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Principal subspaces of higher-level standard $\widehat{\mathfrak{sl}(3)}$-modules

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Abstract

We use the theory of vertex operator algebras and intertwining operators to obtain systems of q-difference equations satisfied by the graded dimensions of the principal subspaces of certain level k standard modules for $\widehat{\mathfrak{sl}(3)}$. As a consequence we establish new formulas for the graded dimensions of the principal subspaces corresponding to the highest weights $i\Lambda_1 + (k-i)\Lambda_2$, where $1 \le i \le k$ and Λ_1 and Λ_2 are fundamental weights of $\widehat{\mathfrak{sl}(3)}$.

MSC: 17B69; 17B65

1. Introduction

The theory of vertex operator algebras ([2,19]; cf. [23]) leads to classical and new combinatorial identities (see e.g. [25–27,24,8,31] and [32]) and q-difference equations (recursions) satisfied by the graded dimensions (characters) of certain substructures of the standard representations of affine Lie algebras (see [9,10,4] and [11]).

This paper is a continuation of [4], to which we refer the reader for background and notation (see also [5] and [6]). In [4] we have derived a complete set of recursions that characterize the graded dimensions of all the principal subspaces of the level 1 standard representations of $\widehat{\mathfrak{sl}(n)}$ with $n \ge 3$. Here we extend this approach to the principal subspaces of the higher-level standard modules for $\widehat{\mathfrak{sl}(3)}$. This work and the work done in [4] can be viewed as a continuation of a program to obtain Rogers–Ramanujan-type recursions, which was initiated by Capparelli, Lepowsky and Milas in [9,10].

In the present paper we continue the study of a relationship between intertwining operators, in the sense of [17] and [12], associated with standard modules and the corresponding principal subspaces of these modules. We consider the definition of the principal subspace $W(\Lambda)$ from [13] and [14], namely, $W(\Lambda) = U(\bar{\mathfrak{n}}) \cdot v_\Lambda$, where Λ and v_Λ are the highest weight and a highest weight vector of the standard $\widehat{\mathfrak{sl}(3)}$-representation $L(\Lambda)$ of level $k > 1$. By $\bar{\mathfrak{n}}$ we mean $\mathfrak{n} \otimes \mathbb{C}[t, t^{-1}]$, where $\mathfrak{n}$ is the subalgebra of $\mathfrak{sl}(3)$ consisting of the strictly upper-triangular matrices. As is well known, the highest weights of the level k standard modules have the form $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$, where $\Lambda_0, \Lambda_1, \Lambda_2$ are the fundamental weights of $\widehat{\mathfrak{sl}(3)}$ and k_0, k_1, k_2 are non-negative integers whose sum is k.

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The main result of our paper is the following theorem, which gives two families of exact sequences that yield linear systems of q-difference equations:

Theorem. *For any integer i with $1 \leq i \leq k$ there are natural sequences*

$$0 \longrightarrow W(i\Lambda_1 + (k-i)\Lambda_2) \longrightarrow$$
$$W(i\Lambda_0 + (k-i)\Lambda_1) \longrightarrow$$
$$W((i-1)\Lambda_0 + (k-i+1)\Lambda_1) \longrightarrow 0$$

and

$$0 \longrightarrow W((k-i)\Lambda_1 + i\Lambda_2) \longrightarrow$$
$$W(i\Lambda_0 + (k-i)\Lambda_2) \longrightarrow$$
$$W((i-1)\Lambda_0 + (k-i+1)\Lambda_2) \longrightarrow 0,$$

and these sequences are exact.

(See Theorem 4.1 for details.)

As a consequence of this theorem we derive a system of q-difference equations that characterize the graded dimensions of the principal subspaces $W(i\Lambda_0 + (k-i)\Lambda_j)$, where $0 \leq i \leq k$ and $j = 1, 2$ (see Theorem 4.2 below). The graded dimensions of these subspaces were previously obtained by Georgiev using a different method. Combining our recursions with the formulas for the graded dimensions of $W(i\Lambda_0 + (k-i)\Lambda_j)$ we obtain the graded dimensions of the principal subspaces $W(i\Lambda_1 + (k-i)\Lambda_2)$ (see Corollary 4.1 in Section 4). These are new results in the process of obtaining graded dimensions of principal subspaces; the method used in [21] did not give an answer for highest weights of type $i\Lambda_1 + (k-i)\Lambda_2$ with $1 \leq i \leq k-1$. Perhaps the main point of our strategy and results, though, is understanding the role of the intertwining operators involved in the construction of the exact sequences.

In this paper we formulate as a conjecture a presentation of the principal subspaces of all the higher-level standard modules for $\widehat{\mathfrak{sl}(3)}$. It appears that in fact one can use an idea developed in [4] to prove this result, but we will instead prove this conjecture in a different way, in a future publication that is part of ongoing joint work with Lepowsky and Milas.

The paper is organized as follows. Section 2 gives background and notation. In Section 3 we discuss the principal subspaces of the standard $\widehat{\mathfrak{sl}(3)}$-modules. In Section 4 we construct exact sequences that give q-difference equations and we obtain the graded dimensions of the principal subspaces $W(i\Lambda_1 + (k-i)\Lambda_2)$ with $1 \leq i \leq k$.

2. Preliminaries

The aim of this section is to recall the vertex operator construction of the higher-level standard modules for the untwisted affine Lie algebra $\widehat{\mathfrak{sl}(3)}$ by using the corresponding constructions of the level 1 standard representations. We use the setting and notation from [4].

In this paper we work with the finite-dimensional complex Lie algebra $\mathfrak{sl}(3)$ that has a standard basis

$$\{x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_1+\alpha_2}, h_{\alpha_1}, h_{\alpha_2}, x_{-\alpha_1}, x_{-\alpha_2}, x_{-\alpha_1-\alpha_2}\}.$$

We fix the Cartan subalgebra $\mathfrak{h} = \mathbb{C}h_{\alpha_1} \oplus \mathbb{C}h_{\alpha_2}$ of $\mathfrak{sl}(3)$. The standard symmetric invariant nondegenerate bilinear form $\langle x, y\rangle = \mathrm{tr}(xy)$ defined for any x and y in $\mathfrak{sl}(3)$ allows us to identify $\mathfrak{h}$ with $\mathfrak{h}^*$. Take α_1 and α_2 to be the (positive) simple roots corresponding to the vectors x_{α_1} and x_{α_2}. Under our identification we have $h_{\alpha_1} = \alpha_1$ and $h_{\alpha_2} = \alpha_2$. The fundamental weights of $\mathfrak{sl}(3)$ are linear functionals λ_1 and λ_2 in the dual space $\mathfrak{h}^*$ (=$\mathfrak{h}$). They are determined by the conditions $\langle \lambda_i, \alpha_j\rangle = \delta_{i,j}$ for $i, j = 1, 2$, so that

$$\lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2 \quad \text{and} \quad \lambda_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2.$$

We denote by $\mathfrak{n}$ the positive nilpotent subalgebra of $\mathfrak{sl}(3)$,

$$\mathfrak{n} = \mathbb{C}x_{\alpha_1} \oplus \mathbb{C}x_{\alpha_2} \oplus \mathbb{C}x_{\alpha_1+\alpha_2},$$

which can be viewed as the subalgebra consisting of the strictly upper-triangular matrices.

Now we consider the untwisted affine Lie algebra associated with $\mathfrak{sl}(3)$,

$$\widehat{\mathfrak{sl}(3)} = \mathfrak{sl}(3) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c, \tag{2.1}$$

where c is a non-zero central element and

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m\langle x, y\rangle \delta_{m+n,0}c \tag{2.2}$$

for any $x, y \in \mathfrak{sl}(3)$ and $m, n \in \mathbb{Z}$. By adjoining the degree operator d ($[d, x \otimes t^m] = m$, $[d, c] = 0$) to the Lie algebra $\widehat{\mathfrak{sl}(3)}$ one obtains the affine Kac–Moody algebra $\widetilde{\mathfrak{sl}(3)} = \widehat{\mathfrak{sl}(3)} \oplus \mathbb{C}d$ (cf. [22]). We introduce the following subalgebras of $\widehat{\mathfrak{sl}(3)}$:

$$\bar{\mathfrak{n}} = \mathfrak{n} \otimes \mathbb{C}[t, t^{-1}], \tag{2.3}$$

$$\bar{\mathfrak{n}}^+ = \mathfrak{n} \otimes \mathbb{C}[t], \tag{2.4}$$

$$\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c, \tag{2.5}$$

and

$$\widehat{\mathfrak{h}}_{\mathbb{Z}} = \coprod_{m\in\mathbb{Z}\setminus 0} \mathfrak{h} \otimes t^m \oplus \mathbb{C}c. \tag{2.6}$$

The latter is a Heisenberg subalgebra of $\widehat{\mathfrak{sl}(3)}$ in the sense that its commutator subalgebra is equal to its center, which is one dimensional. The usual extension of the form $\langle\cdot,\cdot\rangle$ to $\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ will be denoted by the same symbol ($\langle c, c\rangle = 0$, $\langle d, d\rangle = 0$ and $\langle c, d\rangle = 1$). We will identify $\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ with its dual $(\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d)^*$ via this form. The simple roots of $\widehat{\mathfrak{sl}(3)}$ are $\alpha_0, \alpha_1, \alpha_2$. We denote the fundamental weights of $\widehat{\mathfrak{sl}(3)}$ by $\Lambda_0, \Lambda_1, \Lambda_2$. Then

$$\alpha_0 = c - (\alpha_1 + \alpha_2)$$

and

$$\Lambda_0 = d, \qquad \Lambda_1 = \Lambda_0 + \lambda_1, \qquad \Lambda_2 = \Lambda_0 + \lambda_2.$$

We say that a $\widehat{\mathfrak{sl}(3)}$-module has level $k \in \mathbb{C}$ if c acts as multiplication by k. It is well known that any standard module $L(\Lambda)$ with $\Lambda \in (\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d)^*$ has non-negative integral level, given by $\langle \Lambda, c\rangle$ (cf. [22]). We denote by $L(\Lambda_0)$, $L(\Lambda_1)$, $L(\Lambda_2)$ the standard $\widehat{\mathfrak{sl}(3)}$-modules of level 1 with v_{Λ_0}, v_{Λ_1} and v_{Λ_2} highest weight vectors.

We form the induced $\widehat{\mathfrak{h}}$-module

$$M(1) = U(\widehat{\mathfrak{h}}) \otimes_{U(\mathfrak{h}\otimes\mathbb{C}[t]\oplus\mathbb{C}c)} \mathbb{C},$$

such that $\mathfrak{h} \otimes \mathbb{C}[t]$ acts trivially and c acts as identity on the one-dimensional module $\mathbb{C}$. Let $Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ be the root lattice and $P = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$ be the weight lattice of $\mathfrak{sl}(3)$. Denote by $\mathbb{C}[Q]$ and $\mathbb{C}[P]$ the group algebras of the lattices Q and P with bases $\{e^\alpha | \alpha \in Q\}$ and $\{e^\lambda | \lambda \in P\}$. Consider the following vector spaces:

$$V_P = M(1) \otimes \mathbb{C}[P],$$

$$V_Q = M(1) \otimes \mathbb{C}[Q]$$

and

$$V_Q e^{\lambda_i} = M(1) \otimes \mathbb{C}[Q]e^{\lambda_i}, \quad i = 1, 2.$$

It is well known ([18,33]; cf. [19]) that the vector spaces V_P, V_Q, $V_Q e^{\lambda_1}$ and $V_Q e^{\lambda_2}$ admit a natural $\widehat{\mathfrak{sl}(3)}$-module structure via certain vertex operators. Moreover, V_Q, $V_Q e^{\lambda_1}$ and $V_Q e^{\lambda_2}$ are the level 1 standard representations of $\widehat{\mathfrak{sl}(3)}$ with highest weights Λ_0, Λ_1 and Λ_2 and highest weight vectors $v_{\Lambda_0} = 1 \otimes 1$, $v_{\Lambda_1} = 1 \otimes e^{\lambda_1}$ and $v_{\Lambda_2} = 1 \otimes e^{\lambda_2}$. We shall identify

$$V_Q \simeq L(\Lambda_0), \qquad V_Q e^{\lambda_1} \simeq L(\Lambda_1), \qquad V_Q \mathrm{e}^{\lambda_2} \simeq L(\Lambda_2) \tag{2.7}$$

and we shall write

$$v_{\Lambda_0} = 1, \qquad v_{\Lambda_1} = e^{\lambda_1} \quad \text{and} \quad v_{\Lambda_2} = e^{\lambda_2}. \tag{2.8}$$

For any weight $\lambda \in P$, by e^{λ} we mean a vector of V_P or an operator on V_P, depending on the context. The space V_Q has a natural structure of vertex operator algebra and $V_Q e^{\lambda_i}$ are V_Q-modules for $i = 1, 2$ ([2] and [19]). See Section 2 in [4] for further details and background about the vertex operator construction of the level 1 standard $\widehat{\mathfrak{sl}(3)}$-modules.

The highest weights of the level k standard $\widehat{\mathfrak{sl}(3)}$-modules are given by

$$k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2, \tag{2.9}$$

where k_0, k_1, k_2 are non-negative integers such that $k_0 + k_1 + k_2 = k$. Throughout this paper $k > 1$ stands for the level of a representation.

We consider $L(\Lambda)$ a standard $\widehat{\mathfrak{sl}(3)}$-module of level k with highest weight Λ as in (2.9) and a highest weight vector v_Λ. Set

$$V_P^{\otimes k} = \underbrace{V_P \otimes \cdots \otimes V_P}_{k \text{ times}} \tag{2.10}$$

and

$$v_{i_1,\ldots,i_k} = v_{\Lambda_{i_1}} \otimes \cdots \otimes v_{\Lambda_{i_k}} \in V_P^{\otimes k}, \tag{2.11}$$

where exactly k_0 indices are equal to 0, k_1 indices are equal to 1 and k_2 indices are equal to 2. Then of course $v_{i_1,\ldots,i_k}$ is a highest weight vector for $\widehat{\mathfrak{sl}(3)}$, and

$$L(\Lambda) \simeq U(\widehat{\mathfrak{sl}(3)}) \cdot v_{i_1,\ldots,i_k} \subset V_P^{\otimes k} \tag{2.12}$$

(cf. [22]). Thus there is an embedding of $L(\Lambda)$ into $V_P^{\otimes k}$:

$$L(\Lambda) \longrightarrow V_P^{\otimes k}, \tag{2.13}$$

uniquely determined by the identification $v_\Lambda = v_{i_1,\ldots,i_k}$.

The action of $\widehat{\mathfrak{sl}(3)}$ on $V_P^{\otimes k}$ is given by the usual comultiplication

$$a \cdot v = \Delta(a)v = (a \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes a)v \tag{2.14}$$

for $a \in \widehat{\mathfrak{sl}(3)}$ and $v \in V_P^{\otimes k}$ and this action extends in the usual way to $U(\widehat{\mathfrak{sl}(3)})$. Throughout this paper we will write $x(m)$ for the action of $x \otimes t^m$ on any $\widehat{\mathfrak{sl}(3)}$-module, where $x \in \mathfrak{sl}(3)$ and $m \in \mathbb{Z}$. In particular, for any root α and integer m we set $x_\alpha(m)$ for the action of $x_\alpha \otimes t^m \in \widehat{\mathfrak{sl}(3)}$ on $L(\Lambda)$. In this paper we will also use the notation $x(m)$ for the Lie algebra element $x \otimes t^m$. It will be clear from the context if $x(m)$ is an operator or an element of $\widehat{\mathfrak{sl}(3)}$.

The standard $\widehat{\mathfrak{sl}(3)}$-modules have structures of vertex operator algebra and modules and this result is stated below:

Theorem 2.1 (*[20]; cf. [12,28,23]*). *The standard module $L(k\Lambda_0)$ has a natural vertex operator algebra structure. The set of the level k standard $\widehat{\mathfrak{sl}(3)}$-modules provides a complete list of irreducible $L(k\Lambda_0)$-modules (up to equivalence).*

The vertex operator map

$$Y(\cdot, x) : L(k\Lambda_0) \longrightarrow \text{End } L(k\Lambda_0)[[x, x^{-1}]]$$
$$v \mapsto Y(v, x) = \sum_{m\in\mathbb{Z}} v_m x^{-m-1}$$

has the property

$$Y(x_\alpha(-1) \cdot v_{k\Lambda_0}, x) = \sum_{m\in\mathbb{Z}} x_\alpha(m) x^{-m-1} \tag{2.15}$$

(cf. [20,12,28,23]).

We have the following products of operators:

$$e^{\lambda} x_\alpha(m) = x_\alpha(m - \langle\lambda, \alpha\rangle) e^{\lambda} \tag{2.16}$$

for any $\lambda \in P$, α a simple root and $m \in \mathbb{Z}$ and

$$e^{\lambda} h(n) = h(n) e^{\lambda} \tag{2.17}$$

for any $\lambda \in P$, $h \in \mathfrak{h}$ and $n \neq 0$.

3. Principal subspaces

In [13,14] Feigin and Stoyanovsky introduced certain substructures, called principal subspaces, of the standard modules for $\widehat{\mathfrak{sl}(n)}$ with $n \geq 2$. Consider $L(\Lambda)$ a level k standard module for $\widehat{\mathfrak{sl}(3)}$. Then $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$, where $k_0, k_1, k_2 \in \mathbb{N}$ whose sum is k (cf. (2.9)). The principal subspace of $L(\Lambda)$, denoted by $W(\Lambda)$, is defined as follows:

$$W(\Lambda) = U(\bar{\mathfrak{n}}) \cdot v_{\Lambda}, \tag{3.1}$$

where $U(\bar{\mathfrak{n}})$ is the universal enveloping algebra of $\bar{\mathfrak{n}}$ (recall (2.3)) and v_{Λ} is a highest weight vector of $L(\Lambda)$.

Let us consider the natural surjective map:

$$\begin{aligned} f_{\Lambda} : U(\bar{\mathfrak{n}}) &\longrightarrow W(\Lambda) \\ a &\mapsto a \cdot v_{\Lambda}. \end{aligned} \tag{3.2}$$

Denote by I_{Λ} the annihilator of the highest weight vector v_{Λ} in $U(\bar{\mathfrak{n}})$,

$$I_{\Lambda} = \operatorname{Ker} f_{\Lambda}. \tag{3.3}$$

This is a left ideal of $U(\bar{\mathfrak{n}})$. Hence the principal subspace $W(\Lambda)$ is isomorphic (as linear spaces) with the quotient space $U(\bar{\mathfrak{n}})/I_{\Lambda}$.

Recall from Section 2 the operators $x_{\alpha}(m)$ for any root α and $m \in \mathbb{Z}$. We consider the following formal infinite sums:

$$R_t^{[1]} = \sum_{m_1 + \cdots + m_{k+1} = t} x_{\alpha_1}(m_1) \cdots x_{\alpha_1}(m_{k+1}) \tag{3.4}$$

and

$$R_t^{[2]} = \sum_{m_1 + \cdots + m_{k+1} = t} x_{\alpha_2}(m_1) \cdots x_{\alpha_2}(m_{k+1}) \tag{3.5}$$

for any $t \in \mathbb{Z}$. It will be convenient to truncate $R_t^{[1]}$ and $R_t^{[2]}$ as follows:

$$R_{t;m}^{[j]} = \sum_{\substack{m_1, \ldots, m_{k+1} \leq m, \\ m_1 + \cdots + m_{k+1} = t}} x_{\alpha_j}(m_1) \cdots x_{\alpha_j}(m_{k+1}), \tag{3.6}$$

where $j = 1, 2$ and m is a fixed (not necessarily negative) integer. We shall often view $R_{t;m}^{[j]}$ as elements of the algebra $U(\bar{\mathfrak{n}})$, rather than as endomorphisms of a $\widehat{\mathfrak{sl}(3)}$-module. It will be clear from the context when expressions such as (3.6) are understood as elements of $U(\bar{\mathfrak{n}})$ or as operators. Let us denote by J the two-sided ideal of $U(\bar{\mathfrak{n}})$ generated by $R_{t;m}^{[1]}$ and $R_{t;m}^{[2]}$ for all $t, m \in \mathbb{Z}$.

In order to obtain exact sequences and q-difference equations in the next section, it is important to have a description of the ideals $I_{k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2}$. This problem is equivalent to finding a presentation of the principal subspaces $W(k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2)$.

The presentation of the principal subspaces of the standard modules for $\widehat{\mathfrak{sl}(2)}$ was proved in [13,14]. In [14] the authors announced a presentation of the principal subspaces $W(k\Lambda_0)$ of the standard representations $L(k\Lambda_0)$ of $\widehat{\mathfrak{sl}(n)}$ for $n \geq 3$. We have given a precise description of the ideals I_{Λ} corresponding to the level 1 standard modules $L(\Lambda)$ of $\widehat{\mathfrak{sl}(n)}$ for $n \geq 3$ in [4].

Here we conjecture a description of the ideals $I_{k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2}$ as follows:

Conjecture. *We have*

$$I_{k\Lambda_0} = J + U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}^+ \tag{3.7}$$

and

$$I_{k_0\Lambda_0+(k-k_0)\Lambda_j} = J + U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}^+ + U(\bar{\mathfrak{n}})x_{\alpha_j}(-1)^{k_0+1} \tag{3.8}$$

for $0 \le k_0 \le k$ *and* $j = 1, 2$. *More generally,*

$$I_{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2} = J + U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}^+ + U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{k_0+k_2+1} + U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_0+k_1+1} \tag{3.9}$$

for any $k_0, k_1, k_2 \ge 0$ *such that* $k_0 + k_1 + k_2 = k$.

As a consequence of this statement we obtain the discrepancy between the defining ideals of the principal subspaces.

Corollary 3.1. *We have*

$$I_{k_0\Lambda_0+(k-k_0)\Lambda_1} = I_{k\Lambda_0} + U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{k_0+1}, \tag{3.10}$$

$$I_{k_0\Lambda_0+(k-k_0)\Lambda_2} = I_{k\Lambda_0} + U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_0+1} \tag{3.11}$$

and

$$I_{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2} = I_{k\Lambda_0} + U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{k_0+k_2+1} + U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_0+k_1+1}. \tag{3.12}$$

In order to prove the above conjecture we can follow the idea of the proof of a presentation of the principal subspaces of the level 1 standard modules for $\widehat{\mathfrak{sl}(n)}$ for $n \ge 3$ developed in [4]. Since this proof of a presentation of the principal subspaces of higher-level representations is very technical we omit it. We will instead prove this conjecture in [7] by using a different and new approach.

4. Exact sequences and q-difference equations

In this section we prove our main results. Mainly, we construct exact sequences of maps between principal subspaces of certain standard $\widehat{\mathfrak{sl}(3)}$-modules. We derive systems of q-difference equations satisfied by the graded dimensions of the principal subspaces $W(i\Lambda_0 + (k-i)\Lambda_j)$ for $0 \le i \le k$ and $j = 1, 2$. We also obtain new formulas for the graded dimensions of $W(i\Lambda_1 + (k-i)\Lambda_2)$.

Take $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$, where k_0, k_1, k_2 are non-negative integers with $k_0 + k_1 + k_2 = k$ and take $L(\Lambda)$ a standard module of level k. It is known (cf. [20,12,23]) that $L(\Lambda)$ is graded with respect to a standard action of the Virasoro algebra operator $L(0)$ and that

$$L(\Lambda) = \coprod_{s\in\mathbb{Z}} L(\Lambda)_{s+h_\Lambda}, \tag{4.1}$$

where

$$h_\Lambda = \frac{\langle \Lambda, \Lambda + \alpha_1 + \alpha_2\rangle}{2(k+3)} \tag{4.2}$$

and where $L(\Lambda)_{s+h_\Lambda}$ is the weight space of $L(\Lambda)$ of weight $s + h_\Lambda$. This is known as grading by *weight*. The space $L(\Lambda)$ has also gradings by *charge* given by the eigenvalues of the operators λ_1 and λ_2 (thought of as $\lambda_1(0)$ and $\lambda_2(0)$). The gradings by charge are compatible with the weight grading. Thus $L(\Lambda)$ decomposes as

$$L(\Lambda) = \coprod_{r_1,r_2,s\in\mathbb{Z}} L(\Lambda)_{r_1+\langle\lambda_1,\Lambda\rangle, r_2+\langle\lambda_2,\Lambda\rangle; s+h_\Lambda}, \tag{4.3}$$

where $L(\Lambda)_{r_1+\langle\lambda_1,\Lambda\rangle, r_2+\langle\lambda_2,\Lambda\rangle; s+h_\Lambda}$ is the subspace of $L(\Lambda)$ consisting of the vectors of charges $r_1 + \langle\lambda_1, \Lambda\rangle$, $r_2 + \langle\lambda_2, \Lambda\rangle$ and of weight $s + h_\Lambda$.

Now we restrict these gradings to the principal subspace $W(\Lambda) \subset L(\Lambda)$ and thus we have

$$W(\Lambda) = \coprod_{r_1, r_2, s \in \mathbb{N}} W(\Lambda)_{r_1 + \langle \lambda_1, \Lambda \rangle, r_2 + \langle \lambda_2, \Lambda \rangle; s + h_\Lambda}. \tag{4.4}$$

We consider the graded dimension (i.e. the generating function of the dimensions of the homogeneous subspaces) of the space $W(\Lambda)$:

$$\chi_{W(\Lambda)}(x_1, x_2; q) = \dim_*(W(\Lambda), x_1, x_2; q) = \mathrm{tr}|_{W(\Lambda)} x_1^{\lambda_1} x_2^{\lambda_2} q^{L(0)}, \tag{4.5}$$

where x_1, x_2 and q are formal variables. To avoid the factor $x_1^{\langle \lambda_1, \Lambda \rangle} x_2^{\langle \lambda_2, \Lambda \rangle} q^{h_\Lambda}$ we use slightly modified graded dimensions as follows:

$$\chi'_{W(\Lambda)}(x_1, x_2; q) = x_1^{-\langle \lambda_1, \Lambda \rangle} x_2^{-\langle \lambda_2, \Lambda \rangle} q^{-h_\Lambda} \chi_{W(\Lambda)}(x_1, x_2; q). \tag{4.6}$$

Thus we have

$$\chi'_{W(\Lambda)}(x_1, x_2; q) \in \mathbb{C}[[x_1, x_2; q]],$$

and in fact, the constant term of $\chi'_{W(\Lambda)}(x_1, x_2; q)$ is 1. Notice that

$$\chi'_{W(k\Lambda_0)}(x_1, x_2; q) = \chi_{W(k\Lambda_0)}(x_1, x_2; q) \in \mathbb{C}[[x_1, x_2; q]]. \tag{4.7}$$

We shall also use the notation

$$W(\Lambda)'_{r_1, r_2; k} = W(\Lambda)_{r_1 + \langle \lambda_1, \Lambda \rangle, r_2 + \langle \lambda_2, \Lambda \rangle; k + h_\Lambda}. \tag{4.8}$$

We now recall from [4] the automorphisms $e^\lambda : V_P \longrightarrow V_P$ for any weight λ. Consider

$$e^\lambda_{(k)} : V_P^{\otimes k} \longrightarrow V_P^{\otimes k}, \qquad e^\lambda_{(k)} = \underbrace{e^\lambda \otimes \cdots \otimes e^\lambda}_{k \text{ times}}, \tag{4.9}$$

a linear automorphism for any $\lambda \in P$. Thus it follows that

$$e^\lambda_{(k)} \ (x_\alpha(m_1) \cdots x_\alpha(m_r) \cdot v) = x_\alpha(m_1 - \langle \lambda, \alpha \rangle) \cdots x_\alpha(m_r - \langle \lambda, \alpha \rangle) \cdot e^\lambda_{(k)}(v) \tag{4.10}$$

and

$$e^\lambda_{(k)} \ (x_{-\alpha}(m_1) \cdots x_{-\alpha}(m_r) \cdot v) = x_{-\alpha}(m_1 + \langle \lambda, \alpha \rangle) \cdots x_{-\alpha}(m_r + \langle \lambda, \alpha \rangle) \cdot e^\lambda_{(k)}(v) \tag{4.11}$$

for any positive root α, $m_1, \ldots, m_r \in \mathbb{Z}$ and $v \in V_P^{\otimes k}$ (cf. (2.16)). We also have

$$e^\lambda_{(k)} \ (h_1(m_1) \cdots h_1(m_r) \cdot v) = h_1(m_1) \cdots h_r(m_r) \cdot e^\lambda_{(k)}(v) \tag{4.12}$$

if each $m_j \neq 0$ and $h_j \in \mathfrak{h}$ (recall (2.17)).

One can also consider maps of the following type:

$$\mathrm{Id}_{(k_0)} \otimes e^\lambda_{(k-k_0)} : V_P^{\otimes k} \longrightarrow V_P^{\otimes k} \tag{4.13}$$

for any $\lambda \in P$ and $0 \leq k_0 \leq k$, where Id is the identity map.

Let k_0 and k_1 be non-negative integers whose sum is k. The next results show that there exist maps of type (4.9) between certain standard modules, and in particular between their corresponding principal subspaces. Using (2.13) we have:

Lemma 4.1. *Let i be an integer such that $0 \leq i \leq k$.*

1. *The restriction of $e^{\lambda_1}_{(k)}$ to $L(i\Lambda_0 + (k-i)\Lambda_2)$ lies in $L((k-i)\Lambda_0 + i\Lambda_1)$.*

2. *We have an injective linear map between principal subspaces*

$$e^{\lambda_1}_{(k)} : W(i\Lambda_0 + (k-i)\Lambda_2) \longrightarrow W((k-i)\Lambda_0 + i\Lambda_1). \tag{4.14}$$

3. *If* $i = k$ *then* (4.14) *is a linear isomorphism*

$$e^{\lambda_1}_{(k)} : W(k\Lambda_0) \longrightarrow W(k\Lambda_1). \tag{4.15}$$

Moreover, we obtain the following relation between the graded dimensions of the principal subspaces $W(k\Lambda_0)$ *and* $W(k\Lambda_1)$*:*

$$\chi'_{W(k\Lambda_1)}(x_1, x_2; q) = \chi'_{W(k\Lambda_0)}(x_1 q, x_2; q), \tag{4.16}$$

which is equivalent to

$$\chi_{W(k\Lambda_1)}(x_1, x_2; q) = x_1^{2k/3} x_2^{k/3} q^{h_{k\Lambda_1}} \chi_{W(k\Lambda_0)}(x_1 q, x_2; q). \tag{4.17}$$

Proof. 1. We view $L(i\Lambda_0 + (k-i)\Lambda_2)$ and $L((k-i)\Lambda_0 + i\Lambda_1)$ embedded in $V_P^{\otimes k}$ (cf. (2.13)). We have

$$L(i\Lambda_0 + (k-i)\Lambda_2) = U(\widehat{\mathfrak{sl}(3)}) \cdot v_{i_1,\ldots,i_k},$$

where $v_{i_1,\ldots,i_k} = v_{\Lambda_{i_1}} \otimes \cdots \otimes v_{\Lambda_{i_k}}$ with exactly i indices equal to 0 and $k-i$ indices equal to 2. We may and do assume that the first i indices are 0 and the other $k-i$ indices are 2. By using our identifications (2.8) we have

$$e^{\lambda_1} v_{\Lambda_2} = e^{\alpha_1+\alpha_2} = x_{\alpha_1+\alpha_2}(-1) \cdot v_{\Lambda_0},$$
$$x_{\alpha_1+\alpha_2}(-1) \cdot v_{\Lambda_1} = x_{\alpha_1+\alpha_2}(-1) \cdot e^{\lambda_1} = 0$$

and

$$x_{\alpha_1+\alpha_2}(-1)^2 \cdot v_{\Lambda_0} = x_{\alpha_1+\alpha_2}(-1)^2 \cdot 1 = 0$$

(cf. [21,4]). Thus it follows that

$$\begin{aligned} e^{\lambda_1}_{(k)}(v_{i\Lambda_0+(k-i)\Lambda_2}) &= e^{\lambda_1}_{(k)}(\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{i \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{(k-i) \text{ times}}) \\ &= \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{i \text{ times}} \otimes \underbrace{x_{\alpha_1+\alpha_2}(-1) \cdot v_{\Lambda_0} \otimes \cdots \otimes x_{\alpha_1+\alpha_2}(-1) \cdot v_{\Lambda_0}}_{(k-i) \text{ times}} \\ &= a\Delta(x_{\alpha_1+\alpha_2}(-1)^{k-i})(\underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{i \text{ times}} \otimes \underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{(k-i) \text{ times}}), \\ &= a x_{\alpha_1+\alpha_2}(-1)^{k-i} \cdot v_{(k-i)\Lambda_0+i\Lambda_1}, \end{aligned} \tag{4.18}$$

where a is a nonzero constant, so that

$$e^{\lambda_1}_{(k)}(v_{i\Lambda_0+(k-i)\Lambda_2}) \in U(\widehat{\mathfrak{sl}(3)}) \cdot v_{(k-i)\Lambda_0+i\Lambda_1}. \tag{4.19}$$

Now by (4.10)–(4.12) and (4.19) we obtain the linear map

$$e^{\lambda_1}_{(k)} : L(i\Lambda_0 + (k-i)\Lambda_2) \longrightarrow L((k-i)\Lambda_0 + i\Lambda_1). \tag{4.20}$$

2. Let us restrict the map (4.20) to the principal subspace $W(i\Lambda_0 + (k-i)\Lambda_2)$. Using similar arguments with $U(\bar{\mathfrak{n}})$ instead of $U(\widehat{\mathfrak{sl}(3)})$, we obtain a linear map

$$e^{\lambda_1}_{(k)} : W(i\Lambda_0 + (k-i)\Lambda_2) \longrightarrow W((k-i)\Lambda_0 + i\Lambda_1),$$

which is clearly injective.

3. Let us take $i = k$ in (4.14). By (4.18) we have

$$e^{\lambda_1}_{(k)}(v_{k\Lambda_0}) = v_{k\Lambda_1}. \tag{4.21}$$

Since

$$W(k\Lambda_1) = U(\bar{\mathfrak{n}}) \cdot v_{k\Lambda_1} = U(\bar{\mathfrak{n}}) \cdot e^{\lambda_1}_{(k)}(v_{k\Lambda_0}) = e^{\lambda_1}_{(k)}(W(k\Lambda_0))$$

(cf. (4.10)) we obtain that the linear map

$$e_{(k)}^{\lambda_1} : W(k\Lambda_0) \longrightarrow W(k\Lambda_1) \tag{4.22}$$

is surjective and thus it is a linear isomorphism. The isomorphism (4.22) does not preserve weight and charge. Let $W(k\Lambda_0)_{r_1,r_2;s}$ with $r_1, r_2, s \in \mathbb{N}$ be a homogeneous subspace of $W(k\Lambda_0)$. The map (4.22) increases the charge corresponding to λ_j by $\langle k\Lambda_1, \lambda_j\rangle$ for $j = 1, 2$. For any $w \in W(k\Lambda_0)_{r_1,r_2;s}$ the homogeneous element $e_{(k)}^{\lambda_1}(w)$ has weight $s + r_1 + h_{k\Lambda_1}$. Hence we obtain an isomorphism between homogeneous spaces

$$e_{(k)}^{\lambda_1} : W(k\Lambda_0)'_{r_1,r_2;s} \longrightarrow W(k\Lambda_1)'_{r_1,r_2;s+r_1},$$

which gives the relation between the graded dimensions

$$\chi'_{W(k\Lambda_1)}(x_1, x_2; q) = \chi'_{W(k\Lambda_0)}(x_1 q, x_2; q),$$

and so

$$\chi_{W(k\Lambda_1)}(x_1, x_2; q) = x_1^{2k/3} x_2^{k/3} q^{h_{k\Lambda_1}} \chi_{W(k\Lambda_0)}(x_1 q, x_2; q). \quad \square$$

We have a result completely analogous to the previous lemma:

Lemma 4.2. *Let i be an integer such that $0 \le i \le k$.*

1. *The restriction of $e_{(k)}^{\lambda_2}$ to $L(i\Lambda_0 + (k-i)\Lambda_1)$ lies in $L((k-i)\Lambda_0 + i\Lambda_2)$.*
2. *At the level of the principal subspaces we have the following injection:*

$$e_{(k)}^{\lambda_2} : W(i\Lambda_0 + (k-i)\Lambda_1) \longrightarrow W((k-i)\Lambda_0 + i\Lambda_2). \tag{4.23}$$

3. *If $i = k$ then (4.23) is a linear isomorphism*

$$e_{(k)}^{\lambda_2} : W(k\Lambda_0) \longrightarrow W(k\Lambda_2). \tag{4.24}$$

In particular, we obtain

$$\chi'_{W(k\Lambda_2)}(x_1, x_2; q) = \chi'_{W(k\Lambda_0)}(x_1, x_2 q; q), \tag{4.25}$$

which is equivalent to

$$\chi_{W(k\Lambda_2)}(x_1, x_2; q) = x_1^{k/3} x_2^{2k/3} q^{h_{k\Lambda_2}} \chi_{W(k\Lambda_0)}(x_1, x_2 q; q). \quad \square \tag{4.26}$$

Our main goal is to obtain exact sequences of maps between principal subspaces. We have already seen in Lemmas 4.1 and 4.2 that the maps $e_{(k)}^{\lambda_1}$ and $e_{(k)}^{\lambda_2}$ are examples in this direction. In order to construct more general exact sequences and thereby to obtain q-difference equations we introduce maps of type $e_{(k)}^{\lambda}$ with $\lambda \neq \lambda_1$ and $\lambda \neq \lambda_2$ between principal subspaces.

We first consider the weight $\lambda = \alpha_1 - \lambda_1 = \frac{1}{3}\alpha_1 - \frac{1}{3}\alpha_2$ and the linear isomorphism

$$e_{(k)}^{\lambda} : V_P^{\otimes k} \longrightarrow V_P^{\otimes k}. \tag{4.27}$$

The restriction of (4.27) to the principal subspace $W(i\Lambda_1 + (k-i)\Lambda_2)$ is the map

$$e_{(k)}^{\lambda} : W(i\Lambda_1 + (k-i)\Lambda_2) \longrightarrow W(i\Lambda_0 + (k-i)\Lambda_1), \tag{4.28}$$

where $0 \le i \le k$.

Since

$$\lambda + \lambda_1 = \alpha_1, \qquad \lambda + \lambda_2 = \lambda_1$$

and

$$v_{i\Lambda_1+(k-i)\Lambda_2} = \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{i \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{(k-i) \text{ times}},$$

$$\begin{aligned} e^{\lambda}_{(k)}(v_{i\Lambda_1+(k-i)\Lambda_2}) &= \underbrace{x_{\alpha_1}(-1)\cdot v_{\Lambda_0}\otimes\cdots\otimes x_{\alpha_1}(-1)\cdot v_{\Lambda_0}}_{i \text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{(k-i)\text{ times}} \\ &= a\Delta(x_{\alpha_1}(-1)^i)(\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{i\text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{(k-i)\text{ times}}) = ax_{\alpha_1}(-1)^i\cdot v_{i\Lambda_0+(k-i)\Lambda_1}, \end{aligned} \tag{4.29}$$

where a is a nonzero constant (note that $x_{\alpha_1}(-1)\cdot v_{\Lambda_1}=0$ and $x_{\alpha_1}(-1)^2\cdot v_{\Lambda_0}=0$). Thus by (2.16) and (4.29) it follows that

$$\begin{aligned} &e^{\lambda}_{(k)}\left(x_{\alpha_1}(m_{1,1})\cdots x_{\alpha_1}(m_{r_1,1})\cdot v_{i\Lambda_1+(k-i)\Lambda_2}\right) \\ &\quad = ax_{\alpha_1}(m_{1,1}-1)\cdots x_{\alpha_1}(m_{r_1,1}-1)x_{\alpha_1}(-1)^i\cdot v_{i\Lambda_0+(k-i)\Lambda_1} \end{aligned} \tag{4.30}$$

and

$$\begin{aligned} &e^{\lambda}_{(k)}\left(x_{\alpha_2}(m_{1,2})\cdots x_{\alpha_2}(m_{r_2,2})\cdot v_{i\Lambda_1+(k-i)\Lambda_2}\right) \\ &\quad = ax_{\alpha_2}(m_{1,1}+1)\cdots x_{\alpha_2}(m_{r_2,2}+1)x_{\alpha_1}(-1)^i\cdot v_{i\Lambda_0+(k-i)\Lambda_1}, \end{aligned} \tag{4.31}$$

where $a\neq 0$, $r_j>0$, $m_{1,j},\ldots,m_{r_j,j}\in\mathbb{Z}$ and $j=1,2$.

For $\beta=\alpha_2-\lambda_2=-\frac{1}{3}\alpha_1+\frac{1}{3}\alpha_2$ we have the isomorphism

$$e^{\beta}_{(k)}: V_P^{\otimes k}\longrightarrow V_P^{\otimes k}.$$

As before, we obtain a linear map between principal subspaces

$$e^{\beta}_{(k)}: W((k-i)\Lambda_1+i\Lambda_2)\longrightarrow W(i\Lambda_0+(k-i)\Lambda_2) \tag{4.32}$$

for any i with $0\le i\le k$. Thus it follows that

$$\begin{aligned} &e^{\beta}_{(k)}(x_{\alpha_j}(m_{1,j})\cdots x_{\alpha_j}(m_{r_j,j})\cdot v_{(k-i)\Lambda_1+i\Lambda_2}) \\ &\quad = ax_{\alpha_j}(m_{1,j}-\langle\beta,\alpha_j\rangle)\cdots x_{\alpha_j}(m_{r_j,j}-\langle\beta,\alpha_j\rangle)x_{\alpha_2}(-1)^i\cdot v_{i\Lambda_0+(k-i)\Lambda_2} \end{aligned} \tag{4.33}$$

for $a\neq 0$ (cf. (4.29)–(4.31)).

When $k=i=1$ the maps $e^{\lambda}_{(1)}$ and $e^{\beta}_{(1)}$ are exactly the maps e^{λ^1} and e^{λ^2} defined in Section 4 of [4].

Intertwining operators among standard $\widehat{\mathfrak{sl}(3)}$-modules and fusion rules (the dimensions of the vector spaces of intertwining operators of a certain type) are important tools in our work. In this paper we follow [12] for the construction of distinguished intertwining operators and for some of their relevant properties.

Let i be an integer with $1\le i\le k$. By using Chapter 13 of [12] there exists a nonzero operator

$$\mathcal{Y}(v_{(k-1)\Lambda_0+\Lambda_1},x): L(i\Lambda_0+(k-i)\Lambda_1)\longrightarrow L((i-1)\Lambda_0+(k-i+1)\Lambda_1)\{x\} \tag{4.34}$$

corresponding to an intertwining operator of type

$$\begin{pmatrix} L((i-1)\Lambda_0+(k-i+1)\Lambda_1) \\ L((k-1)\Lambda_0+\Lambda_1)\quad L(i\Lambda_0+(k-i)\Lambda_1)\end{pmatrix}. \tag{4.35}$$

In fact, the dimension of the vector space of the intertwining operators of type (4.35) (i.e. fusion rule) is one (cf. [12, 20,30]; see also [3,15] and [16]). In this work we use only intertwining operators whose fusion rules equal one.

We use the notation

$$h_1=h_{(k-1)\Lambda_0+\Lambda_1},\qquad h_2=h_{i\Lambda_0+(k-i)\Lambda_1}\quad\text{and}\quad h_3=h_{(i-1)\Lambda_0+(k-i+1)\Lambda_1}$$

(recall (4.2)). We have

$$\mathcal{Y}(v_{(k-1)\Lambda_0+\Lambda_1},x)\in x^{h_3-h_1-h_2}(\mathrm{Hom}\,(L(i\Lambda_0+(k-i)\Lambda_1),L((i-1)\Lambda_0+(k-i+1)\Lambda_1)))[[x,x^{-1}]]$$

(cf. [17,20,30]). As a consequence of the Jacobi identity we obtain

$$[\mathcal{Y}(v_{(k-1)\Lambda_0+\Lambda_1},x),U(\bar{\mathfrak{n}})]=0, \tag{4.36}$$

so that the coefficients of $\mathcal{Y}(v_{(k-1)\Lambda_0+\Lambda_1}, x)$ commute with the action of $U(\bar{\mathfrak{n}})$ (cf. [12]; see also [21,9,10,4]).

Now we take the constant term of $x^{-h_3+h_1+h_2}\mathcal{Y}(v_{(k-1)\Lambda_0+\Lambda_1}, x)$ and we denote it by $\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1}, x)$. The restriction of $\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1}, x)$ to $W(i\Lambda_0 + (k-i)\Lambda_1)$ is a linear map

$$\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1}, x) : W(i\Lambda_0 + (k-i)\Lambda_1) \longrightarrow W((i-1)\Lambda_0 + (k-i+1)\Lambda_1) \tag{4.37}$$

with the following property:

$$\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1}, x)\ (v_{i\Lambda_0+(k-i)\Lambda_1}) = bv_{(i-1)\Lambda_0+(k-i+1)\Lambda_1}, \tag{4.38}$$

where b is a nonzero scalar. By (4.36) we immediately have

$$[\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1}, x), U(\bar{\mathfrak{n}})] = 0, \tag{4.39}$$

so that $\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1}, x)$ commutes with the action of $U(\bar{\mathfrak{n}})$.

Similarly, there exists a nonzero intertwining operator of type

$$\begin{pmatrix} L((i-1)\Lambda_0 + (k-i+1)\Lambda_2) \\ L((k-1)\Lambda_0 + \Lambda_2) \quad L(i\Lambda_0 + (k-i)\Lambda_2) \end{pmatrix}, \tag{4.40}$$

and there also exists a linear map associated with the intertwining operator $\mathcal{Y}(v_{(k-1)\Lambda_0+\Lambda_2}, x)$ of type (4.40),

$$\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_2}, x) : W(i\Lambda_0 + (k-i)\Lambda_2) \longrightarrow W((i-1)\Lambda_0 + (k-i+1)\Lambda_2), \tag{4.41}$$

satisfying similar properties to those of the map (4.37).

The standard modules $L(k\Lambda_0)$, $L(k\Lambda_1)$ and $L(k\Lambda_2)$ are "group-like" elements in the fusion ring at level k. Such modules are also called simple currents (cf. [29]).

When $k = i = 1$, the maps (4.37) and (4.41) are the maps $\mathcal{Y}_c(e^{\lambda_1}, x)$ and $\mathcal{Y}_c(e^{\lambda_2}, x)$ used in [4].

We now prove our main theorem that gives two families of i exact sequences of maps between principal subspaces for $1 \le i \le k$.

Theorem 4.1. *Consider the maps $e^{\lambda}_{(k)}$, $e^{\beta}_{(k)}$, $\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1}, x)$ and $\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_2}, x)$ introduced above (recall (4.28), (4.32), (4.37) and (4.41)). Then for any i with $1 \le i \le k$ the following sequences:*

$$\begin{aligned} 0 \longrightarrow W(i\Lambda_1 + (k-i)\Lambda_2) &\xrightarrow{e^{\lambda}_{(k)}} W(i\Lambda_0 + (k-i)\Lambda_1) \xrightarrow{\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1}, x)} \\ &W((i-1)\Lambda_0 + (k-i+1)\Lambda_1) \longrightarrow 0 \end{aligned} \tag{4.42}$$

and

$$\begin{aligned} 0 \longrightarrow W((k-i)\Lambda_1 + i\Lambda_2) &\xrightarrow{e^{\beta}_{(k)}} W(i\Lambda_0 + (k-i)\Lambda_2) \xrightarrow{\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_2}, x)} \\ &W((i-1)\Lambda_0 + (k-i+1)\Lambda_2) \longrightarrow 0 \end{aligned} \tag{4.43}$$

are exact.

Proof. We will prove that the sequence (4.42) is exact. The proof of the exactness of the sequence (4.43) is completely analogous and we omit it.

We already know by (4.30) and (4.31) that $e^{\lambda}_{(k)}$ maps $W(i\Lambda_1 + (k-i)\Lambda_2)$ to $W(i\Lambda_0 + (k-i)\Lambda_1)$. This map is clearly injective. Since $\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1}, x)$ maps the highest weight vector $v_{i\Lambda_0+(k-i)\Lambda_1}$ to a nonzero multiple of the highest weight vector $v_{(i-1)\Lambda_0+(k-i+1)\Lambda_1}$ (recall (4.38)) and since this map commutes with the action of $U(\bar{\mathfrak{n}})$ (recall (4.39)) we have that $\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1}, x)$ is surjective.

Let $w \in \mathrm{Im}\, e^{\lambda}_{(k)}$. By (4.30) and (4.31) we have

$$w = vx_{\alpha_1}(-1)^i \cdot v_{i\Lambda_0+(k-i)\Lambda_1}$$

with $v \in U(\bar{\mathfrak{n}})$. Hence

$$\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1}, x)(w) = bvx_{\alpha_1}(-1)^i \cdot v_{(i-1)\Lambda_0+(k-i)\Lambda_1}, \tag{4.44}$$

where b is a nonzero constant (cf. (4.38) and (4.39)). Now (4.44) combined with

$$x_\alpha(-1)\cdot v_{\Lambda_1}=0 \quad \text{and} \quad x_\alpha(-1)^2\cdot v_{\Lambda_0}=0$$

implies

$$\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1},x)(w)=0,$$

and thus it gives the inclusion

$$\operatorname{Im} e^{\lambda}_{(k)}\subset \operatorname{Ker}\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1},x). \tag{4.45}$$

It remains to prove the inclusion

$$\operatorname{Ker}\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1},x)\subset \operatorname{Im} e^{\lambda}_{(k)}. \tag{4.46}$$

In order to show this inclusion we first characterize the vector spaces Ker $\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1},x)$ and Im $e^{\lambda}_{(k)}$. Let $w\in$ Ker $\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1},x)$. In particular $w\in W(i\Lambda_0+(k-i)\Lambda_1)$, so that

$$w=f_{i\Lambda_0+(k-i)\Lambda_1}(u)$$

for $u\in U(\bar{\mathfrak{n}})$ (cf. (3.2)). By using again (4.38) and (4.39) we obtain

$$\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1},x)(f_{i\Lambda_0+(k-i)\Lambda_1}(u))=0\Leftrightarrow f_{(i-1)\Lambda_0+(k-i+1)\Lambda_1}(u)=0.$$

We have just shown that

$$w=f_{i\Lambda_0+(k-i)\Lambda_1}(u)\in \operatorname{Ker}\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1},x)\Leftrightarrow u\in I_{(i-1)\Lambda_0+(k-i+1)\Lambda_1}. \tag{4.47}$$

Let $w\in$ Im $e^{\lambda}_{(k)}$. Then by (4.30) and (4.31) we have

$$w=vx_{\alpha_1}(-1)^i\cdot v_{i\Lambda_0+(k-i)\Lambda_1}=f_{i\Lambda_0+(k-i)\Lambda_1}(vx_{\alpha_1}(-1)^i), \tag{4.48}$$

where $v\in U(\bar{\mathfrak{n}})$. On the other hand, since $w\in$ Im $e^{\lambda}_{(k)}\subset W(i\Lambda_0+(k-i)\Lambda_1)$,

$$w=f_{i\Lambda_0+(k-i)\Lambda_1}(u) \tag{4.49}$$

with $u\in U(\bar{\mathfrak{n}})$. Now (4.48) and (4.49) imply

$$u-vx_{\alpha_1}(-1)^i\in I_{i\Lambda_0+(k-i)\Lambda_1}.$$

Thus we have obtained

$$w=f_{i\Lambda_0+(k-i)\Lambda_1}(u)\in \operatorname{Im} e^{\lambda}_{(k)}\Leftrightarrow u\in I_{i\Lambda_0+(k-i)\Lambda_1}+U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^i. \tag{4.50}$$

We also have

$$I_{(i-1)\Lambda_0+(k-i+1)\Lambda_1}=I_{k\Lambda_0}+U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^i \tag{4.51}$$

and

$$I_{i\Lambda_0+(k-i)\Lambda_1}=I_{k\Lambda_0}+U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{i+1}, \tag{4.52}$$

so that

$$I_{(i-1)\Lambda_0+(k-i+1)\Lambda_1}=I_{k\Lambda_0}+U(\bar{\mathfrak{n}}_-)x_{\alpha_1}(-1)^i\subset I_{i\Lambda_0+(k-i)\Lambda_1}+U(\bar{\mathfrak{n}}_-)x_{\alpha_1}(-1)^i \tag{4.53}$$

(recall Corollary 3.1). Now (4.47), (4.50) and (4.53) prove the inclusion

$$\operatorname{Ker}\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1},x)\subset \operatorname{Im} e^{\lambda}_{(k)}, \tag{4.54}$$

and thus the exactness of the sequence (4.42). This concludes our theorem. □

Remark 4.1. The chain property of the sequence (4.42) follows from (4.47) and (4.50)–(4.52). The chain property follows also by a different argument, as we have just seen in the proof of the previous theorem. We do not have an elementary proof that does not use the description of the ideals $I_{i\Lambda_0+(k-i)\Lambda_1}$ and $I_{(i-1)\Lambda_0+(k-i+1)\Lambda_1}$ for the exactness of the sequence (4.42). Similar assertions are of course true for the sequence (4.43).

Recall from Lemmas 4.1 and 4.2 the equations

$$\chi'_{W(k\Lambda_1)}(x_1, x_2; q) = \chi'_{W(k\Lambda_0)}(x_1 q, x_2; q) \tag{4.55}$$

and

$$\chi'_{W(k\Lambda_2)}(x_1, x_2; q) = \chi'_{W(k\Lambda_0)}(x_1, x_2 q; q). \tag{4.56}$$

Given the exact sequences from Theorem 4.1 we now derive a system of q-difference equations that characterize the graded dimensions of $W(i\Lambda_0 + (k-i)\Lambda_j)$ for $1 \le i \le k$ and $j = 1, 2$.

Theorem 4.2. *For any i such that $1 \le i < k$ we have the following q-difference equations:*

$$\begin{aligned}&\chi'_{W(i\Lambda_0+(k-i)\Lambda_1)}(x_1, x_2; q) - \chi'_{W((i-1)\Lambda_0+(k-i+1)\Lambda_1)}(x_1, x_2; q)\\ &\quad + x_1^i x_2^{i-k} q^k \chi'_{W((k-i-1)\Lambda_0+(i+1)\Lambda_2)}(x_1 q^2, x_2 q^{-2}; q)\\ &\quad - x_1^i x_2^{i-k} q^k \chi'_{W((k-i)\Lambda_0+i\Lambda_2)}(x_1 q^2, x_2 q^{-2}; q) = 0\end{aligned} \tag{4.57}$$

and

$$\begin{aligned}&\chi'_{W(i\Lambda_0+(k-i)\Lambda_2)}(x_1, x_2; q) - \chi'_{W((i-1)\Lambda_0+(k-i+1)\Lambda_2)}(x_1, x_2; q)\\ &\quad + x_1^{i-k} x_2^{i} q^k \chi'_{W((k-i-1)\Lambda_0+(i+1)\Lambda_1)}(x_1 q^{-2}, x_2 q^{2}; q)\\ &\quad - x_1^{i-k} x_2^{i} q^k \chi'_{W((k-i)\Lambda_0+i\Lambda_1)}(x_1 q^{-2}, x_2 q^{2}; q) = 0.\end{aligned} \tag{4.58}$$

If $i = k \ge 1$ then we have

$$\begin{aligned}&\chi'_{W((k-1)\Lambda_0+\Lambda_1)}(x_1, x_2; q) - \chi'_{W((k-1)\Lambda_0+\Lambda_2)}(x_1, x_2; q)\\ &\quad + (x_1 q)^k \chi'_{W(k\Lambda_1)}(x_1 q, x_2 q^{-1}; q) - (x_2 q)^k \chi'_{W(k\Lambda_2)}(x_1 q^{-1}, x_2 q; q) = 0.\end{aligned} \tag{4.59}$$

Proof. We first prove that the following q-difference equations hold:

$$\chi'_{W(i\Lambda_0+(k-i)\Lambda_1)}(x_1, x_2; q) = x_1^i q^i \chi'_{W(i\Lambda_1+(k-i)\Lambda_2)}(x_1 q, x_2 q^{-1}; q) + \chi'_{W((i-1)\Lambda_0+(k-i+1)\Lambda_1)}(x_1, x_2; q) \tag{4.60}$$

and

$$\chi'_{W(i\Lambda_0+(k-i)\Lambda_2)}(x_1, x_2; q) = x_2^i q^i \chi'_{W((k-i)\Lambda_1+i\Lambda_2)}(x_1 q^{-1}, x_2 q; q) + \chi'_{W((i-1)\Lambda_0+(k-i+1)\Lambda_2)}(x_1, x_2; q). \tag{4.61}$$

The q-equation (4.60) follows by using the exact sequence (4.42). Formula (4.61) is obtained similarly from the exactness of the sequence (4.43).

Let us consider the homogeneous subspace $W(i\Lambda_1 + (k-i)\Lambda_2)'_{r_1,r_2;s}$ (recall our notation (4.8)). From (4.30) and (4.31) we notice that the map $e^{\lambda}_{(k)}$ increases the charge corresponding to λ_1 by i and preserves the charge corresponding to λ_2. Also, for any $w \in W(i\Lambda_1 + (k-i)\Lambda_2)'_{r_1,r_2;s}$ the vector $e^{\lambda}_{(k)}(w)$ has weight $s + r_1 - r_2 + i$. Hence we have

$$e^{\lambda}_{(k)} : W(i\Lambda_1 + (k-i)\Lambda_2)'_{r_1,r_2;s} \longrightarrow W(i\Lambda_0 + (k-i)\Lambda_1)'_{r_1+i,r_2;s+r_1-r_2+i} \tag{4.62}$$

or equivalently,

$$e^{\lambda}_{(k)} : W(i\Lambda_1 + (k-i)\Lambda_2)'_{r_1-i,r_2;s-r_1+r_2} \longrightarrow W(i\Lambda_0 + (k-i)\Lambda_1)'_{r_1,r_2;s}. \tag{4.63}$$

By (4.38) and (4.39) we obtain

$$\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1}, x) : W(i\Lambda_0 + (k-i)\Lambda_1)'_{r_1,r_2;s} \longrightarrow W((i-1)\Lambda_0 + (k-i+1)\Lambda_1)'_{r_1,r_2;s}. \tag{4.64}$$

The exactness of the sequence (4.42) implies the Eq. (4.60) for any i with $1 \le i \le k$.

Easy computations show that (4.60) and (4.61) yield the following equations:

$$\begin{aligned}&\chi'_{W(i\Lambda_1+(k-i)\Lambda_2)}(x_1, x_2; q)\\ &\quad = x_1^{-i} \chi'_{W(i\Lambda_0+(k-i)\Lambda_1)}(x_1 q^{-1}, x_2 q; q) - x_1^{-i} \chi'_{W((i-1)\Lambda_0+(k-i+1)\Lambda_1)}(x_1 q^{-1}, x_2 q; q)\end{aligned} \tag{4.65}$$

and

$$\chi'_{W((k-i)\Lambda_1+i\Lambda_2)}(x_1,x_2;q)$$
$$= x_2^{-i}\chi'_{W(i\Lambda_0+(k-i)\Lambda_2)}(x_1q,x_2q^{-1};q) - x_2^{-i}\chi'_{W((i-1)\Lambda_0+(k-i+1)\Lambda_2)}(x_1q,x_2q^{-1};q) \tag{4.66}$$

for all i with $1 \le i \le k$.

Now assume that $1 \le i < k$. We take i instead of $k-i$ in (4.66) and equate the expression of $\chi'_{W(i\Lambda_1+(k-i)\Lambda_2)}(x_1,x_2;q)$ from the resulting formula with the expression of $\chi'_{W(i\Lambda_1+(k-i)\Lambda_2)}(x_1,x_2;q)$ from (4.65) and thus we obtain

$$x_1^{-i}\chi'_{W(i\Lambda_0+(k-i)\Lambda_1)}(x_1q^{-1},x_2q;q) - x_1^{-i}\chi'_{W((i-1)\Lambda_0+(k-i+1)\Lambda_1)}(x_1q^{-1},x_2q;q)$$
$$= x_2^{-k+i}\chi'_{W((k-i)\Lambda_0+i\Lambda_2)}(x_1q,x_2q^{-1};q) - x_2^{-k+i}\chi'_{W((k-i-1)\Lambda_0+(i+1)\Lambda_2)}(x_1q,x_2q^{-1};q).$$

By multiplying all terms of this expression by x_1^i and substituting x_1 with x_1q and x_2 with x_2q^{-1} we obtain the four-term q-difference equation (4.57).

In a similar way, we obtain (4.58) by taking $k-i$ instead of i in (4.65), equating the expressions of $\chi'_{W((k-i)\Lambda_1+i\Lambda_2)}(x_1,x_2;q)$ and doing the corresponding substitutions for x_1 and x_2. Note that (4.58) can be derived from (4.57) by using an isomorphism induced from the Dynkin diagram automorphism of $\mathfrak{sl}(3)$ that interchanges α_1 and α_2, and thus it interchanges Λ_1 and Λ_2.

We now assume that $k = i$ in (4.65) and (4.66). We first multiply (4.65) by x_1^k and (4.66) by x_2^k and then we make the substitutions $x_1 \mapsto x_1q$, $x_2 \mapsto x_2q^{-1}$ in (4.65) and $x_1 \mapsto x_1q^{-1}$, $x_2 \mapsto x_2q$ in (4.66). By eliminating the term $\chi'_{W(k\Lambda_0)}(x_1,x_2;q)$ we obtain (4.59). □

The particular case $k = i = 1$ of Theorem 4.2 is listed below.

Remark 4.2. By taking $k = i = 1$ in (4.60) and (4.61) and by using (4.55) and (4.56) for $k = 1$ we recover the recursions satisfied by the graded dimension of the principal subspace $W(\Lambda_0)$ of the level 1 vacuum representation of $\widehat{\mathfrak{sl}(3)}$ obtained in [4] (see also [5]):

$$\chi_{W(\Lambda_0)}(x_1,x_2;q) = \chi_{W(\Lambda_0)}(x_1q,x_2;q) + x_1q\chi_{W(\Lambda_0)}(x_1q^2,x_2q^{-1};q) \tag{4.67}$$

and

$$\chi_{W(\Lambda_0)}(x_1,x_2;q) = \chi_{W(\Lambda_0)}(x_1,x_2q;q) + x_2q\chi_{W(\Lambda_0)}(x_1q^{-1},x_2q^2;q) \tag{4.68}$$

(recall from [4] that $\chi'_{W(\Lambda_0)}(x_1,x_2;q) = \chi_{W(\Lambda_0)}(x_1,x_2;q)$). These recursions are equivalent to

$$\chi_{W(\Lambda_0)}(x_1q,x_2;q) - \chi_{W(\Lambda_0)}(x_1,x_2q;q) + x_1q\chi_{W(\Lambda_0)}(x_1q^2,x_2q^{-1};q) - x_2q\chi_{W(\Lambda_0)}(x_1q^{-1},x_2q^2;q) = 0.$$

Graded dimensions of the principal subspaces $W(i\Lambda_0+(k-i)\Lambda_j)$ with $0 \le i \le k$ and $j = 1, 2$ were computed in [21] by constructing quasiparticle bases of these principal subspaces:

$$\chi'_{W(i\Lambda_0+(k-i)\Lambda_j)}(x_1,x_2;q)$$
$$= \sum_{\substack{0\le M_k\le\cdots\le M_1,\\ 0\le N_k\le\cdots\le N_1}} \frac{q^{\sum_{t=1}^{k}(M_t^2+N_t^2-M_tN_t)+\sum_{t=1}^{k}(M_t\delta_{1,j_t}+N_t\delta_{2,j_t})}}{(q)_{M_1-M_2}\cdots(q)_{M_k}(q)_{N_1-N_2}\cdots(q)_{N_k}} x_1^{\sum_{t=1}^{k}M_t} x_2^{\sum_{t=1}^{k}N_t}, \tag{4.69}$$

where $j_t = 0$ for $0 \le t \le i$ and $j_t = j$ for $i < t \le k$, $j = 1, 2$, and where we use the notation

$$(q)_m = (1-q)\cdots(1-q^m)$$

for any non-negative integer m. Following an argument similar to the one developed in section 7.3 in [1] one can show that (4.69) satisfies the system of Eqs. (4.55)–(4.59).

The q-difference equations (4.65) and (4.66) combined with formulas (4.69) give the graded dimensions of the principal subspaces corresponding to highest weights of the form $i\Lambda_1+(k-i)\Lambda_2$ for any i with $1 \le i \le k$.

We first observe that by taking $k = i$ in (4.65) and using the formulas for $\chi'_{W(k\Lambda_0)}(x_1q^{-1}, x_2q; q)$ and $\chi'_{W((k-1)\Lambda_0+\Lambda_1)}(x_1q^{-1}, x_2q; q)$ we recover the graded dimension $\chi'_{W(k\Lambda_1)}(x_1, x_2; q)$.
Now assume that $1 \le i \le k-1$ in (4.65).

Corollary 4.1. *We have*

$$\chi'_{W(i\Lambda_1+(k-i)\Lambda_2)}(x_1, x_2; q) = \sum_{\substack{0\le M_k\le\cdots\le M_1,\\ 0\le N_k\le\cdots\le N_1}} \frac{q^{\sum_{t=1}^{k}(M_t^2+N_t^2-M_tN_t)+\sum_{t=i+1}^{k}M_t}(1-q^{M_i})q^{\sum_{t=1}^{k}(N_t-M_t)}}{(q)_{M_1-M_2}\cdots(q)_{M_k}(q)_{N_1-N_2}\cdots(q)_{N_k}} \sum_{\substack{0\le M_k\le\cdots\le M_1,\\ 0\le N_k\le\cdots\le N_1}} x_1^{\sum_{t=1}^{k}M_t} x_1^{-i} x_2^{\sum_{t=1}^{k}N_t}. \tag{4.70}$$

Proof. The statement follows by using (4.65) together with formula (4.69) for the graded dimensions of $W(i\Lambda_0 + (k-i)\Lambda_1)$ and $W((i-1)\Lambda_0 + (k-i+1)\Lambda_1)$. □

It is interesting to consider the problem of computing the graded dimension of $W(k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2)$, where k_0, k_1, k_2 are any positive integers whose sum is k, as a combination of the graded dimensions already obtained. One attempt is given by the next theorem, whose proof is completely analogous to that of Theorem 4.1.

Let $k_0, k_1, k_2 \in \mathbb{N}$ such that $k_1, k_2 \ge 1$ and $k_0 + k_1 + k_2 = k$. We consider intertwining operators of types

$$\begin{pmatrix} & L((k_1-1)\Lambda_0+(k_0+k_2+1)\Lambda_1) & \\ L((k_1+k_2-1)\Lambda_0+(k_0+1)\Lambda_1) & & L((k_0+k_1)\Lambda_0+k_2\Lambda_1)\end{pmatrix} \tag{4.71}$$

and

$$\begin{pmatrix} & L((k_2-1)\Lambda_0+(k_0+k_1+1)\Lambda_2) & \\ L((k_1+k_2-1)\Lambda_0+(k_0+1)\Lambda_2) & & L((k_0+k_2)\Lambda_0+k_1\Lambda_2)\end{pmatrix}. \tag{4.72}$$

By [12] (see also [20,30]) we have that the dimension of the vector spaces of intertwining operators of type (4.71) and of type (4.72), respectively, equals one. Consider the constant terms of the intertwining operators $\mathcal{Y}(v, x)$ of type (4.71) and $\mathcal{Y}(v', x)$ of type (4.72):

$$\mathcal{Y}_c(v, x) : W((k_0+k_1)\Lambda_0+k_2\Lambda_1) \longrightarrow W((k_1-1)\Lambda_0+(k_0+k_2+1)\Lambda_1)$$

and

$$\mathcal{Y}_c(v', x) : W((k_0+k_2)\Lambda_0+k_1\Lambda_2) \longrightarrow W((k_2-1)\Lambda_0+(k_0+k_1+1)\Lambda_2),$$

where $v = v_{(k_1+k_2-1)\Lambda_0+(k_0+1)\Lambda_1}$ and $v' = v_{(k_1+k_2-1)\Lambda_0+(k_0+1)\Lambda_2}$ are highest weight vectors of $L((k_1+k_2-1)\Lambda_0 + (k_0+1)\Lambda_1)$ and of $L((k_1+k_2-1)\Lambda_0+(k_0+1)\Lambda_2)$.
Recall the linear maps (4.13), (4.28) and (4.32). Consider

$$\mathrm{Id}_{(k_0)} \otimes e^{\lambda}_{(k-k_0)} : W(k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2) \longrightarrow W((k_0+k_1)\Lambda_0+k_2\Lambda_1)$$

and

$$\mathrm{Id}_{(k_0)} \otimes e^{\beta}_{(k-k_0)} : W(k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2) \longrightarrow W((k_0+k_2)\Lambda_0+k_1\Lambda_2).$$

Theorem 4.3. *There are natural exact sequences*

$$0 \longrightarrow W(k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2) \overset{\mathrm{Id}_{(k_0)}\otimes e^{\lambda}_{(k-k_0)}}{\longrightarrow} W((k_0+k_1)\Lambda_0+k_2\Lambda_1) \overset{\mathcal{Y}_c(v,x)}{\longrightarrow} W((k_1-1)\Lambda_0+(k_0+k_2+1)\Lambda_1) \longrightarrow 0 \tag{4.73}$$

and

$$0 \longrightarrow W(k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2) \overset{\mathrm{Id}_{(k_0)}\otimes e^{\beta}_{(k-k_0)}}{\longrightarrow} W((k_0+k_2)\Lambda_0+k_1\Lambda_2) \overset{\mathcal{Y}_c(v',x)}{\longrightarrow} W((k_2-1)\Lambda_0+(k_0+k_1+1)\Lambda_2) \longrightarrow 0. \quad \square \tag{4.74}$$

However, the maps $\mathrm{Id}_{(k_0)} \otimes e^{\lambda}_{(k-k_0)}$ and $\mathrm{Id}_{(k_0)} \otimes e^{\beta}_{(k-k_0)}$, restricted to the corresponding homogeneous spaces, do not shift the weight of a homogeneous element by a scalar, so that our method used before in Theorem 4.2 to obtain q-difference equations does not extend to this case.

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