A Fixed Point Theorem with Application to an Infectious Disease Model

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1. Summary of Results

Following Krasnosel'skii [3], we say that a map $A: K \rightarrow K$ of the cone $K$ of an ordered Banach space is a compression of the cone if $A(0) = 0$ and if there exist numbers $R > r > 0$ such that

$$Ax \leq x \text{ if } x \in K, \|x\| \leq r, \text{ and } x \neq 0 \quad (1.1)$$

and, for all $\epsilon > 0$,

$$(1 + \epsilon)x \leq Ax \text{ if } x \in K \text{ and } \|x\| \geq R. \quad (1.2)$$

Krasnosel'skii showed that if $A$ is a compression of the cone $K$ and is completely continuous on $K$, then $A$ has at least one nonzero fixed point $x$ in $K$ with $r \leq \|x\| \leq R$ (see [3]). It is not difficult to show that Krasnosel'skii's result remains valid if (1.1) and (1.2) are replaced by the weaker conditions

$$Ax \leq x \text{ if } x \in K \text{ and } \|x\| = r \quad (1.1)'$$

and, for all $\epsilon > 0$,

$$(1 + \epsilon)x \leq Ax \text{ if } x \in K \text{ and } \|x\| = R. \quad (1.2)'$$

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The compression of the cone theorem has been applied to nonlinear operators by several authors (e.g., [1–3]). While condition (1.2)' (or (1.2)) may be easily verified for a large class of nonlinear integral operators, (1.1) (or (1.1)) is a rather stringent condition and is usually not easily verified. In this paper we improve the compression of the cone theorem by replacing (1.1)' with the weaker condition

\[ Ax \leq x \quad \text{if} \quad x \in K(u) \quad \text{and} \quad \| x \| = r, \quad (1.3) \]

where \( u \) is a fixed element of \( K \setminus \{0\} \) and

\[ K(u) = \{ x \in K \mid \alpha x \geq u \text{ for some positive number } \alpha \}. \]

In addition to applying to a larger class of operators, our result has the advantage of usually being easier to apply even when the compression of the cone theorem is also applicable.

To illustrate the advantage in restricting attention to a set of the form \( K(u) \), we consider an operator \( A \) defined on \( C^+([0, 1]) \), the cone of real-valued, non-negative, continuous functions on \([0, 1]\), by

\[ A x(t) = \int_0^1 K(t, s) f(s, x(s)) \, ds. \]

Assume that \( K \) is nonnegative and continuous on \([0, 1] \times [0, 1]\), that \( K(t, s) \geq m > 0 \) for \((t, s) \in I \times I\) for some subinterval \( I = [a, b] \) in \([0, 1]\), and that \( f \) is continuous and satisfies \( f(s, x) \geq x^{1/2} \) for all \( s \) in \( I \) and \( 0 \leq x \leq 1 \). Let \( u \) be a function in \( C^+([0, 1]) \) which is positive on \([0, 1]\). Then it is easy to show that for \( \| x \| \) small enough, \( Ax \leq x \) provided \( x \in K(u) \). In fact, choose a positive number \( r < 1 \) which is less than \( [(b - a) m]^2 \), and suppose \( x \in K(u) \) and \( \| x \| = r \). If \( Ax \) assumes its minimum on \( I \) at \( t \), then

\[ Ax(t) \geq (b - a) (m) \min_{s \in I} \{ [x(s)]^{1/2} \} > \min_{s \in I} \{ [x(s)] \}. \]

Therefore \( Ax \leq x \). Condition (1.1)' cannot be shown to hold under the given conditions.

In Section 3 we apply our result to the nonlinear integral equation

\[ x(t) - Ax(t) = \int_{t-\tau}^t f(s, x(s)) \, ds, \quad (1.4) \]

which can be interpreted as a model for the spread of certain infectious diseases with periodic contact rate that varies seasonally (see Cooke and Kaplan [2]). In Eq. (1.4), \( x(t) \) represents the proportion of infectives in the population at time \( t \), \( f(t, x(t)) \) is the proportion of new infectives per unit time \( (f(t, 0) = 0) \), and \( \tau \) is the length of time an individual remains infectious. Cooke and Kaplan
consider (1.4) for functions $f(t, x)$ which generalize the function $f(t, x) = a(t) x(1 - x)$, where $a(t)$ is the effective contact rate at time $t$. By assuming that the effective contact rate is uniformly larger than $\tau^{-1}$, those authors were able to apply the compression of the cone theorem to prove the existence of a periodic solution of (1.4). (See Nussbaum [4] for a different approach to (1.4).) In Section 3 we apply our fixed point result to show that the occurrence of a disease modeled by (1.4) may be periodic even if the average contact rate is small (even zero) during some seasons. It is required that the average effective contact rate during the remaining seasons be large enough to “offset” the smaller contact rates in the sense that a certain product of average seasonal contact rates be greater than one.

2. Definitions and Fixed Point Theorems

Let $E$ be a real Banach space. A closed, convex set $K$ in $E$ is called a (positive cone) if the following conditions are satisfied:

(i) if $x \in K$, then $\lambda x \in K$ for $\lambda \geq 0$;

(ii) if $x \in K$ and $-x \in K$, then $x = 0$.

A cone $K$ in $E$ induces a partial ordering $\leq$ in $E$ by

$$x \leq y \quad \text{if and only if} \quad y - x \in K.$$  

(We shall write $x \leq y$ if $y - x \in K$.) A Banach space $E$ with a partial ordering $\leq$ induced by a cone $K$ is called an ordered Banach space. By a completely continuous map we mean a continuous function which takes bounded sets into relatively compact sets. In this section we consider completely continuous maps which take some subset $K_{c_{1}}$, $0 < c_{1} < 00$, of a cone $K$ back into $K$, where

$$K_{c} = \{ x \in K \mid \| x \| \leq c \}, \quad 0 < c < \infty, \quad \text{and} \quad K_{\infty} = K.$$  

Our improvement of the compression of the cone theorem is a consequence of the following result.

**Theorem 1.** Let $A : K_{R} \rightarrow K$ be a completely continuous operator with $A(0) = 0$, and suppose the following conditions are satisfied:

for all $c > 0$, \quad \(1 + c\) \(x \leq Ax \quad \text{for} \quad x \in K, \quad \| x \| = R; \quad (2.1)\)

there exists a nullsequence $\{u_{n}\}$ in $K_{R}$ such that if $x_{n} \geq u_{n}$, $n = 1, 2, 3, \ldots$, and $x_{n} \rightarrow 0$, then there exists a subsequence $\{x_{n_{k}}\}$ such that $Ax_{n_{k}} \leq x_{n_{k}}$, $k = 1, 2, \ldots$.  

Then $A$ has a nonzero fixed point in $K_{R}$.  


Proof. Define a sequence $A_n$ of operators on $K$ by

$$A_n x = Ax + 4\beta \left( \frac{R}{2} - \|x\| \right) \frac{u_n}{R} \quad \text{if} \quad x \in K_R,$$

$$= A \left( \frac{Rx}{\|x\|} \right) \quad \text{if} \quad x \in K \setminus K_R,$$

where $\beta(s) = \max\{0, s\}$. Then each $A_n$ maps $K$ continuously into a relatively compact subset of $K$ and hence has a fixed point $x_n'$ by virtue of Schauder’s theorem. If $\|x_n\| > R$, then

$$A(Rx_n/\|x_n\|) = A_n x_n = x_n - \left( \frac{\|x_n\|}{R} \right) \left( \frac{Rx_n}{\|x_n\|} \right) - (1 + \epsilon) \left( \frac{Rx_n}{\|x_n\|} \right),$$

which contradicts (2.1). Hence $\|x_n\| \leq R$ for each $n$ and

$$x_n = A_n x_n = Ax_n + 4\beta(R/2 - \|x_n\|) u_n/R.$$

Since $\{x_n\}$ is a bounded sequence (in $K_R$) and $A$ is completely continuous on $K_R$, there is a subsequence (call it $\{x_{n_k}\}$) such that $\{Ax_{n_k}\}$ converges to some $y \in K$. Then

$$\lim x_n = \lim [Ax_n + 4\beta(R/2 - \|x_n\|) u_n/R] = \lim Ax_n = y,$$

so that $y \in K_R$ and $Ay = \lim Ax_n = y$.

Assume $y = 0$. Then $\{x_n\}$ converges to 0 and for large $n$,

$$x_n = Ax_n + 4\beta(R/2 - \|x_n\|) u_n/R \geq 4\beta(R/4) u_n/R = u_n.$$

Since

$$x_n = Ax_n + 4\beta(R/2 - \|x_n\|) u_n/R \geq Ax_n, \quad n = 1, 2, 3, \ldots,$$

condition (2.2) cannot be satisfied. Thus $y \neq 0$ and the proof is complete. \qed

We are now prepared to prove the result discussed in Section 1.

**Theorem 2.** Let $A: K_R \to K$ be a completely continuous operator with $A(0) = 0$. Suppose there exist a number $r$, $0 < r < R$, and a vector $u \in K \setminus \{0\}$ such that

$$Ax \leq x \quad \text{if} \quad x \in K(u) \quad \text{and} \quad \|x\| = r. \quad (2.3)$$

Suppose further that for each $\epsilon > 0$,

$$(1 + \epsilon) x \leq Ax \quad \text{if} \quad x \in K \quad \text{and} \quad \|x\| = R. \quad (2.4)$$

Then $A$ has a fixed point $x$ in $K$ with $r \leq \|x\| \leq R$. 
Proof. Define $B: K_R \to K$ by

$$
Bx = \begin{cases} 
Ax & \text{if } r \leq \|x\| \leq R, \\
\|x\| r^{-1} A(r \|x\|^{-1} x) & \text{if } 0 < \|x\| < r, \\
o & \text{if } x = 0.
\end{cases}
$$

It is easy to verify that $B$ is completely continuous on $K_R$. Condition (2.1) holds for $B$ since $Bx = Ax$ for $\|x\| = R$. We shall show that $B$ satisfies condition (2.2) of Theorem 1. Theorem 2 will then follow, for suppose $Bx = x$ and $x \neq 0$. If $r \leq \|x\| \leq R$, then $x = Bx = Ax$, and if $0 < \|x\| < r$, then

$$
x = Bx = \|x\| r^{-1} A(r \|x\|^{-1} x),
$$

which implies that $A(y) = y$, where $y = r \|x\|^{-1} x$.

We may assume that $u \in K_R$. Define a sequence $\{u_n\}$ in $K_R$ by $u_n = u/n$, $n = 1, 2, 3, \ldots$, and suppose $x_n$ is a nullsequence with $x_n \geq u_n$. For large $n$, $\|x_n\| < r$, so that $Bx_n = \|x_n\| r^{-1} A(r \|x_n\|^{-1} x_n)$. Now $r \|x_n\|^{-1} x_n \in K(u)$ since $x_n \in K(u)$, and $\|r \|x_n\|^{-1} x_n\| = r$, so that

$$
A(r \|x_n\|^{-1} x_n) \leq (r \|x_n\|^{-1} x_n)
$$

and therefore

$$
Bx_n = \|x_n\| r^{-1} A(r \|x_n\|^{-1} x_n) \leq \|x_n\| r^{-1} r \|x_n\|^{-1} x_n = x_n.
$$

This proves condition (2.2).

3. An Application

In this section we apply Theorem 2 to Eq. (1.4). We assume that $\tau$ and $\omega$ are positive constants, and we make the following assumptions concerning $f$ and $a$:

H1. The function $f(t, x)$ is continuous from $(-\infty, \infty) \times [0, \infty)$ into $[0, \omega)$.

H2. For each $t \in \mathbb{R}$ and $x \geq 0$, $f(t, x) = f(t + \omega, x)$ and $f(t, 0) = 0$.

H3. There exists $R > 0$ such that $f(t, x) \leq R/\tau$ for all $(t, x) \in [0, \omega] \times [0, R]$.

H4. For each $t$, $a(t) = \lim_{x \to 0} f(t, x/x)$, and for each $k \in (0, 1)$ there exists $\epsilon_k > 0$ such that $f(t, x) \geq ka(t) x$, $t \in \mathbb{R}$, $0 \leq x \leq \epsilon_k$.

Assumptions H1 and H2 are precisely as in [2, 4]. Instead of H3, Cooke and Kaplan [2] require that $f$ be bounded above and Nussbaum [4] requires that
\[ \lim_{x \to 0} (f(t, x)/x) = 0. \] Instead of \( H_4 \), both [2] and [4] require that \( a \) be the uniform limit of \( f(t, x)/x \) and that \( a \) be bounded away from zero. These assumptions are more restrictive than \( H_4 \). For example, if \( f(t, x) \geq a(t) x \) for all \( t \in \mathbb{R}, x \geq 0 \), then \( H_4 \) is satisfied and \( a(t) \) may be zero on some intervals.

**Theorem 3.** Suppose \( H_1 - H_4 \) are satisfied and \( N \) is the smallest integer such that \( \omega/N \leq \tau/2 \). For \( j = 0, 1, \ldots, N \), set \( I_j = (((j - 1)/N) \omega, (j/N) \omega] \). If \( \prod_{j=1}^{N} \int_{I_j} a(s) \, ds > 1 \), then Eq. (1.4) has a nonzero solution.

**Proof.** Let \( E \) be the sup-normed Banach space of continuous real-valued functions on \( \mathbb{R} \) which are \( \omega \)-periodic, and let \( K \) be the cone of nonnegative functions in \( E \). Define the operator \( A: K \to K \) by

\[ Ax(t) = \int_{t-\tau}^{t} f(s, x(s)) \, ds. \]

It is easy to see, using the Arzela–Ascoli Theorem, that \( A \) is completely continuous. Note that if \( x \in K_R \), then for \( t \in \mathbb{R} \),

\[ Ax(t) = \int_{t-\tau}^{t} f(s, x(s)) \, ds \leq \int_{t-\tau}^{t} R \tau \, ds = R, \]

so that \( A \) maps \( K_R \) into \( K_R \). Thus \( A \) satisfies condition (2.4) of Theorem 2.

Let \( u(t) \equiv 1 \), and choose \( h < 1 \) such that

\[ k^N \prod_{j=1}^{N} \int_{I_j} a(s) \, ds > 1. \]

Choose \( r < R \) small enough so that

\[ f(t, x) \geq ha(t) x, \quad t \in \mathbb{R}, \ 0 \leq x \leq r. \]

Assume that for some \( x \in K(u) \) with \( \| x \| = r \), \( Ax \leq x \). For \( 1 \leq j \leq N \) and \( t \in I_j \), \( I_{j-1} \) is a subset of \( [t - \tau, t] \). Hence

\[ \int_{I_j} a(t) x(t) \, dt \geq \int_{I_j} a(t) Ax(t) \, dt \]

\[ - \int_{I_j} \left( a(t) \int_{t-\tau}^{t} f(s, x(s)) \, ds \right) \, dt \geq \int_{I_j} a(t) \left( \int_{I_{j-1}} f(s, x(s)) \, ds \right) \, dt \]

\[ \geq k \left( \int_{I_j} a(t) \, dt \right) \left( \int_{I_{j-1}} a(s) x(s) \, ds \right). \]
Therefore
\[
\int_{I_N} a(t) x(t) \, dt \geq k^N \left( \prod_{j=1}^N \int_{I_j} a(s) \, ds \right) \left( \int_{I_0} a(t) x(t) \, dt \right)
\]

\[
= k^N \left( \prod_{j=1}^N \int_{I_j} a(s) \, ds \right) \left( \int_{I_N} a(t) x(t) \, dt \right).
\]

This is a contradiction since \( k^N \prod_{j=1}^N \int_{I_j} a(s) \, ds > 1 \) and \( \int_{I_N} a(t) x(t) \, dt \neq 0 \). Thus \( Ax \leq x \) and condition (2.3) of Theorem 2 is satisfied. It now follows that \( A \) has a nonzero fixed point \( y \in K_R \).

Our result indicates that the disease may recur periodically even if the contact rate is zero over some short time intervals. In fact, the hypotheses of Theorem 3 allow \( f(t, x) \) to be zero for all \( x > 0 \) and some small intervals of the variable \( t \). Hence the operator \( A \) may map functions of any norm to zero, so that the compression of the cone theorem does not apply.

**References**