Stability of linear multistep methods for delay integro-differential equations

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Abstract

This paper is concerned with the numerical solution of delay integro-differential equations. The adaptation of linear multistep methods is considered. The emphasis is on the linear stability of numerical methods. It is shown that every A-stable, strongly 0-stable linear multistep method of Pouzet type can preserve the delay-independent stability of the underlying linear systems. In addition, some delay-dependent stability conditions for the stability of numerical methods are also given.

Keywords: Delay integro-differential equations; Linear multistep methods of Pouzet type; Asymptotic stability

1. Introduction

In this paper, we study the asymptotic stability of numerical methods for delay integro-differential equations of the form

\[ y'(t) = f(t, y(t), y(t - \tau), \int_{t-\tau}^{t} g(t, v, y(v))dv), \quad t > 0, \]

with the initial condition

\[ y(t) = \psi(t), \quad t \in [-\tau, 0], \]

where \( \tau \) is a positive number, \( y \in \mathbb{C}^d \) is an unknown vector of complex function, \( f : \mathbb{R} \times \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d \) and \( g : \mathbb{R} \times \mathbb{R} \times \mathbb{C}^d \to \mathbb{C}^d \) are given vectors of complex functions with appropriate domains of definition. This type of equations have widely occurred in many biological and control problems (see [1,2] and their references therein). For the convergence of numerical methods for (1.1) or equations of more general forms, we refer to [3–6]. In recent years, many papers have also studied the stability of numerical methods. In the case of scalar model equations with real coefficients, Baker and Ford [7] first investigated the stability of linear multistep methods. Their result shows that...
some methods have a stability region similar to the analytical stability region. Luzynina, Engelborghs and Roose [8] further gave a series of stability results for linear multistep methods under a sufficient condition for the stability of model equations, where the integral part is discretized by a quadrature method based on Lagrange interpolation and a Gauss–Legendre quadrature rule. Huang and Vandewalle [9,10] recently proved that all the Gauss methods of Pouzet type can completely retain the asymptotic stability of such a model equation. Relevant to the case of linear multidimensional systems, Koto [11] proved that every $A$-stable Runge–Kutta–Pouzet method preserves the delay-independent stability of the analytical solution. Zhang and Vandewalle [12] further studied the analytical stability of linear systems of neutral type as well as the stability of Runge–Kutta and linear multistep methods, where the integrals are discretized by a compound quadrature rule. In addition, the nonlinear stability of numerical methods was also considered (see, e.g., [13,14]). However, in the literature concerning the stability of linear multistep methods for delay integro-differential equations mentioned above, the schemes of Pouzet type are not discussed.

In this paper, we investigate the stability of linear multistep methods of Pouzet type for multidimensional linear systems. It will be proven that strongly 0-stable, $A$-stable linear multistep methods can completely preserve the delay-independent stability. This is in accordance with the result on Runge–Kutta methods given in [11]. In addition, some other sufficient conditions for the stability of the schemes will be also studied.

2. Multistep methods for delay integro-differential equations

Since $\tau$ is constant, we can consider a simple adaptation of multistep methods to problem (1.1) and (1.2). Let stepsize $h$ be submultiple of the delay $\tau$, i.e.

$$h = \frac{\tau}{m},$$

(2.1)

where $m$ is a positive integer, let $\sum_{j=0}^{k} \alpha_j \xi_j$ and $\sum_{j=0}^{k} \beta_j \xi_j$ be generating polynomials of an ordinary linear $k$-step method, which are assumed to have no common divisors, and let $t_n = nh, n \in \mathbb{Z}$, denote grid points. Approximating the differential part in (1.1) with the multistep method, we obtain the scheme

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f(t_{n+j}, y_{n+j}, y_{n+j-m}, z_{n+j}),$$

(2.2)

where $y_n$ is an approximation to the exact solution $y(t_n)$ and $z_n$ is an approximation to the integral

$$\int_{t_n-\tau}^{t_n} g(t_n, v, y(v))dv.$$

Next, we consider how to approximate this integral. Define

$$G_n(t) = \int_{t_n-\tau}^{t_n} g(t, v, y(v))dv, \quad t \geq t_n,$$

then,

$$\int_{t_n-\tau}^{t_n} g(t, v, y(v))dv = G_n(t_n) - G_{n-m}(t_n).$$

Let $\tilde{G}_{m}(t), \ldots, \tilde{G}_{k-1}(t)$ denote given approximations to $G_{m}(t), \ldots, G_{k-1}(t)$. Then we can define approximations $\tilde{G}_{n+k}(t)$ to $G_{n+k}(t)$ for $n \geq 0$ by

$$\sum_{j=0}^{k} \alpha_j \tilde{G}_{n+j}(t) = h \sum_{j=0}^{k} \beta_j g(t, t_{n+j}, y_{n+j}), \quad t \geq t_{n+k},$$

(2.3)

which gives

$$z_n = \tilde{G}_n(t_n) - \tilde{G}_{n-m}(t_n).$$

(2.4)
We shall always assume that the starting functions $\tilde{G}_{-m}(t), \ldots, \tilde{G}_{k-1}(t)$ are defined by a quadrature

$$\tilde{G}_n(t) = h \sum_{j=-m}^{k-1} w_{nj} g(t, t_j, y_j), \quad -m \leq n \leq k-1, t \geq t_k,$$

(2.5)

where $y_0, \ldots, y_{k-1}$ are the starting values and

$$y_n = \psi(t_n), \quad -m \leq n \leq 0.$$

Throughout the paper, we assume that the quadrature is exact for the integral of constant, i.e.

$$\sum_{j=-m}^{k-1} w_{nj} = n + m, \quad -m \leq n \leq k - 1.$$  

(2.6)

Pouzet [15] first studied Adams-type methods for Volterra integral equations. For the theory of linear multistep methods for integral equations, we refer to Brunner and Van der Houwen [16], and Lubich [17]. Here we further adapt them to delay integro-differential equations.

3. Stability analysis

Firstly, we introduce some notations. $\sigma(\cdot)$ and $\rho(\cdot)$ designate the spectrum and spectral radius of a matrix, $I_d$ is the $d \times d$ identity matrix and other notations include

$$\mathbb{C}^+ = \{z \in \mathbb{C} \mid \text{Re } z > 0\}, \quad \mathbb{C}^0 = \{z \in \mathbb{C} \mid \text{Re } z = 0\}, \quad \mathbb{C}^- = \{z \in \mathbb{C} \mid \text{Re } z < 0\},$$

$$D = \{z \in \mathbb{C} \mid |z| < 1\}, \quad \Gamma = \{z \in \mathbb{C} \mid |z| = 1\}.$$

In order to get insight into the stability behaviour of numerical methods for problem (1.1), Koto [11] introduced the model equation

$$y'(t) = Ly(t) + My(t - \tau) + N \int_{t-\tau}^{t} y(\nu) d\nu, \quad t > 0,$$

(3.1)

with the constant coefficient matrices $L, M, N \in \mathbb{C}^{d \times d}$. For the asymptotic stability of (3.1), Koto [11] proved the following result.

**Proposition 3.1.** System (3.1) is asymptotically stable for every $\tau > 0$ if and only if

1. $\det[L + M + \tau N] \neq 0$ for every $\tau \geq 0$,
2. $\det[z^2 I_d - zL - N] = 0, z \neq 0 \Rightarrow z \in \mathbb{C}^-$,
3. $\rho[(z^2 I_d - zL - N)^{-1}(zM - N)] < 1$ for any $z \in \mathbb{C}^0$ with $z \neq 0$.

The conditions in Proposition 3.1 are called the delay-independent stability conditions for (3.1). Nevertheless, for a fixed delay $\tau$, those conditions are not necessary. The following proposition provides another stability condition, which is a natural extension of the analytical stability result given in [8] and which is also a special case of the result on neutral systems given in [12].

**Proposition 3.2.** System (3.1) is asymptotically stable if

$$\sigma(L + M\xi_1 + N\tau \xi_2) \subset \mathbb{C}^-, \quad \text{for every } \xi_1 \in D \cup \Gamma, \xi_2 \in D \cup \Gamma.$$

(3.2)

Applying method (2.2)–(2.4) to the test problem (3.1), one arrives at the system of difference equations

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j \left[Ly_{n+j} + My_{n+j-m} + Nz_{n+j}\right],$$

(3.3a)

$$\sum_{j=0}^{k} \alpha_j z_{n+j} = h \sum_{j=0}^{k} \beta_j (y_{n+j} - y_{n+j-m}),$$

(3.3b)
where
\[ z_n = h \sum_{j=-m}^{k-1} (w_{nj} - w_{n-m,j}) y_j, \quad n = 0, \ldots, k - 1. \] (3.3c)

Note that (3.3a) and (3.3b) are not independent of each other because of the constraint (3.3c). However, if the difference equations (3.3a) and (3.3b) are asymptotically stable when they are regarded to be independent of each other, i.e. they are asymptotically stable for any initial values \( y_{-m}, \ldots, y_{k-1} \) and \( z_{-m}, \ldots, z_{k-1} \), then the system (3.3) is asymptotically stable, too.

When (3.3a) and (3.3b) are regarded to be independent of each other, the corresponding characteristic equation is given by
\[ \det Q(\xi) = 0, \] (3.4)
where
\[ Q(\xi) = \begin{bmatrix} \sum_{j=0}^{k} \alpha_j \xi^j I_d - h \sum_{j=0}^{k} \beta_j \xi^j (L + \xi^{-m} M) & -h \sum_{j=0}^{k} \beta_j \xi^j N \\ -h \sum_{j=0}^{k} \beta_j \xi^j (1 - \xi^{-m}) I_d & \sum_{j=0}^{k} \alpha_j \xi^j I_d \end{bmatrix}. \]

Now we recall some stability concepts of multistep methods for ordinary differential equations.

**Definition 3.3.** A linear multistep method is said to be 0-stable if \( \sum_{j=0}^{k} \alpha_j \xi^j \) is a Von Neumann polynomial, i.e. all of its roots are in the closed unit disk \( D \cup \Gamma \) and all of its roots on the unit circle \( \Gamma \) are simple roots. The method is said to be strongly 0-stable if it is 0-stable and 1 is the only root of the polynomial \( \sum_{j=0}^{k} \alpha_j \xi^j \) on the unit circle \( \Gamma \).

**Definition 3.4.** The stability polynomial of a linear multistep method is defined by
\[ R(\xi, \bar{h}) = \sum_{j=0}^{k} \xi^j (\alpha_j - \beta_j \bar{h}), \quad \bar{h} \in \mathbb{C}. \]
The stability region of the method for ODEs is the set
\[ S_M = \{ \bar{h} \in \mathbb{C} \text{ s.t. all the roots of } R(\xi, \bar{h}) \text{ lie in the open unit disk } D \}. \]
The method is said to be A-stable if its stability region satisfies
\[ S_M \supset \mathbb{C}^{-}. \]

Next we state and prove the main result of the paper, which shows that most of the A-stable methods can preserve the delay-independent stability of the analytical solution of (3.1).

**Theorem 3.5.** Suppose that system (3.1) is asymptotically stable for every \( \tau > 0 \) and that the underlying method is strongly 0-stable. Then the difference equation (3.3) is asymptotically stable for all stepsize \( h > 0 \) with the constraint (2.1) if and only if the underlying multistep method is A-stable for ordinary differential equations.

**Proof.** In order to prove the “if” part, we will prove by contradiction that the characteristic equation (3.4) has no solution \( \xi \) with \( |\xi| \geq 1 \). Suppose that there exists \( \xi \in \mathbb{C} \) such that \( \det(Q(\xi)) = 0 \) and \( |\xi| \geq 1 \). We first prove
\[ \sum_{j=0}^{k} \beta_j \xi^j \neq 0. \]
In fact, if \( \sum_{j=0}^{k} \beta_j \xi^j = 0 \), then (3.4) implies
\[ \sum_{j=0}^{k} \alpha_j \xi^j = 0, \]
which contradicts with the assumption that \( \sum_{j=0}^{k} \alpha_j \xi^j \) and \( \sum_{j=0}^{k} \beta_j \xi^j \) have no common divisor. Hence, the characteristic equation is equivalent to

\[
\det \left[ \begin{array}{ccc}
  z I_d - L - \xi^{-m} M & -N \\
  -(1 - \xi^{-m}) I_d & z I_d \\
\end{array} \right] = 0,
\]

where

\[
z = \frac{\sum_{j=0}^{k} \alpha_j \xi^j}{h \sum_{j=0}^{k} \beta_j \xi^j}.
\]

From the definition of stability region, we have \( hz \in \mathbb{C} \setminus S_M \). Therefore, \( A \)-stability implies \( \Re z \geq 0 \).

First, we consider the case \( z \neq 0 \). From (3.5) and the relation that

\[
\left[ \begin{array}{ccc}
  z I_d - L - \xi^{-m} M & -N \\
  -(1 - \xi^{-m}) I_d & z I_d \\
\end{array} \right] \left[ \begin{array}{ccc}
  z I_d \\
  (1 - \xi^{-m}) I_d \\
\end{array} \right] = \left[ \begin{array}{ccc}
  z^2 I_d - zL - N - \xi^{-m} (zM - N) & -N \\
  -(1 - \xi^{-m}) I_d & z I_d \\
\end{array} \right],
\]

it follows that

\[
\det [z^2 I_d - zL - N - \xi^{-m} (zM - N)] = 0. \tag{3.6}
\]

By the second condition of Proposition 3.1, we obtain that the matrix \( z^2 I_d - zL - N \) is nonsingular because \( \Re z \geq 0 \) and \( z \neq 0 \). Hence,

\[
\det [I_d - \xi^{-m} (z^2 I_d - zL - N)^{-1} (zM - N)] = 0, \tag{3.7}
\]

which directly contradicts with the third condition of Proposition 3.1 when \( \Re z = 0 \) and \( z \neq 0 \). When \( \Re z > 0 \), by the maximum modulus principle for subharmonic functions (see (18) in Koto [11]) and the third condition of Proposition 3.1, we have

\[
\sup_{\Re z > 0} \rho [z^2 I_d - zL - N]^{-1} (zM - N) < \sup_{\Re z = 0} \rho [z^2 I_d - zL - N]^{-1} (zM - N) \leq 1,
\]

which contradicts again with (3.7). Therefore, it is impossible that \( \Re z \geq 0 \) and \( z \neq 0 \).

Second, we consider the case \( z = 0 \). In this case, we have

\[
\sum_{j=0}^{k} \alpha_j \xi^j = 0,
\]

which, combined with the strong 0-stability of the method, gives

\[
\xi = 1,
\]

i.e. the difference equation (3.3) has nonzero constant solution. Let \( y_{n+j} = u \) and \( z_{n+j} = v \). Then (3.3a) and (3.3b) imply

\[
\sum_{j=0}^{k} \alpha_j u = h \sum_{j=0}^{k} \beta_j ((L + M)u + Nv),
\]

which gives

\[(L + M)u + Nv = 0.\]

On the other hand, (3.3c) implies

\[v = h \sum_{j=-m}^{k-1} (w_n j - w_{n-m,j}) u,\]
which, combined with (2.6), gives
\[ v = mhu = \tau u. \]

Therefore,
\[ (L + M + N\tau)u = 0. \]

Considering the first condition of Proposition 3.1, we have
\[ u = v = 0. \]

This contradicts with the assumption that the difference equation has nonzero constant solution.

For the “only if” part, we set \( d = 1, M = N = 0 \) in (3.1). Hence, \( A \)-stability follows immediately from the asymptotic stability of the difference equation (3.3). \( \square \)

**Remark 3.6.** In the case of Pouzet-type Runge–Kutta methods with a constraint mesh, Koto [11] has proven that every \( A \)-stable method preserves the delay-independent stability of the exact solution. Here we obtain a similar result for linear multistep methods. In the case of nonconstraint mesh, however, a similar result is possibly no longer valid. This has been confirmed for the \( \theta \)-methods by Koto [18].

**Remark 3.7.** The assumption of strong 0-stability has been used in the stability analysis for functional-differential and functional equations by Huang and Chang [19]. As pointed out in that reference, this assumption can lead to sharper results on numerical stability. For example, we consider delay equations of the form
\[ y'(t) = ay(t) + by(t - \tau), \]
where \( a, b \in \mathbb{R} \). The well-known \( P \)-stability result implies that the numerical solution generated by an \( A \)-stable method is asymptotically stable whenever \( |b| + a < 0 \), but it says nothing about the case of \( a = b \) and \( a < 0 \), where the exact solution is still asymptotically stable. Strong 0-stability, combined with \( A \)-stability, can also guarantee the asymptotic stability of the numerical solution in the latter case. At last but not least, most known multistep methods, including Adams-type methods and Backward Differentiation Formulae, obviously satisfy this assumption.

Since (3.2) is another analytical stability condition, a natural question is whether \( A \)-stable methods also preserve the stability region determined by (3.2). To answer this question, we first prove a general result.

**Theorem 3.8.** Suppose that the underlying linear multistep method is strongly 0-stable,
\[
\left| \frac{(1 - \xi^{-1}) \sum_{j=0}^{k} \beta_j \xi^j}{\sum_{j=0}^{k} \alpha_j \xi^j} \right| \leq 1, \quad \text{for every } \xi \in \Gamma, \tag{3.8}
\]
and
\[
\sigma (h(L + M\xi_1 + N\tau\xi_2)) \subset S_M, \quad \text{for every } \xi_1 \in D \cup \Gamma, \ \xi_2 \in D \cup \Gamma, \tag{3.9}
\]
then, the difference equation (3.3) is asymptotically stable.

**Proof.** We only need to prove that the characteristic equation (3.5) has no solution \( \xi \) with \( |\xi| \geq 1 \). Suppose that there exists \( \xi \in \mathbb{C} \) with \( |\xi| \geq 1 \) such that (3.5) holds. From the definition of \( S_M \), we have
\[ 0 \notin S_M, \]
which, combined with (3.9), gives
\[ \det(L + M + N\tau) \neq 0. \]
Therefore, from the proof of Theorem 3.5, it follows that

\[ z = \frac{\sum_{j=0}^{k} \alpha_j \xi^j}{h \sum_{j=0}^{k} \beta_j \xi^j} \neq 0. \]

Hence, (3.6) holds, i.e.

\[
\det \left[ h z I_d - h \left( L + \xi^{-m} M + \frac{1 - \xi^{-m}}{z} N \right) \right] = 0, \quad (3.10)
\]

where \( h z \in \mathbb{C} \setminus S_M \). On the other hand, by (3.8) and the maximum modulus principle, one has

\[
\left| \frac{(1 - \xi^{-1}) \sum_{j=0}^{k} \beta_j \xi^j}{\sum_{j=0}^{k} \alpha_j \xi^j} \right| \leq 1, \quad \text{for every } |\xi| \geq 1,
\]

where we have used the strong 0-stability of the method and the fact that \( \xi = 1 \) is a removable singularity. Considering

\[
\frac{1 - \xi^{-m}}{z} = \frac{(1 - \xi^{-1}) \sum_{j=0}^{k} \beta_j \xi^j}{\sum_{j=0}^{k} \alpha_j \xi^j} \times \frac{1 + \xi^{-1} + \cdots + \xi^{-m+1}}{m} \times mh,
\]

we have

\[
\left| \frac{1 - \xi^{-m}}{z} \right| \leq \tau,
\]

which, combined with (3.9) and the assumption \( |\xi| \geq 1 \), gives

\[
\sigma \left( h \left( L + \xi^{-m} M + \frac{1 - \xi^{-m}}{z} N \right) \right) \subset S_M.
\]

This contradicts with (3.10). The proof is completed. \( \square \)

For \( A \)-stable methods, we have the following result.

**Theorem 3.9.** Suppose that the underlying method is \( A \)-stable and strongly 0-stable, and (3.8) and (3.2) hold. Then, the difference equation (3.3) is asymptotically stable.

Finally, we give some examples which satisfy (3.8). For the \( \theta \)-methods, we have

\[ k = 1, \quad \alpha_0 = 1, \quad \alpha_1 = 1, \quad \beta_0 = (1 - \theta), \quad \beta_1 = \theta, \]

which gives

\[
\left| \frac{(1 - \xi^{-1}) \sum_{j=0}^{k} \beta_j \xi^j}{\sum_{j=0}^{k} \alpha_j \xi^j} \right| = |\theta + (1 - \theta)\xi^{-1}| \leq 1, \quad \text{for every } \xi \in \Gamma, \theta \in [0, 1].
\]

Therefore, we have the following result.

**Theorem 3.10.** All the \( \theta \)-methods with \( \theta \in [0, 1] \) satisfy (3.8). Hence, every \( \theta \)-method with \( \theta \in [1/2, 1] \) preserves the stability region defined by (3.2).
Next, we consider the $k$-step Backward Differentiation Formula (BDF) which is given by

$$\sum_{j=0}^{k} \alpha_j \xi^j = \sum_{j=1}^{k} \frac{1}{j} (\xi - 1)^j \xi^{k-j}, \quad \sum_{j=0}^{k} \beta_j \xi^j = \xi^k.$$ 

See Li [20]. Hence,

$$\left| \frac{(1 - \xi^{-1}) \sum_{j=0}^{k} \beta_j \xi^j}{\sum_{j=0}^{k} \alpha_j \xi^j} \right| = \frac{1}{\sum_{j=1}^{k} \frac{1}{j} (1 - \xi^{-1})^{j-1}}.$$ 

We draw these curves for $k = 2, 3, 4$ and $\xi \in \Gamma$ in Fig. 1 which shows the following result.

**Theorem 3.11.** The $k$-step BDFe satisfy (3.8) for $k = 1, \ldots, 4$. Hence, in the case of $k = 1$ or $2$, the formula preserves the stability region defined by (3.2).

**References**