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On the Quasi-Heredity of Birman–Wenzl Algebras[1]

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In this paper we consider the Birman–Wenzl algebras over an arbitrary field and prove that they are cellular in the sense of Graham and Lehrer. Furthermore, we determine for which parameters the Birman–Wenzl algebras are quasi-hereditary. So the general theory of cellular algebras and quasi-hereditary algebras applies to Birman–Wenzl algebras. As a consequence, we can determine all irreducible representations of the Birman–Wenzl algebras by linear algebra methods. We prove also that the new Hecke algebras induced from Birman–Wenzl algebras are Frobenius over a field (but not always cellular). © 2000 Academic Press

Key Words: Birman–Wenzl algebra; Brauer algebra; Hecke algebra; quasi-hereditary algebra; cellular algebra.

1. INTRODUCTION

As an algebraic formulation of some new link polynomials in knot theory, Birman and Wenzl introduced in [2] a family of algebras, which are called nowadays the Birman–Wenzl algebras or the Birman–Murakami–Wenzl algebras. These algebras have an intimate connection with various mathematical subjects: Kauffman link invariants, topological quantum field theory, Hecke algebras, Brauer algebras, and braid groups. In fact, Birman–Wenzl algebras play also an important role in the study of subfactors and quantum groups (see [19 and 8]).

Geometrically, Birman–Wenzl algebras are defined in a similar way to Brauer algebras. As a linear basis for a Brauer algebra one uses certain diagrams, and the multiplication is taken just as the natural concatenation of diagrams, here the symmetric groups enter in the role. Birman–Wenzl algebras can be considered as a deformation of Brauer algebras, just by replacing the symmetric groups by their Hecke algebras. It is well known that the Brauer algebras are cellular in the sense of Graham and Lehrer [9]. The purpose of this paper is to study the cell structure of Birman–Wenzl algebras. We show that the basis constructed in [16] (see also [7]) for the

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Birman–Wenzl algebra is in fact a cellular basis. (We should note that not all canonical bases in the sense of Kazhdan and Lusztig are cellular.) Thus the Birman–Wenzl algebras are cellular. This enables us to determine the irreducible representations of Birman–Wenzl algebras over arbitrary field by using the standard methods in the theory of cellular algebras. As another application, we can describe precisely for which parameters the corresponding Birman–Wenzl algebra is quasi-hereditary in the sense of [4]; thus for those Birman–Wenzl algebras, the finite dimensional left modules form a highest weight category with many important homological properties [18].

In Section 2 we recall definitions and collect some necessary facts. In this section we define also a new class of Hecke algebras and prove that they are Frobenius. Unfortunately, they are not always cellular. (So they are different from the usual Hecke algebras.) In Section 3 we prove the main result that Birman–Wenzl algebras are cellular. In Section 4 we first recall the definition of quasi-hereditary algebras and then determine when a Birman–Wenzl algebra is quasi-hereditary.

2. BASIC DEFINITIONS AND FACTS

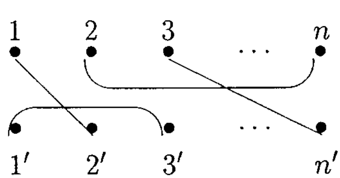
In this section we shall recall the definition of Brauer algebras, Hecke algebras, and Birman–Wenzl algebras and collect some known facts from the literature, which we shall need later. We also define a new class of Hecke algebras and show that they are Frobenius.

2.1. *Brauer Algebras*

Birman–Wenzl algebras are deformations of Brauer algebras. In this section we recall the definition of Brauer algebras and introduce some notations for the later use.

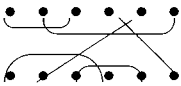
Let n be a natural number, and let $\mathbb{Z}[\delta]$ be the polynomial ring in one variable δ over the integers.

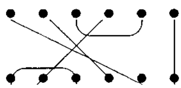
The Brauer algebra $B_{\mathbb{Z}[\delta]}(n, \delta)$, or written briefly $B(n, \delta)$, has as $\mathbb{Z}[\delta]$-linear basis the set of all partitions of the set $S := \{1, 2, ..., n, 1', 2', ..., n'\}$ into two-element subsets (here the cardinality of S is 2n). As usual, we shall represent the basis element by a diagram in a rectangle of the plane, for example,

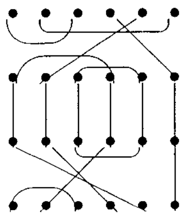
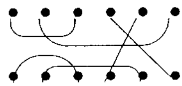


where the top row has n vertices marked by $1, 2, ..., n$; and the bottom row is numbered by $1', 2', ..., n'$. If i and j are in the same subset we draw a line between i and j. We call the corresponding diagram a Brauer n-diagram and denote by B_n the set of all Brauer n-diagrams. The multiplication in the Brauer algebra $B(n, \delta)$ is just the concatenation of two diagrams with a coefficient counting the number of cycles produced by forming the concatenation. The following example explains the definition.

For example $n=6$,

$d_1 =$ 

$d_2 =$ 

$d_1 d_2 =$  $= \delta^1$ 

We define an order on S by $1<2< \cdots <n, n'<(n-1)'< \cdots <2'<1'$, and $i<j'$ for all $1 \leqslant i, j \leqslant n$. For $d \in B_n$ we write $\{i, j\} \in d$ for the line in d and define $x(d)$ to be the number of pairs of $\{i, j\}, \{k, l\} \in d$ such that $i<k<j<l$. Also we write $v(d)$ for the number of pairs $\{i, j\}$ in d such that i and j are in different rows. This is just the number of vertical lines in the diagram d.

Generally, given an arbitrary ring R with identity and an element $\delta \in R$, we can define the Brauer algebra $B_R(n, \delta)$ over R by using the Brauer n-diagrams as R-basis. The following result is well known in [9]; see also [12].

Theorem 2.1. *The Brauer algebra $B_R(n, \delta)$ is cellular for any ring R and $\delta \in R$.*

2.2. *Birman–Wenzl Algebras*

Now we recall the definition of Birman–Wenzl algebras. There are two ways to define Birman–Wenzl algebras. The usual one is given in terms of generators and relations.

Let n be a natural number, and let R be the quotient ring $\mathbb{Z}[\lambda, \lambda^{-1}, z, \delta]/(\lambda^{-1} - \lambda - z(\delta - 1))$.

Definition 2.2. (see [2]) The Birman–Wenzl algebra $BW_R(n, \lambda, z, \delta)$, or simply denoted by $BW(n, \lambda, z, \delta)$ or BW_n, is the quotient of the free algebra over R with generators $g_1^{\pm 1}, g_2^{\pm 1}, ..., g_{n-1}^{\pm 1}$ and $e_1, e_2, ..., e_{n-1}$ modulo the ideal generated by the relations

(1) Kauffman skein relations: $g_i - g_i^{-1} = z(1 - e_i)$;

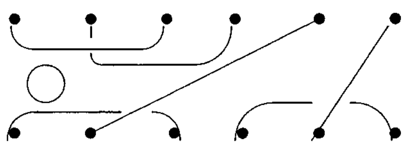
(2) Braid relations: $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ and $g_i g_j = g_j g_i$ for $|i-j| > 1$;

(3) Delooping relations: $g_i e_i = e_i g_i = \lambda e_i$ and $e_i g_{i-1}^{\pm 1} e_i = \lambda^{\mp 1} e_i$.

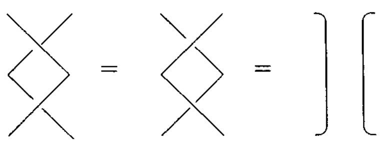
For convenience, we adopt the geometric formulation in terms of tangles (see [10 and 17]). In fact, it is proved in [17] that these two definitions give the same algebra (up to isomorphism).

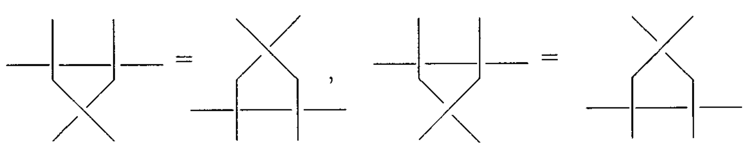
DEFINITION 2.3. An (m, n)-tangle is a piece of knot diagram in a rectangle in the plane, consisting of arcs and closed cycles, such that the end points of the arcs consist of m points at the top of the rectangle and n points at the bottom, in some standard position. An n-tangle is defined to be an (n, n)-tangle.

Thus an n-tangle can be presented as two rows of n vertices, one above the other, and n strands that connect vertices in such a way that each vertex is incident precisely one strand, here over and undercrossings are indicated. We also allow an n-tangle to contain finitely many closed cycles. Strands that connect vertices in the same row are called horizontal, and strands that connect vertices in different rows are called vertical. For example, the following is a 6-tangle:



The Reidemeister moves of types II and III are

(II)  ,

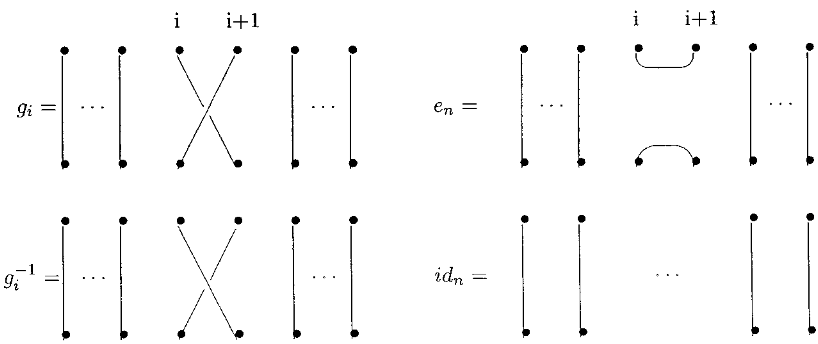
(III) 

DEFINITION 2.4. Two tangles are ambient isotopic if they are related by a sequence of Reidemeister's moves, together with isotopies of the rectangle fixing its boundary. They are regularly isotopic if only (II) and (III) are applied.

Thus regular isotopy is an equivalent relation among tangles, it is given by applying these moves (II) and (III) to tangles by isolating one of these crossings in an open disk in a tangle and applying the relations. We shall take n-tangles to be their equivalence classes under the Reidemeister moves (II) and (III). By T_n we denote the set of n-tangles.

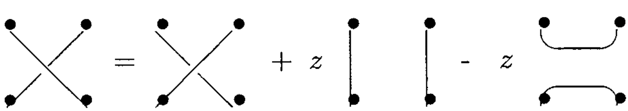
If $t_1, t_2 \in T_n$, we define $t_1 t_2$ to be (the equivalence class of) the tangle obtained by concatenating t_1 and t_2 (placing t_1 above t_2 and identifying the vertices in the top row of t_2 with the corresponding vertices in the bottom row of t_1).

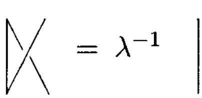
The concatenation product makes the set T_n of all n-tangles into a monoid. There are special elements in T_n defined by



Let M_n be the submonoid of T_n generated by $\{\mathrm{id}_n\} \cup \{g_i^{\pm 1}, e_i \mid 1 \leqslant i \leqslant n-1\}$.

Definition 2.5. The Birman–Wenzl algebra $BW(n, \lambda, z, \delta)$ is the quotient R-algebra of the monoid algebra $R[M_n]$ of M_n over R modulo the following relations:

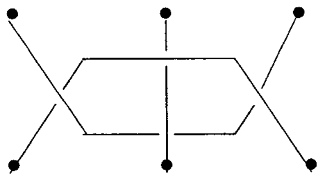
(Q1) 

(Q2) 

(Q3)  $= \lambda$

(Q4) $\bigcirc \cup T = \delta\, T$, where T is an n-tangle in M_n.

As noted in [10], not all n-tangles in T_n are in M_n. For example, the following 3-tangle



is not an element in M_3. Thus it does not represent an element in $BW(3, \lambda, z, \delta)$. An n-tangle in M_n is called a BW_n-diagram. We say that a BW_n-diagram is reachable if no two strands cross more than once, no strand crosses itself and it contains no closed cycles. It is shown that reachable diagrams represent elements of $BW(n, \lambda, z, \delta)$.

Note that the Birman–Wenzl algebra defined in the above geometric way is isomorphic to the one defined algebraically at the beginning of this subsection, this is proved in [17].

For a Brauer n-diagram $d \in B_n$ we define an n-tangle T_d as follows: T_d is a diagram with the same vertices as d and we require the following: the line $\{i, j\}$ in d passes over $\{k, l\}$ if $i < k < j < l$ in S. Thus we have a map $T: B_n \to BW(n, \lambda, z, \delta)$ given by $d \mapsto T_d$. By [16] or [17] (see also Theorem 3.13 in [8]), we have the following fact:

Lemma 2.6. *$\{T_d \mid d \in B_n\}$ is an R-basis for the algebra $BW_R(n, \lambda, z, \delta)$.*

2.3. *Hecke Algebras of Type A*

In this subsection let R be an arbitrary commutative ring with 1. For a given natural number n, we denote by Σ_n the symmetric group on n letters. Recall that for each $q \in R$ there is a Hecke algebra $\mathscr{H}_n(q)$ over R defined as follows: $\mathscr{H}_n(q)$ is a free R-module with the basis $\{\tau_w \mid w \in \Sigma_n\}$, and the multiplication is given by

$$\tau_{s_i} \tau_w = \begin{cases} \tau_{s_i w} & \text{if } \ell(s_i w) = \ell(w) + 1, \\ (q-1)\, \tau_w + q \tau_{s_i, w}, & \text{otherwise,} \end{cases}$$

where $s_i = (i, i+1)$ is a transposition in Σ_n and ℓ is the usual length function.

The following result is well known. For the definition of cellular algebras we refer to the next section.

LEMMA 2.7. *Let $R = \mathbb{Z}[q, q^{-1}]$. Then the R-algebra $\mathscr{H}_n(q)$ is a cellular algebra.*

In this paper we shall use a different version of Hecke algebras.

Let z be an element in R. Let $H_n(z)$ be a free R-module with the basis $\{t_w \mid w \in \Sigma_n\}$. We define the multiplication on $H_n(z)$ by

$$t_{s_i} t_w = \begin{cases} t_{s_i w} & \text{if } \ \ell(s_i w) = \ell(w) + 1, \\ z t_w + t_{s_i w}, & \text{otherwise}, \end{cases}$$

where $s_i = (i, i+1)$ is a transposition in Σ_n and ℓ is the usual length function.

There is a close relation of the two definitions. If there is an invertible element $q \in R$ with $z = q - q^{-1}$, then $\mathscr{H}_n(q^{-2}) \cong H_n(z)$ as R-algebras.

Suppose R is a field. Then we can prove that $H_n(z)$ is a Frobenius-algebra for all $z \in R$. Recall that a finite dimensional R-algebra A is called Frobenius (respectively, symmetric) if ${}_A A \cong {}_A(DA)$ (respectively, ${}_A A_A \cong {}_A(DA)_A$), where D is the usual R-duality.

LEMMA 2.8. *Let R be a field. Then the Hecke algebra $H_n(z)$ is Frobenius.*

Proof. To prove the lemma, it is sufficient to show that there is an R-linear map $f\colon A \to R$ such that the kernel of f contains neither non-zero left ideals nor non-zero right ideals in A. (Notice that the following proof works for Hecke algebras of any type.)

We define a linear map f by $f(\sum_w \lambda_w t_w) = \lambda_{w_0}$, where w_0 is the unique longest element in Σ_n and λ_w are the coefficients in R. Suppose L is a left ideal contained in the kernel of f. We pick an element $h \in L$ and write it as $h = h_0 + h_1 + \cdots + h_{\ell(w_0)}$, where $h_j = \sum_{w \in J_i} \lambda_w t_w$ and J_i is the subset of all w in Σ_n with $\ell(w) = i$. It is clear that $\lambda_{w_0} = 0$ and $h_{\ell(w_0)} = 0$. Suppose that $h_{\ell(w_0)-i} = 0$ for $i = 1, 2, \ldots, j-1$. Now let $w \in J_{\ell(w_0)-j}$. Then there are transpositions $s_j, \ldots, s_2, s_1$ such that $s_j \cdots s_2 s_1 w = w_0$ and $\ell(w_0) = \ell(w) + j$. Now consider the element $t_{s_j \cdots s_2 s_1} h$. The multiplication in $H_n(z)$ shows that t_{w_0} can not appear in $t_{s_j \cdots s_2 s_1}(h_0 + \cdots + h_{\ell(w_0)-j-1})$. The coefficient of t_{w_0} in $t_{s_j \cdots s_2 s_1} h_{\ell(w_0)-j}$ is λ_w. Thus $\lambda_w = 0$, because $t_{s_j \cdots s_2 s_1} h$ is in the kernel of f. This shows that $h_{\ell(w_0)-j} = 0$. By induction, we know that $h = 0$. Similarly, we can show that there is no non-zero right ideal contained in the kernel of f. This finishes the proof.

Note that if t_i stands for the element t_{s_i} then $t_i^2 = zt_i + 1$ and there is a surjective R-algebra-homomorphism from $BW(n, \lambda, z, \delta)$ to $H_n(-z)$ given by $g_i \mapsto t_i$ and $e_i \mapsto 0$.

Remark. In general, $H_n(z)$ is not a cellular algebra over a field (for the involution we take the usual one, namely, $t_w \mapsto t_{w^{-1}}$). For instance, $R = \mathbb{Q}$ and $n = 2$. Then the Hecke algebra $H_2(1)$ is a finite field extension of $\mathbb{Q}$. Since the dimension of $H_2(1)$ is 2, we can easily deduce that $H_2(1)$ is not cellular. Note that over any field, $\mathscr{H}_n(z)$ is cellular for all $z \neq 0$.

It is not known whether this algebra $H_n(z)$ is symmetric.

3. THE CELLULAR STRUCTURE FOR BW_n

In this section we recall first the definition of cellular algebras in the sense of Graham and Lehrer and an equivalent definition given in [11]. Then we apply the idea in [21] to prove that the Birman–Wenzl algebra BW_n is cellular. In fact, our proof shows also that the canonical basis constructed in ([16, see also 7]) is a cellular basis for the Birman–Wenzl algebra. (Note that not all canonical-like bases are cellular bases.) Using the result in this section we are able to determine the quasi-hereditary Birman–Wenzl algebras in the next section.

Definition 3.1 (Graham and Lehrer, [9]). Let R be a commutative ring. An associative R–algebra A is called a *cellular algebra* with cell datum (I, M, C, i) if the following conditions are satisfied:

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has an R-basis $C^\lambda_{S,T}$ where (S, T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is an R-linear anti-automorphism of A with $i^2 = id$ which sends $C^\lambda_{S,T}$ to $C^\lambda_{T,S}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC^\lambda_{S,T}$ can be written as $(\sum_{U \in M(\lambda)} r_a(U, S)\, C^\lambda_{U,T}) + r'$ where r' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficients $r_a(U, S) \in R$ do not depend on T.

The basis $\{C^\lambda_{S,T}\}$ of a cellular algebra A is called a cellular basis. With this basis there is a bilinear form Φ_λ, for each $\lambda \in \Lambda$, which is defined by

$$C^\lambda_{S,T} C^\lambda_{U,V} \equiv \Phi_\lambda(T, U)\, C^\lambda_{S,V}$$

modulo the ideal generated by all basis elements with upper index μ strictly smaller than λ. Graham and Lehrer proved in [9] that the isomorphism classes of simple modules are parametrized by the set $\Lambda_0 = \{\lambda \in \Lambda \mid \Phi_\lambda \neq 0\}$.

Note that if R' is another commutative ring and $f: R' \to R$ is a ring homomorphism, then the R'-algebra $R' \otimes_R A$ is a cellular algebra if the R-algebra A is cellular.

In the following, an R-linear anti-automorphism i of A with $i^2 = \mathrm{id}$ will be called an involution.

Definition 3.2. Let A be an R–algebra where R is a commutative Noetherian integral domain. Assume there is an antiautomorphism i on A with $i^2 = \mathrm{id}$. A two–sided ideal J in A is called a *cell ideal* if and only if $i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that Δ is finitely generated and free over R and that there is an isomorphism of A-bimodules $\alpha: J \simeq \Delta \otimes_R i(\Delta)$ (where $i(\Delta) \subset J$ is the i-image of Δ) making the following diagram commutative:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_R i(\Delta) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle x \otimes y \mapsto i(y) \otimes i(x)} \\ J & \xrightarrow{\alpha} & \Delta \otimes_R i(\Delta). \end{array}$$

The algebra A (with the involution i) is called *cellular* if and only if there is an R-module decomposition $A = J'_1 \oplus J'_2 \oplus \cdots \oplus J'_n$ (for some n) with $i(J'_j) = J'_j$ for each j and such that setting $J_j = \bigoplus_{l=1}^{j} J'_l$ gives a chain of two-sided ideals of $A: 0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ (each of them fixed by i) and for each j ($j = 1, \ldots, n$) the quotient $J'_j = J_j / J_{j-1}$ is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j-1}.

The Δ's obtained from each section J_j/J_{j-1} are called standard modules of the cellular algebra A. Note that all simple modules are obtained from standard modules [9]. (Standard modules are called cell modules or Weyl modules in [9].)

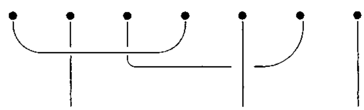
In [11] it is proved that the two definitions of cellular algebras are equivalent. The first definition can be used to check concrete examples. The second definition, however, is often more handy for theoretical and structural purposes.

Typical examples of cellular algebras are the following: Group algebras of symmetric groups, or more general Hecke algebras of type A or even of Ariki–Koike type (i.e., cyclotomic Hecke algebras), Brauer algebras of types B and C [9] (see also [12] for another proof), partition algebras [21] and various kinds of Temperley–Lieb algebras [9]. We shall prove that Birman–Wenzl algebras are also cellular. Before doing this, let us now introduce further notion and notation.

DEFINITION 3.3. A flat (n, k)-dangle is a partition of $\{1, 2, ..., n\}$ into k one-element subsets; and $(n-k)/2$ two-element subsets, here k must be a natural number in $\{n, n-2, n-4, ...\}$. An (n, k)-dangle is similar to a flat (n, k)-dangle, but we must indicate the overcrossing and the undercrossing, also we allow some number of closed cycles (the vertical strands do not cross). An (n, k)-dangle is called reachable if two strands cross at most once and there are no closed cycles.

Geometrically, we can represent a flat dangle d by a diagram in the plane such that i is joined to j by an edge if $\{i, j\} \in d$ and there is a vertical line starting from i if $i \notin d$. The following are examples of a flat $(7, 3)$-dangle and a $(7, 3)$-dangle, respectively.

flat $(7,3)$-dangle



$(7,3)$-dangle

We denote by $FD(n, k)$ the set of all flat (n, k)-dangles, and by $D(n, k)$ the set of all (n, k)-dangles, modulo the equivalence relation generated by Reidemeister moves (II) and (III). Note that on the flat dangles we do not require Reidemeister-like equivalence relations since they are defined by set-theoretical language. We define $V(n, k)$ to be the R-module spanned by the reachable (n, k)-dangles in $D(n, k)$ modulo the relations (Q1)–(Q4). If $d \in FD(n, k)$, we define an (n, k)-dangle D_d by requiring that $\{i, j\} \in d$ passes over $\{k, l\} \in d$ if $i<k<j<l$, and that vertical lines pass under the horizontal lines. We define $V'(n, k)$ to be a copy of $V(n, k)$, but draw the pictures dangling upward rather than down, and label the vertices by $1', 2', ..., n'$. Dually, given a flat dangle d, we have an (n, k)-dangle D'_d in $V'(n, k)$ by requiring that the vertical line passes over the horizontal lines and that $\{i', j'\}$ in d passes under the line $\{k', l'\}$ if $i<k<j<l$. The following lemma is easily deduced from definition.

LEMMA 3.4. *The set* $\{D_d \mid d \in FD(n, k)\}$ *is a basis for the R-module* $V(n, k)$. *Dually,* $\{D'_d \mid d \in FD(n, k)\}$ *is a basis of* $V'(n, k)$.

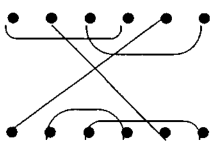
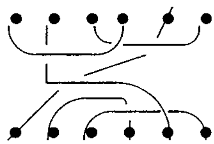
From now on we denote by R the ring $\mathbb{Z}[\lambda, \lambda^{-1}, z, \delta]/(\lambda^{-1} - \lambda - z(\delta - 1))$. We shall use dangles to describe the basis elements in Lemma 2.6.

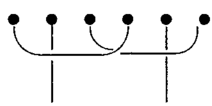
LEMMA 3.5. *For each element $d \in B_n$, we can write T_d uniquely as an element in $V(n,k) \otimes_R V'(n,k) \otimes_R H_n(z)$.*

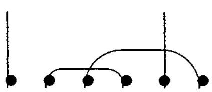
Proof. Suppose there are k vertical lines in $d \in B_n$. Then we have two flat (n,k)-dangles d_1, d_2 (by cutting off all vertical lines) and a permutation $\pi(d) \in \Sigma_k$. These data $(d_1, d_2, \pi(d))$ are uniquely determined by d, where $\pi(d)$ is obtained in the following way: numerate the top ends of the vertical lines in d from left to right by $1, 2, ..., k$, and the bottom ends of the vertical lines in d from left to right by $1', 2', ..., k'$. If i and j' are joined by a vertical line in d, then we define $\pi(d)(i) = j$. In this way we have a permutation $\pi(d)$ in Σ_k.

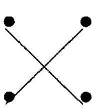
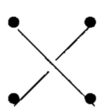
Thus we can write T_d as $D_{d_1} \otimes D'_{d_2} \otimes T_{\pi(d)}$. Clearly, if we are given $d_1, d_2 \in FD(n,k)$, and $\pi \in \Sigma_k$, we have a unique element $d \in B_n$ with the data (d_1, d_2, π); this gives us a unique element T_d which can be written as $D_{d_1} \otimes D'_{d_2} \otimes T_\pi$. Thus T_d corresponds uniquely to $D_{d_1} \otimes D'_{d_2} \otimes T_{\pi(d)}$.

For example, $n = 6$, $k = 2$.

$d =$  $T_d =$ 

$d_1 =$  $D_{d_1} =$ 

$d_2 =$  $D'_{d_2} =$ 

$\pi =$  $T_\pi =$ 

Now we want to define an R-bilinear form φ_k from $V'(n,k) \otimes_R V(n,k)$ to $H_k(z)$. Given two basis elements D'_{d_1}, D_{d_2} with $d_1, d_2 \in FD(n,k)$. Then, by 3.5, we form the element $X_j := D_{d_j} \otimes D'_{d_j} \otimes 1$ in $V(n,k) \otimes V'(n,k) \otimes H_k(z)$ for $j = 1, 2$. Suppose the product $X_1 X_2$ is expressed as an R-linear combination of the basis elements in $\{T_d \mid d \in B_n\}$, say $\sum_j f'_j(\lambda, z, \delta)\, T_{c_j}$, where $f'_j(\lambda, z, \delta)$ are elements in R. It is clear from the multiplication in BW_n that this expression of $X_1 X_2$ can be rewritten as $\sum f_i(\lambda, z, \delta)\, T_{c_i} + a$ with $v(c_i) = k$ and a in the R-spanning of those basis elements T_c with $v(c) < k$. Now we rewrite T_{c_i} as the form $D_{c_i 1} \otimes D'_{c_i 2} \otimes T_{\pi_i}$. Moreover, for those T_{c_i} with $v(c_i) = k$, we know from the concatenation of two tangles that they are of the form $D_{d_1} \otimes D'_{d_2} \otimes T_{\pi_i}$. Hence $X_1 X_2$ can be written finally as $\sum D_{d_1} \otimes D'_{d_2} \otimes f_j(\lambda, z, \delta)\, T_{\pi_j} + a$. Now we define $\varphi_k(D'_{d_1}, D_{d_2}) := \sum_i f_i(\lambda, z, \delta)\, T_{\pi_i} \in H_k(z)$. Note that $\varphi_k(D'_d, D_d) = \delta^{(n-k)/2} \in H_k(z)$ for $d \in FD(n, k)$ with $d = \bullet\!-\!\bullet \cdots \bullet\!-\!\bullet \quad \bullet \cdots \bullet$.

Now we define J_k to be the R-module generated by basis elements T_d with $v(d) \leqslant k$. It is clear that $J_k \subset J_{k+1}$ and J_k is an ideal in BW_n for all k.

LEMMA 3.6. *If* $T_c = D_{c_1} \otimes D'_{c_2} \otimes T_\sigma$ *and* $T_d = D_{d_1} \otimes D'_{d_2} \otimes T_\pi$ *with* $d_1, d_2, c_1, c_2 \in FD(n, k)$ *and* $\sigma, \pi \in \Sigma_k$, *then* $T_c T_d = D_{c_1} \otimes D'_{d_2} \otimes T_\sigma \varphi_k (D'_{c_2}, D_{d_1})\, T_\pi \pmod{J_{k-2}}$.

Proof. Given a dangle D_b with $b \in FD(n, k)$, we may consider D_b as a natural (n, k)-tangle, which we denote again by D_b; also we may consider D'_b as a (k, n)-tangle. Now we consider the concatenating of the two tangles D'_{c_2} and D_{d_1}, this gives us a k-tangle T. If we write this tangle T as a linear combination of the basis elements $\{T_x \mid x \in B_k\}$, then we see that $T = I_k \otimes I_k \otimes \varphi_k(D'_{c_2}, D_{d_1}) + a'$, where $\varphi_k(D'_{c_2}, D_{d_1})$ is the above defined bilinear form, $a' \in J_{k-2}$ and $I_k = id_k$ is the (k, k)-dangle with k-vertical strands. So the product $T_c T_d$ is formed by a series of concatenations:

$$D_{c_1} \cdot T_{\pi(c)} \cdot D'_{c_2} \cdot D_{d_1} \cdot T_{\pi(d)} \cdot D'_{d_2}.$$

Thus we have that $T_c T_d = D_{c_1} \cdot T_{\pi(c)} \cdot \varphi_k(D'_{c_2}, D_{d_1}) \cdot T_{\pi(d)} \cdot D'_{d_2} + a$ with $a \in J_{k-2}$. This implies that

$$T_c T_d \equiv D_{c_1} \otimes D'_{d_2} \otimes T_{\pi(c)} \varphi_k(D'_{c_2}, D_{d_1})\, T_{\pi(d)} \qquad (\mathrm{mod}\ J_{k-2}).$$

This proves the lemma.

By 3.5, we have an R-module decomposition: $BW(n, \lambda, z, \delta) = V(n, n) \otimes V'(n, n) \otimes H_n(z) \oplus V(n, n-2) \otimes V'(n, n-2) \otimes H_{n-2}(z) \oplus V(n, n-4) \otimes V'(n, n-4) \otimes H_{n-4}(z) \oplus \cdots$. The following lemma tells us how to get an ideal in BW_n from an ideal in Hecke algebras.

LEMMA 3.7. *Let* I *be an ideal in* $H_k(z)$. *Then* $J_{k-2} + V(n, k) \otimes V'(n, k) \otimes I$ *is an ideal in* BW_n.

Proof. To prove the lemma, it is sufficient to show that for each $T_d = D_{d_1} \otimes D'_{d_2} \otimes T_{\pi(d)}$ with $d \in B_n$ and $v(d) = l > k$, and $T_c = D_{c_1} \otimes D'_{c_2} \otimes T_{\pi(c)}$ with $c \in B_n$ and $v(c) = k$, the following property holds:

$$(D_{d_1} \otimes D'_{d_2} \otimes T_{\pi(d)})(D_{c_1} \otimes D'_{c_2} \otimes T_{\pi(c)}) \equiv D_b \otimes D'_{c_2} \otimes aT_{\pi(c)} \qquad (\mathrm{mod}\ J_{k-2})$$

for some $b \in FD(n, k)$; and the element a is an element in $H_k(z)$ which is independent of $T_{\pi(c)}$.

However, this property follows again by considering the composition of tangles as in the proof of 3.6. In fact, the concatenation $D_{d_1} \cdot T_{\pi(d)} \cdot D'_{d_2} \cdot D_{c_1}$ is an (n, m)-tangle with $m \leqslant k$. If $m < k$, then the composition of T_d and T_c is in J_{k-2}, so we are done. Suppose $m = k$, then this concatenation $D_{d_1} \cdot T_{\pi(d)} \cdot D'_{d_2} \cdot D_{c_1}$ is of the form $D_b \cdot a + a'$, where a is in $H_k(z)$ and a' is a linear combination of (n, j)-tangles with $j < k$. If we concatenate this further with $T_{\pi(c)} \cdot D'_{c_2}$, then we get the desired statement.

Let us now define an involution on BW_n. Note that the word involution means involutory anti-automorphism in this paper.

DEFINITION 3.8. We write $i\colon BW_n \to BW_n$ for the involution defined by $i(g_j) = g_j$ and $i(e_j) = e_j$ for $j = 1, 2, \ldots, n-1$.

Geometrically, i is a natural symmetry given by rotating a tangle through the horizontal axis. Note also that there is an involution i on $H_n(z)$ defined by $i(t_j) = t_j$. Moreover, we have that $i(t_w) = t_{w^{-1}}$. The following lemma is well known (see for example [9], or [15]).

LEMMA 3.9. *If q is an invertible element in a ring Λ, then the Hecke algebra $H_n(q - q^{-1})$ over Λ is a cellular algebra with respect to the involution i.*

The following lemma describes the effect of i on a BW-diagram. Their proofs can be seen from the geometric realization of the involution i and the tangle concatenations.

LEMMA 3.10. (1) *If $T_d = D_{d_1} \otimes D'_{d_2} \otimes T_{\pi(d)}$ with $d = (d_1, d_2, \pi(d)) \in B_n$, then $i(T_d) = D_{d_2} \otimes D'_{d_1} \otimes i(T_{\pi(d)})$.*

(2) *The involution i on $H_k(z)$ and the bilinear form φ_k have the following property: $i\varphi_k(D'_c, D_d) = \varphi_k(D'_d, D_c)$ for all $c, d \in FD(n, k)$.*

Now let us prove the following main result in this section.

THEOREM 3.11. *Suppose that Λ is a commutative noetherian ring which contains R as a subring with the same identity. If q is invertible in Λ, then the Birman–Wenzl algebra $BW(n, \lambda, q - q^{-1}, \delta)$ over the ring Λ is cellular with respect to the involution i.*

Proof. We use the idea in [21] to prove the theorem. For this we need the following lemma in [21].

LEMMA 3.12. *Let A be a Λ-algebra with an involution i. Suppose there is a decomposition*

$$A = \bigoplus_{j=1}^{m} V_j \otimes_{\Lambda} V_j \otimes_{\Lambda} B_j \qquad (\textit{direct sum of } \Lambda\textit{-modules})$$

where V_j is a free Λ-module of finite rank and B_j is a cellular Λ-algebra with respect to an involution σ_j and a cell chain $J_1^{(j)} \subset \cdots \subset J_{s_j}^{(j)} = B_j$ for each j. Define $J_t = \bigoplus_{j=1}^{t} V_j \otimes_\Lambda V_j \otimes_\Lambda B_j$. Assume that the restriction of i on $V_j \otimes_\Lambda V_j \otimes_\Lambda B_j$ is given by $w \otimes v \otimes b \to v \otimes w \otimes \sigma_j(b)$. If for each j there is a bilinear form ϕ_j: $V_j \otimes_\Lambda V_j \to B_j$ such that $\sigma_j(\phi_j(w, v)) = \phi_j(v, w)$ for all $w, v \in V_j$ and that the multiplication of two elements in $V_j \otimes V_j \otimes B_j$ is governed by ϕ_j modulo J_{j-1}; that is, for $x, y, u, v \in V_j$, and $b, c \in B_j$, we have

$$(x \otimes y \otimes b)(u \otimes v \otimes c) = x \otimes v \otimes b\phi_j(y, u)\, c$$

modulo the ideal J_{j-1}, and if $V_j \otimes V_j \otimes J_l^{(j)} + J_{j-1}$ is an ideal in A for all l and j, then A is a cellular algebra.

Since we know that $V(n, k)$ and $V'(n, k)$ have the same rank, we can apply the above lemma to the algebra BW_n. We put $J_{-1} = 0$, $H_0(q - q^{-1}) = \Lambda$, and $B_k = H_k(q - q^{-1})$. Then the Birman–Wenzl algebra has a decomposition

$$BW_n = V(n, n) \otimes_\Lambda V'(n, n) \otimes_\Lambda B_n \oplus V(n, n-2) \otimes_\Lambda V'(n, n-2)$$
$$\otimes_\Lambda B_{n-2} \oplus \cdots \oplus V(n, \varepsilon) \otimes_\Lambda V'(n, \varepsilon) \otimes_\Lambda B_\varepsilon,$$

where ε is zero if n is even, and 1 if n is odd. The above discussion shows that for this decomposition the conditions of Lemma 3.12 are satisfied. Hence, we see that BW_n is a cellular algebra. This finishes the proof of the theorem.

As a consequence, we have the following parametrization of standard modules. Here, for a given n, we denote by I the set $\{(k, \mu) \mid k$ is a non-negative integer with $0 \leqslant k \leqslant n$ and $n - k \in 2\mathbb{Z}$; μ is a partition of $k\}$.

COROLLARY 3.13. *Under the assumption of Theorem* 3.11, *the standard modules over BW_n are*

$$\{\Delta_k(\mu) := V(n, k) \otimes v_k \otimes \Delta(\mu) \mid (k, \mu) \in I\},$$

where v_k is an arbitrary non-zero element in $V'(n, k)$, where $\Delta(\mu)$ is a standard module of the Hecke algebra $H_k(q-q^{-1})$ corresponding to the partition μ of k.

In the following we assume that R is a field and that $\lambda \neq 0$, and that $z = q - q^{-1}$ and δ are elements in R satisfying $\lambda - \lambda^{-1} = z(\delta - 1)$. We denote by $e(q)$ the smallest positive integer m such that $1 + q + q^2 + \cdots + q^{m-1} = 0$. If such a number does not exist, we set $e(q) = \infty$.

COROLLARY 3.14. *Let $BW(n, \lambda, q-q^{-1}, \delta)$ be the Birman–Wenzl algebra over a field R. If $\delta \neq 0$ then the non-isomorphic simple modules are parametrized by the set $\{(k, \mu) \in I \mid \mu$ is an $e(q^{-2})$-restricted partition of $k\}$. In the case of $\delta = 0$, the above assertion is also valid, except $k = 0$.*

Recall that a partition $\mu = (\mu_1, \mu_2, ..., \mu_m)$ is called e-restricted if $\mu_j - \mu_{j+1} < e$, for all j.

Proof. It follows from 3.13 that the simple BW_n-modules are parametrized by $\{(k, \mu) \mid \Phi_{(k,\mu)} \neq 0\}$. Suppose $\delta \neq 0$. If $k \neq 0$, then it follows from 3.6 and an easy computation that $\Phi_{(l,\mu)} \neq 0$ if and only if the corresponding linear form Φ_μ for the cellular algebra $H_k(z)$ is not zero. (Here we use the fact that $\varphi_k(D'_c, D_c) = \delta^{(n-k)/2}$ for some special flat (n, k)-dangle c.) Now it follows from [6], (7.6), or [15] that $\Phi_\mu \neq 0$ if and only if μ is an $e(q^{-2})$-restricted partition of k. If $k = 0$, then $\Phi_{(k,\mu)} \neq 0$ if and only if $\delta \neq 0$. Hence the statement follows.

Now suppose $\delta = 0$. Since the bilinear form $\Phi_{(0,\mu)}$ is zero for $k = 0$, we consider the case $k \neq 0$ and take c and d in $FD(n, k)$ as follows:

$c :=$ 

$d :=$ 

Then we see that $\Phi_{(k,\mu)} \neq 0$ if and only if Φ_μ is not zero, and this means that μ is $e(q^{-2})$-restricted. Hence the result follows.

The following corollary is a consequence of Theorem 3.11 and a result in [13] on cellular algebras:

COROLLARY 3.15. *Under the assumption of Theorem* 3.11, *the determinant of the Cartan matrix C of the Birman–Wenzl algebra is a positive integer, where the entries of C are by definition the multiplicities of composition factors in indecomposable projective modules.*

As a consequence of 3.11 and Theorem 3.8 in [9], we have the following corollary.

COROLLARY 3.16. *Under the assumption of Theorem* 3.11, *the Birman–Wenzl algebra is semisimple if and only if the standard modules are simple and pairwise non-isomorphic.*

Remark. In Theorem 3.11, we choose $z = q - q^{-1}$ for some element $q \in R$ to ensure that the Hecke algebra $H_n(z)$ is cellular. This choice includes the most cases in the literature. But if $H_n(z)$ is not cellular, the Birman–Wenzl algebra may not be cellular. For example, we take $R = \mathbb{Q}$, $\lambda = z = \delta = 1$, and $n = 2$. Then the Birman–Wenzl algebra BW_2 is a commutative algebra of dimension 3, in fact, it is isomorphic with $\mathbb{Q}[t]/(t^3 - 2t + 1)$. Since $t^3 - 2t + 1 = (t-1)(t^2 + t - 1)$, we know that the algebra is isomorphic to a product of $\mathbb{Q}[t]/(t-1)$ and $\mathbb{Q}[t]/(t^2 + t - 1)$. It is clear that this algebra is not cellular by [11], Proposition 3.4.

The non-cellularity in this example is due to splitting. It can be resolved by extension of scalars.

4. QUASI-HEREDITY OF BW_n OVER A FIELD

In this section we assume that R is a field and that λ, q, δ are elements in R with $\lambda \neq 0 \neq q$ such that $\lambda^{-1} - \lambda = z(\delta - 1)$, where $z = q - q^{-1}$. We are interested in the quasi-heredity of the R-algebra BW_n. First of all, let us recall the definition of quasi-hereditary algebras introduced in [4].

DEFINITION 4.1 (Cline, Parshall, and Scott [4]). Let A be a finite dimensional R-algebra. An ideal J in A is called a *heredity ideal* if J is idempotent, $J(\mathrm{rad}(A))\,J = 0$ and J is a projective left (or right) A-module. The algebra A is called *quasi-hereditary* provided there is a finite chain $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ of ideals in A such that J_j/J_{j-1} is a heredity ideal in A/J_{j-1} for all j. Such a chain is then called a heredity chain of the quasi-hereditary algebra A.

Examples of quasi–hereditary algebras are blocks of category $\mathcal{O}$ [1] and Schur algebras [18]. The precise relation to highest weight categories is described in [4]. The following result shows that we can also get quasi-hereditary algebras from Birman–Wenzl algebras.

Recall that, given a non-zero element $q \in R$, we have defined $e(q)$ to be the smallest positive integer m such that $1 + q + q^2 + \cdots + q^{m-1} = 0$ if it exists; otherwise, we set $e(q) = \infty$. It is known that the Hecke algebra $H_n(z)$ is semisimple if and only if $e(q^{-2}) > n$. For this fact, one may again see the paper [6] or [15].

THEOREM 4.2. *The Birman–Wenzl algebra $BW(n, \lambda, q-q^{-1}, \delta)$ is quasi-hereditary if and only if $e(q^{-2})>n$ and either $\delta \neq 0$ or n is odd.*

Proof. Suppose $BW_n = BW(n, \lambda, z, \delta)$ is quasi-hereditary. Note that a self-injective algebra is quasi-hereditary if and ony if it is semisimple. Since $H_n(z)$ is a factor algebra of BW_n by the ideal J_{n-2} which appears in a cell chain, we know by [13] that any cell chain is a heredity chain and that $H_n(z)$ is a hereditary algebra. Thus $H_n(z)$ is semisimple, that is, we have $e(q^{-2})>n$. If $\delta=0$ and n is even, then we shall show that J_0 is nilpotent. This will imply the only if part of the theorem by the result in [13]. In fact, we show that φ_0 is zero. Take $c, d \in FD(n, 0)$, form the dangles D'_c, and D_d and make the concatenation $D'_c \cdot D_d$ of them. What we get is a link. Now we employ the relations (Q1), (Q2), and (Q3) to resolve the overcrossings and undercrossings. At the end we always have a closed cycle with some scalar in R (we can prove this by induction on the number of crossings); this shows that $\varphi_0 = 0$, since a closed cycle is resolved by $\delta = 0$.

Conversely, suppose the conditions in the theorem are satisfied. Then we know that all Hecke algebras $H_k(z)$, $k \leqslant n$ are semisimple. To prove that under our assumption the algebra BW_n is quasi-hereditary, we need to show by [13] that the square of $V(n,k) \otimes V'(n,k) \otimes H_k(z)$ is not zero modulo J_{k-2}. We proceed this just as in [13]. Let $\{C^{\mu}_{S,T} \mid \mu$ is a partition of k and S, T are standard tableaux of type $\mu\}$ be a cellular basis of the semisimple cellular algebra $H_k(z)$. Then there are two elements $C^{\mu}_{S,T}$ and $C^{\mu}_{U,V}$ such that $C^{\mu}_{S,T} C^{\mu}_{U,V}$ is not zero modulo the span of all $C^{\gamma}_{S,T}$ with γ strictly smaller than μ. We take an element $c \in FD(n,k)$ and consider the product of $D_c \otimes D'_c \otimes C^{\mu}_{S,T}$ and $D_c \otimes D'_c \otimes C^{\mu}_{U,V}$. By Lemma 3.6, we have

$$\begin{aligned} x &:= (D_c \otimes D'_c \otimes C^{\mu}_{S,T})(D_c \otimes D'_c \otimes C^{\mu}_{U,V}) \\ &\equiv D_c \otimes D'_c \otimes C^{\mu}_{S,T}\, \delta^{(n-k)/2} C^{\mu}_{U,V} \\ &\equiv D_c \otimes D'_c \otimes (\delta^{(n-k)/2} C^{\mu}_{S,T} C^{\mu}_{U,V}) \qquad (\text{mod } J_{k-2}) \end{aligned}$$

If $\delta \neq 0$, then x is non-zero modulo J_{k-2}. If δ is zero, then n is odd by our assumption. Now we take c and d again as in the proof of 3.14. Then $(D_c \otimes D'_c \otimes C^{\mu}_{S,T})(D_b \otimes D'_b \otimes C^{\mu}_{U,V}) \equiv D_c \otimes D'_c \otimes C^{\mu}_{S,T} C^{\mu}_{U,V} \pmod{J_{k-2}}$. Again we can get a non-zero element modulo J_{k-2}. Hence our claim is proved. This also finishes the proof of the theorem.

As a consequence, we have the following corollary.

COROLLARY 4.3. *If $e(q^{-2})>n$ and either $\delta \neq 0$ or n is odd, then*

(1) *BW_n has finite global dimension.*

(2) *The Cartan determinant of BW_n is* 1.

(3) *The simple BW_n-modules can be parametrized by the set of all pairs (k, μ) with k a non-negative integer such that $(n-k)/2$ is a non-negative integer and μ a partition of k.*

Theorem 4.2 tells us that for $\delta=0$ and $n=2m$ the Birman–Wenzl algebra $BW(n, \lambda, q-q^{-1}, 0)$ itself is not quasi-hereditary, but we can get a maximal quasi-hereditary quotient. The following corollary follows from the proof of Theorem 4.2.

COROLLARY 4.4. *Let n be an even number, and let J_0 be the ideal in $B(n, \lambda, q-q^{-1}, \delta)$ generated by the BW-diagrams without vertical strands. If $e(q^{-2})>n$, then $BW(n, \lambda, q-q^{-1}, 0)/J_0$ is quasi-hereditary.*

Finally, let us remark that one of the interesting cases is that we take R to be the field of complex numbers. If we pick a number $q \neq 0$ and define $\lambda=q^{-m}$, $z=q-q^{-1}$ and $\delta=(q^m-q^{-m}+q-q^{-1})/(q-q^{-1})$, then the Birman–Wenzl algebra is just the algebra $C(q^m, q)$ in the notation of [19]. This case is closely related to the quantum group of so_{m-1}. Hence Theorem 4.2 says that for certain choices of q we might use the properties of quasi-hereditary algebras to investigate the module category of Birman–Wenzl algebras, and further to understand certain indecomposable representations of quantum groups like tilting modules, Ringel duality and so on.

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