(X_i) = f(X_i) \left( i = 1, 2, \ldots, n \right) \). That is, \( \hat{x} \) is the dual greedy solution associated with the dual greedy basis \( (X_i \mid i = 1, 2, \ldots, n) \).

In the following we assume

(A2) For any \( w \in \mathbb{R}_+^k \) Dual_Greedy_Algorithm works.

Remark 3. Sohoni [19] developed a theory of shapes in a more general setting and showed a proposition [19, Proposition 2.1.12] without the greedy (or upper-triangular) basis matrix structure, which includes Theorem 2.1 as a special case. The collection of all the dual greedy bases forms a shape in Sohoni’s sense. Related arguments were also made by Narayanan [16]. In the present paper we are interested in the dual greediness of system (2.1).
Remark 4. The function \( \hat{\mu} \) is an extension of the set function \( f: \mathcal{A} \to \mathbb{R} \), which is a generalization of the so-called Lovász extension of a set function on \( 2^E \). As is the case for the Lovász extension of a submodular function on \( 2^E \), the convexity of the extension \( \hat{\mu} \) completely characterizes the primal feasibility of dual greedy solutions.

Moreover,

Theorem 2.2. Under Assumption (A2) system (2.1) of inequalities is totally dual integral.

Proof. If \( w \) is an integral vector, the coefficients \( \lambda_{ij} \) in (2.5) determined by the dual greedy algorithm are integers and form an optimal (dual) solution of (2.4).

We shall also investigate the primal feasibility of the dual greedy solution in Section 4. In the next section we shall examine properties of the choice function \( C \).

3. Choice functions and abstract convex geometries

Let us call an ordering \((e_1,e_2,\ldots,e_n)\) generated by Dual_Greedy_Algorithm an admissible ordering. It follows from Dual_Greedy_Algorithm that the set of admissible orderings \((e_1,e_2,\ldots,e_n)\) for all nonnegative weight functions \( w \) coincides with the set of orderings \((e_1,e_2,\ldots,e_n)\) that can be generated by the following procedure:

Admissible_Ordering

Put \( X \leftarrow E \).

For each \( i = 1,2,\ldots,n \) do the following:

Choose \( e_i \in C(X) \) and put \( X \leftarrow X \setminus \{e_i\} \).

Define \( \mathcal{F} \subseteq 2^E \) by

\[
\mathcal{F} = \{ \{e_i,e_{i+1},\ldots,e_n\} \mid i = 1,2,\ldots,n, \ (e_1,e_2,\ldots,e_n) \text{ an admissible ordering} \}.
\] (3.1)

By restricting the choice function \( C \) to \( \mathcal{F} \) we regard \( C \) as a function from \( \mathcal{F} \) onto \( \mathcal{A} \).

Example 1 (Antichains of a poset; [4-6,14,1]). For any partially ordered set (poset) \( \mathcal{P}=(E,\preceq) \) let \( \mathcal{F} \subseteq 2^E \) be the set of all the (lower) ideals of \( \mathcal{P} \), where \( I \subseteq E \) is a (lower) ideal of \( \mathcal{P} \) if and only if \( e_1 \preceq e_2 \in I \) implies \( e_1 \in I \). For each ideal \( I \in \mathcal{F} \) define \( C(I) \) to be the set of all maximal elements of \( I \) in poset \( \mathcal{P} \). Note that the set of \( C(I) \) \( (I \in \mathcal{F}) \) coincides with that of antichains of \( \mathcal{P} \).

Example 2 (Extreme points of an (abstract) convex geometry; [2,11,12]). Let \( (E,\mathcal{F}) \) be an (abstract) convex geometry on \( E \) with a family \( \mathcal{F} \) of closed sets, i.e., (1) \( \emptyset \in \mathcal{F} \), (2) \( \mathcal{F} \) is closed with respect to set intersection, and (3) the length of each maximal chain of \( \mathcal{F} \), considered as a lattice, is equal to \( \mid E \mid \). For each \( X \in \mathcal{F} \) let \( C(X) \) be the set of extreme points of \( X \). Recall that \( e \in X (\in \mathcal{F}) \) is an extreme point of \( X \) if and only if \( X \setminus \{e\} \in \mathcal{F} \).

We further introduce the following assumption on the dual greedy bases determined by the dual greedy algorithm. (Note that this is the case for Examples 1 and 2 given above.)

(A3) The dual greedy basis matrix \( A = [a_{ij}] \) of (2.6) satisfies the following property:

(c) each column of \( A \) has 1’s consecutively, i.e., if \( a_{ij} = 1 = a_{i'}j \) for \( i < i' \), then \( a_{i''j} = 1 \) for any \( i'' \) with \( i \leq i'' \leq i' \).

Note that under this assumption the sequence of elements \( e_1,e_2,\ldots,e_n \) found by Dual_Greedy_Algorithm gives the ordering of the columns of the matrix \( A \) such that Properties (a), (b) (in Remark 1), and (c) above hold. In particular, we have \( \{e_i\} = X_i \setminus X_{i+1} \ (i = 1,2,\ldots,n) \) where \( X_{n+1} = \emptyset \).

We can easily see the following property of choice function \( C \).
Lemma 3.1. Under Assumption (A3), the choice function $C: \mathcal{F} \rightarrow \mathscr{A}$ satisfies the following:

(S) For any nonempty $X \in \mathcal{F}$ and any $e \in C(X)$,

$$C(X) \setminus \{e\} \subseteq C(X \setminus \{e\}).$$

(3.2)

**Proof.** Property (S) immediately follows from the consecutive 1’s property (c) of (A3).

It should be noted that a choice function $C: \mathcal{F} \rightarrow \mathscr{A}$ having property (S) completely characterizes the collection of basis matrices $A$ with properties (a)–(c) where $X_i$ is determined through $C$ by set $\bigcup \{X_i \mid k = i, i+1, \ldots, n\}$. Property (S) shows a kind of substitutability of choice function $C$ [15].

**Theorem 3.2.** Consider a choice function $C: \mathcal{F} \rightarrow \mathscr{A}$ satisfying (S) in Lemma 3.1, where $\mathcal{F}$ is defined by (3.1). Then, the pair $(E, C)$ is a convex geometry with a family $\mathcal{F}$ of closed sets.

**Proof.** First note that the length of any maximal chain of $\mathcal{F}$ is equal to $|E|$. Suppose that $X \in \mathcal{F}$ and $e, e' \in C(X)$ where $e \neq e'$. It suffices to show that $X \setminus \{e, e'\} \in \mathcal{F}$ (see, e.g., [12, Lemma 1.2]). Let $(\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n)$ be an admissible ordering such that $X = E \setminus \{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\}$ for $k = |E \setminus X|$. Note that from the assumption we have $e \in C(X)$ and furthermore, $e' \in C(X \setminus \{e\})$ due to (S). It follows from Procedure Admissible Ordering that there exists an admissible ordering of the form $(\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_k, e, e', \ldots)$. Hence we have $X \setminus \{e, e'\} \in \mathcal{F}$.

**Remark 5.** Theorem 3.2 is an easy consequence of Property (S), but it seems to be new in the choice function theory (cf. [13]).

**Remark 6.** It was shown by Kashiwabara and Okamoto [11] that the polyhedra for convex geometries defined in [11] are dual greedy polyhedra defined here. It follows from Theorem 3.2 that Assumption (A3) with choice function $C$ validates the dual greedy algorithm (Assumption (A2)) and the unique representability (Assumption (A1)).

Hence we have

**Theorem 3.3.** The class of dual greedy systems (or dual greedy polyhedra) under Assumptions (A0) and (A3) coincides with the one considered by Kashiwabara and Okamoto [11] for convex geometries.

4. Adjacency in dual greedy polyhedra for convex geometries

There is a one-to-one correspondence between the set of admissible orderings and that of dual greedy bases. Let $(e_1, e_2, \ldots, e_n)$ be an admissible ordering. Then we have a corresponding dual greedy basis formed by

$$X_i = C(\{e_i, e_{i+1}, \ldots, e_n\}) \in \mathscr{A} \quad (i = 1, 2, \ldots, n)$$

(4.1)

that determines the basis matrix $A = [a_{ij}]$ and a vertex, say $v$, in $P(f)$.

For any $k \in \{1, 2, \ldots, n\}$ remove the $k$th row from $A$ and consider the following system of equations:

$$x(X_i) = f(X_i) \quad (i = 1, \ldots, k-1, k+1, \ldots, n).$$

(4.2)

The set of solutions of (4.2) is a line (denoted by $L_n^a$) through $v$. Let $d$ be a $\{0, \pm 1\}$-valued solution of (4.2) with $f(X_i)$ replaced by zero for each $i = 1, \ldots, k-1, k+1, \ldots, n$. Here, note that the consecutive 1’s property of the coefficient matrix guarantees the existence of a $\{0, \pm 1\}$-valued solution. If $L_n^a$ determines an edge vector $z$ from $v$ to one of its adjacent vertices, say $u$ (possibly a point at infinity), then $z$ must be equal, up to a positive multiple, to the $\{0, \pm 1\}$-vector $d = \chi_{F_1} - \chi_{F_2}$ such that $F_1, F_2 \subseteq \{e_1, \ldots, e_k\}$, $F_1 \cap F_2 = \emptyset$, $e_k \in F_2$, and

$$|F_1 \cap X_i| - |F_2 \cap X_i| = 0 \quad (i = 1, \ldots, k-1, k+1, \ldots, n).$$

(4.3)

See Fig. 1, where $d = \chi_{F_1} - \chi_{F_2}$ with $F_1 = \{e_1, e_4, e_5\}$ and $F_2 = \{e_2, e_6\}$.

Define

$$\hat{a} = \sup \{x \mid x > 0, \ v + x(\chi_{F_1} - \chi_{F_2}) \in P(f)\}.$$
Theorem 4.1. The following three statements hold:

(i) $\hat{x} = +\infty$ if and only if $F_1 = \emptyset$ and $F_2 = \{e_k\}$.

(ii) If $0 < \hat{x} < +\infty$, then the vertex $u$ adjacent from $v$ in the direction of $\chi_{F_1} - \chi_{F_2}$ corresponds to the admissible ordering

$$
(e_1, \ldots, e_{k-2}, e_k, e_{k-1}, e_{k+1}, \ldots, e_n).
$$

The sequence

$$
X_1, \ldots, X_{k-1}, C\{e_{k-1}, e_{k+1}, \ldots, e_n\}, X_{k+1}, \ldots, X_n \in \mathcal{A}
$$

determines the vertex $u$ adjacent to $v$, and we have

$$
\hat{x} = f(C\{e_{k-1}, e_{k+1}, \ldots, e_n\}) - v(C\{e_{k-1}, e_{k+1}, \ldots, e_n\}).
$$

(iii) We have $\hat{x} = 0$ if and only if sequence (4.6) gives the same vertex $v$, so that

$$
v(C\{e_{k-1}, e_{k+1}, \ldots, e_n\}) = f(C\{e_{k-1}, e_{k+1}, \ldots, e_n\}).
$$

Proof. (i) If $e_k \notin C\{e_{k-1}, e_{k+1}, \ldots, e_n\}$, then because of Assumption (A3) $n(e_k)$ does not appear in (4.2) explicitly. Hence we have $F_1 = \emptyset$, $F_2 = \{e_k\}$, and $\hat{x} = +\infty$. On the other hand, if $e_k \in C\{e_{k-1}, e_{k+1}, \ldots, e_n\}$, then $\{e_{k-1}, e_{k+1}, \ldots, e_n\} \in \mathcal{F}$ and hence $C\{e_{k-1}, e_{k+1}, \ldots, e_n\}$ is defined. It follows from Assumption (A3) that $e_{k-1} \in C\{e_{k-1}, e_{k+1}, \ldots, e_n\}$, while $e_k \notin C\{e_{k-1}, e_{k+1}, \ldots, e_n\}$. Since we have $F_1 \cup F_2 \subseteq \{e_1, e_2, \ldots, e_k\}$ and $(F_1 \cup F_2) \cap C\{e_{k-1}, e_{k+1}, \ldots, e_n\} = \{e_{k-1}\}$, the difference

$$
f(C\{e_{k-1}, e_{k+1}, \ldots, e_n\}) - v(C\{e_{k-1}, e_{k+1}, \ldots, e_n\})
$$

gives an upper bound for $\hat{x}$. Hence $\hat{x} < +\infty$. Note that in this case we have $e_{k-1} \in F_1$.

(ii) As shown in the proof of (i), if $\hat{x} < +\infty$, there is an admissible ordering (4.5) and the corresponding dual greedy basis matrix is determined by (4.6), where $X_k$ has been replaced by $C\{e_{k-1}, e_{k+1}, \ldots, e_n\}$. Since these two are the only possible dual greedy basis matrices that have $X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n$ in common, statement (ii) holds due to Assumption (A2) (also see Theorem 3.2).

(iii) The present statement follows from the proof of (ii). $\square$

We say that the dual greedy basis given by (4.6) is adjacent to the dual greedy basis $(X_1, X_2, \ldots, X_n)$. It should be noted that the set of all the dual greedy bases is connected with respect to this adjacency relation.

Remark 7. As shown by Sohoni [19] for arbitrary shapes in a more general setting, a shape, the collection of all the dual greedy bases considered here, determines a simplicial division of the intersection of the unit sphere in $\mathbb{R}^L$ and the nonnegative orthant $\mathbb{R}_+^L$. The adjacency of the simplices in the division coincides with the adjacency of dual greedy bases.

Lemma 4.2. Without Assumption (A2), suppose that there exists at least one dual greedy solution belonging to $P(f)$. Then, for any nonnegative weight vector $w \in \mathbb{R}_+^L$, the dual greedy algorithm finds an optimal solution if and only if for
each admissible ordering \((e_1, e_2, \ldots, e_n)\) and its associated dual greedy solution \(v\) we have
\[
v(C(\{e_{k-1}, e_{k+1}, \ldots, e_n\})) \leq f(C(\{e_{k-1}, e_{k+1}, \ldots, e_n\}))
\] (4.9)
for each \(k \in \{1, 2, \ldots, n\}\) such that \(e_k \in C(\{e_{k-1}, e_{k}, \ldots, e_n\})\).

**Proof.** The ‘only if’ part is trivial. Hence we prove the ‘if’ part. It follows from the proof of (ii) of Theorem 4.1 with the present assumption (4.9) that there is no hyperplane \(x(C(X)) = f(C(X))\) with \(X \in \mathcal{F}\) that separates any two adjacent dual greedy solutions. Hence we see from the assumption and the connectedness of the set of all the dual greedy bases that any dual greedy solution is primal feasible. \(\square\)

**Remark 8.** Kashiwabara and Okamoto [11] gave a kind of submodularity condition on \(f\) for the primal feasibility of dual greedy solutions.

**Remark 9.** In the case of a polymatroid, since \(C(X) = X\) for any \(X \subseteq E\), inequality (4.9) can be rewritten as
\[
f(\{e_{k-1}, e_{k+1}, \ldots, e_n\}) \geq f(C(\{e_{k-1}, e_{k+1}, \ldots, e_n\}))
\]
= \(v(\{e_{k-1}, e_{k+1}, \ldots, e_n\})\)
= \(v(e_{k-1}) + v(\{e_{k+1}, \ldots, e_n\})\)
= \(f(\{e_{k-1}, e_{k+1}, \ldots, e_n\}) - f(\{e_{k+1}, \ldots, e_n\}) + f(\{e_{k+1}, \ldots, e_n\})\). (4.10)
As is well known, this is equivalent to the submodularity of \(f\) on \(2^E\).

## 5. Submodularity in dual greedy polyhedra

In this section we examine the structure of the set of edge vectors, which will reveal a submodularity structure behind dual greedy polyhedra for convex geometries.

**Lemma 5.1.** Consider an edge vector \(\chi_{F_1} - \chi_{F_2}\) determined by (4.2). Then,
\[
|F_1 \cap X_i| = |F_2 \cap X_i| = 0 \text{ or } 1 \quad (i = 1, 2, \ldots, k - 1).
\] (5.1)

**Proof.** The present lemma easily follows from properties of the dual greedy basis matrix that is upper-triangular and has the consecutive 1’s in columns. \(\square\)

Here recall that \(F_1 \cup F_2 \subseteq \{e_1, e_2, \ldots, e_n\}\). Put \(Z_i = \{e_{i+1}, \ldots, e_n\}\) (\(i = 1, 2, \ldots, n\)) and define a face \(F(Z_i)\) of \(P(f)\) by \(F(Z_i) = \{x | x \in P(f), \forall i \in \{k + 1, k + 2, \ldots, n\} : x(X_i) = f(X_i)\}\). (5.2)

Also define a face, determined by face \(F(Z_i)\) and a supporting hyperplane \(x(X_i) = f(X_i)\) for a positive integer \(l\) with \(1 \leq l \leq k\), by
\[
F(Z_i, l) = \{x | x \in F(Z_i), x(X_i) = f(X_i)\}.
\] (5.3)

From Lemma 5.1 we have

**Theorem 5.2.** Let \(k\) and \(l\) be positive integers such that \(1 \leq l < k \leq n\). For a face \(F(Z_k, l)\) of \(F(Z_k)\), the projection of \(F(Z_k)\) into the subspace \(\mathbb{R}^{X_i}\) along \(\mathbb{R}^{E(X_i) \setminus Z_k}\) is a base polyhedron associated with a submodular function on \(2^{X_i}\).

**Proof.** After the projection of \(F(Z_k)\) into \(\mathbb{R}^{X_i}\) we have \(x(X_i) = f(X_i)\). It follows from Lemma 5.1 that each edge vector of the projected face is one of the forms \(\chi_{Z_{e'}} - \chi_{Z_{e}}\) \((e, e' \in X_i, e \neq e')\). Hence the projected face is a base polyhedron associated with a submodular function on \(2^{X_i}\), due to Tomizawa (see, e.g., [8, Theorem 3.26; 9, Appendix]). \(\square\)

## 6. Convex geometries associated with faces

Given a nonnegative weight vector \(w \in \mathbb{R}^E\), consider the LP problem \((P_w)\) in (2.3). In the description of Dual_Greedy, Algorithm in Section 2 define
\[
C^w(C(X)) = \{e' | e' \in X_i, w'(e') = \min \{w'(e) | e \in X_i\}\},
\] (6.1)
where \(X_i = C(X)\). Note that \(C^w\) and the composition \(C^w \circ C\) are choice functions.
Lemma 6.1. The choice function $C^w$ satisfies (S) in Lemma 3.1.

Proof. If $|C^w(C(X))| = 1$, then (S) holds. If $|C^w(C(X))| \geq 2$, then for any chosen $e_i \in C^w(C(X))$ and an updated new $w'$ we have

$$e \in C^w(C(X)) \setminus \{e_i\} \Rightarrow w'(e) = 0.$$  \hfill (6.2)

It follows that if $e \in C^w(C(X)) \setminus \{e_i\}$, then we have $e \in \arg\min\{w'(e') | e' \in C(X \setminus \{e_i\})\} = C^w(C(X \setminus \{e_i\}))$. \hfill \Box

We see from this lemma and Theorem 3.2 that $C^w$ defined by (6.1) determines a convex geometry associated with the face of optimal primal solutions of $(P_w)$. It should be noted that the convex geometry is determined by $w$ but not by the face of optimal primal solutions of $(P_w)$. Distinct weight vectors $w_1$ and $w_2$ may determine the same face of optimal solutions and distinct choice functions $C_{w_1}$ and $C_{w_2}$, which occurs only if $P(f)$ is degenerate.

7. An extension

In the previous sections any dual greedy polyhedron $P(f)$ has its characteristic cone (or recession cone) $\mathbb{R}^E$. We extend the class of dual greedy polyhedra to that of polyhedra having more general characteristic cones, which has not been considered in the literature.

Consider a choice function $C_1$ that satisfies the substitutability property (S) in Lemma 3.1. Also, let $C_2$ be a choice function such that the composition $C_2C_1$ satisfies Property (S), and let $\mathcal{F}$ be the family of closed sets of the convex geometry associated with $C_2C_1$. Then, consider the following system of inequalities:

$$x(C_1(X)) \leq f(C_1(X)) \quad (X \in \mathcal{F}),$$  \hfill (7.1)

where $f : \mathcal{F} \to \mathbb{R}$ is a function such that the dual greedy algorithm based on the choice function $C_2C_1$ works for any nonnegative weight function $w$ satisfying $C_2(C_1(X)) \cap C_1^w(C_1(X)) \neq \emptyset \ (X \in \mathcal{F})$. Here, $C_1^w$ is the choice function defined by (6.1) with $C$ being replaced by $C_1$. Denote by $P(f)$ the set of all the feasible solutions of (7.1).

Remark 10. Note that $\mathcal{F}$ in (7.1) is defined from $C_2C_1$ but not from $C_1$. This makes a great difference between (7.1) considered here and (2.1) in Section 2. An admissible ordering for $C_2C_1$ is admissible for $C_1$ but the converse is not true in general.

The following two examples show dual greedy polyhedra with unbounded faces of maximal vectors (also see Fig. 2).

Example 3. For a poset $(E, \leq)$ suppose that $C_1(X) = X \ (X \subseteq E)$ and let $C_2(X)$ be the set of all the maximal elements of $X \subseteq E$. Then the family of closed sets associated with $C_2C_1$ is the family, denoted by $\mathcal{D}$, of all the (lower) ideals of poset $(E, \leq)$. The set $P(f)$ of all the feasible solutions of (7.1) is the so-called submodular polyhedron associated with
a submodular system $(\mathcal{D}, f)$, where $f$ is a submodular function on $\mathcal{D}$ (see [8]). Note that in this example $\mathcal{D}$ is closed with respect to set union as well as set intersection. Also note that the characteristic cone of $P(f)$ is generated by vectors $\chi_e - \chi_{e'}$ for all arcs $(e, e')$ of the Hasse diagram of poset $(E, \leq)$ and vectors $-\chi_e$ for all minimal elements $e$ of $(E, \leq)$. It is different from $\mathbb{R}^E$ in general.

**Example 4.** Let $(E, \mathcal{F})$ be a convex geometry with a family $\mathcal{F}$ of closed sets. Then consider $C_1, C_2 : \mathcal{F} \to 2^E$ given by

$$C_1(X) = X, \quad C_2(X) = ex(X) \quad (X \in \mathcal{F}),$$

(7.2)

where $ex(X)$ denotes the set of extreme points of $X$ in the convex geometry $(E, \mathcal{F})$.

It should be noted that the choice function $C_2^*$ defined by (6.1) (with $C$ being replaced by $C_2$) for any nonnegative weight vector $w$ satisfies the condition for $C_2$. Hence each face of $P(f)$ for (2.1) also gives an example for (7.1).

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**References**


