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The Hyperbolic Derivative in the Poincaré Ball Model of Hyperbolic Geometry

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The generic Möbius transformation of the complex open unit disc induces a binary operation in the disc, called the Möbius addition. Following its introduction, the extension of the Möbius addition to the ball of any real inner product space and the scalar multiplication that it admits are presented, as well as the resulting geodesics of the Poincaré ball model of hyperbolic geometry. The Möbius gyrovector spaces that emerge provide the setting for the Poincaré ball model of hyperbolic geometry in the same way that vector spaces provide the setting for Euclidean geometry. Our summary of the presentation of the Möbius ball gyrovector spaces sets the stage for the goal of this article, which is the introduction of the hyperbolic derivative. Subsequently, the hyperbolic derivative and its application to geodesics uncover novel analogies that hyperbolic geometry shares with Euclidean geometry.

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1. INTRODUCTION

In the 20th century the notions of group and vector space dominated analysis, geometry, and physics to the present days. However, their generalization to gyrogroups and gyrovector spaces, which sprung from the soil of Einstein's special theory of relativity, does not seem to have fired the interest of physicists to the same extent despite their compelling application in hyperbolic geometry and in relativistic physics. Thus, despite the fascination of gyrocommutative gyrogroups (which, following [6], are also known as K-loops) in non-associative algebra [4] for over a decade, it is fair to say that they still await universal acceptance. This is not to say that there have not been valiant attempts to find appropriate uses for them. One can point to their impact on special relativity theory and hyperbolic geometry [9–17]. The theories of gyrogroups and gyrovector spaces provide a new avenue for investigation, leading to a new approach to hyperbolic geometry [12] and, subsequently, to new (as yet to be discovered) physics.

The discovery of the first gyrogroup [7] followed the exposition of the mathematical regularity that the Thomas precession stores [5]. The Thomas precession of relativity physics is a rotation that has no classical counterpart. It has been extended in [9] by abstraction to the so-called *Thomas gyration* which, in turn, suggests the prefix "gyro" that we extensively use to emphasize analogies with classical notions. Thomas gyration is an isometry of hyperbolic geometry that any two points of the geometry generate, enabling novel analogies shared by Euclidean and hyperbolic geometry to be exposed. The novel analogies, in turn, allow the unification of Euclidean and hyperbolic geometry and trigonometry [14, 15].

The aim of this article is (i) to extend the differential operation in vector spaces to a differential operation in gyrovector spaces; and (ii) to study its application to geodesics.

2. GYROGROUPS AND GYROVECTOR SPACES

Gyrogroups are generalized groups that share remarkable analogies with groups [9]. In full analogy with groups

(1) gyrogroups are classified into gyrocommutative gyrogroups and non-gyrocommutative gyrogroups; and

(2) some gyrocommutative gyrogroups admit scalar multiplication, turning them into gyrovector spaces;

(3) gyrovector spaces, in turn, provide the setting for hyperbolic geometry in the same way that vector spaces provide the setting for Euclidean geometry, thus enabling the two geometries to be unified.

In the study of Möbius transformations patterns and interesting concepts emerge from time to time as, for instance, in [2, 13, 16]. A most intriguing motivation for the introduction of the notion of a gyrogroup is, indeed, provided by the Möbius transformation group of the complex open unit disc.

The most general Möbius transformation of the complex open unit disc $\mathbb{D} = \{z : |z| < 1\}$ in the complex z-plane [1, 3],

$$z \mapsto e^{i\theta} \frac{a + z}{1 + \bar{a}z} = e^{i\theta}(a \oplus z) \tag{2.1}$$

induces the Möbius addition $\oplus$ in the disc, allowing the Möbius transformation of the disc to be viewed as a Möbius left *gyrotranslation*

$$z \mapsto a \oplus z = \frac{a + z}{1 + \bar{a}z} \tag{2.2}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $a, z \in \mathbb{D}$, and $\bar{a}$ is the complex conjugate of a. Möbius addition $\oplus$ is neither commutative nor associative. The breakdown of commutativity in Möbius addition is "repaired" by the introduction of gyration, $\mathrm{gyr} : \mathbb{D} \times \mathbb{D} \to \mathrm{Aut}(\mathbb{D}, \oplus)$, given by the equation

$$\mathrm{gyr}[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\bar{b}}{1 + \bar{a}b}, \tag{2.3}$$

where $\mathrm{Aut}(\mathbb{D}, \oplus)$ is the automorphism group of the groupoid $(\mathbb{D}, \oplus)$. We recall that a groupoid $(G, +)$ is a nonempty set G with a binary operation $+$, and an automorphism of a groupoid $(G, +)$ is a bijective self-map of the groupoid G which respects its binary operation $+$. The set of all automorphisms of a groupoid $(G, +)$ forms a group under bijection composition, denoted $\mathrm{Aut}(G, +)$.

The *gyrocommutative law* of Möbius addition $\oplus$ follows from the definition of gyr in (2.3),

$$a \oplus b = \mathrm{gyr}[a, b](b \oplus a). \tag{2.4}$$

Coincidentally, the gyration $\mathrm{gyr}[a, b]$ that repairs the breakdown of the commutative law of $\oplus$ in (2.4), repairs the breakdown of the associative law of $\oplus$ as well, giving rise to the respective *left and right gyroassociative* *laws*

$$
\begin{aligned}
a \oplus (b \oplus z) &= (a \oplus b) \oplus \mathrm{gyr}[a, b]z \\
(a \oplus b) \oplus z &= a \oplus (b \oplus \mathrm{gyr}[b, a]z)
\end{aligned}
\tag{2.5}
$$

for all $a, b, z \in \mathbb{D}$.

Guided by analogies with groups we take key features of the Möbius addition as a model of a gyrogroup, obtaining the following

DEFINITION 2.1 (Gyrogroups). The groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element, 0, called a left identity, satisfying

(G1) $0 \oplus a = a$, Left Identity

for all $a \in G$. There is an element $0 \in G$ satisfying axiom (G1) such that for each a in G there is an element $\ominus a$ in G, called a left inverse of a, satisfying

(G2) $\ominus a \oplus a = 0$, Left Inverse.

Moreover, for any $a, b, z \in G$ there exists a unique element $\mathrm{gyr}[a, b]z \in G$ such that

(G3) $a \oplus (b \oplus z) = (a \oplus b) \oplus \mathrm{gyr}[a, b]z$, Left Gyroassociative Law.

If $\mathrm{gyr}[a, b]$ denotes the map $\mathrm{gyr}[a, b]: G \to G$ given by $z \mapsto \mathrm{gyr}[a, b]z$ then

(G4) $\mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)$, Gyroautomorphism

and $\mathrm{gyr}[a, b]$ is called the Thomas gyration, or the gyroautomorphism of G, generated by $a, b \in G$. Finally, the gyroautomorphism $\mathrm{gyr}[a, b]$ generated by any $a, b \in G$ satisfies

(G5) $\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$, Left Loop Property.

DEFINITION 2.2 (Gyrocommutative Gyrogroups). A gyrogroup $(G, \oplus)$ is gyrocommutative if it satisfies

(G6) $a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$, Gyrocommutative Law.

Grouplike gyrogroup theorems that follow from Definitions 2.1 and 2.2 are presented in [9]. These theorems ensure, for instance, that there exists a unique identity (which is both left and right) and a unique inverse (which is both left and right). Gyrocommutativity, for instance, is equivalent to the validity of the automorphic inverse law, as shown in the following

THEOREM 2.3. *A gyrogroup $(G, \oplus)$ is gyrocommutative if and only if*

(G7) $\ominus(a \oplus b) = \ominus \mathbf{a} \ominus \mathbf{b}$, *Automorphic Inverse Law.*

Furthermore, the left gyroassociative law and the left loop property are accompanied by right counterparts,

(G8) $(a \oplus b) \oplus z = a \oplus (b \oplus \mathrm{gyr}[b, a]z)$, Right Gyroassociative Law

(G9) $\mathrm{gyr}[a, b] = \mathrm{gyr}[a, b \oplus a]$, Right Loop Property

and the left cancellation law is valid,

(G10) $a \oplus (\ominus a \oplus b) = b$, Left Cancellation Law.

3. THE MÖBIUS GYROGROUPS AND GYROVECTOR SPACES

DEFINITION 3.1 (The Möbius Addition). Let $\mathbb{V}$ be a real inner product space, and let $\mathbb{V}_c = \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\| < c\}$ be the open c-ball of $\mathbb{V}$ of radius $c > 0$. Möbius addition $\oplus_M$ in the ball $\mathbb{V}_c$ is a binary operation in $\mathbb{V}_c$ given by the equation [8]

$$\mathbf{u} \oplus_M \mathbf{v} = \frac{\left(1 + (2/c^2)\mathbf{u}\cdot\mathbf{v} + (1/c^2)\|\mathbf{v}\|^2\right)\mathbf{u} + \left(1 - (1/c^2)\|\mathbf{u}\|^2\right)\mathbf{v}}{1 + (2/c^2)\mathbf{u}\cdot\mathbf{v} + (1/c^4)\|\mathbf{u}\|^2\|\mathbf{v}\|^2}, \quad (3.1)$$

where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{V}_c$ inherits from its space $\mathbb{V}$.

To justify calling $\oplus_M$ in Definition 3.1 a Möbius addition we will show in (3.4) below that $\oplus_M$ of (3.1) in two dimensions, when $\mathbb{V} = \mathbb{R}^2$ and $\mathbb{V}_c = \mathbb{R}^2_c$, is equivalent to $\oplus_M$ of (2.2) in the complex unit disc $\mathbb{D}$. For this sake we identify vectors in $\mathbb{R}^2$ with complex numbers in the usual way,

$$\mathbf{u} = (u_1, u_2) = u_1 + iu_2 = u. \quad (3.2)$$

The inner product and the norm in $\mathbb{R}^2$ then become the real numbers

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= \mathrm{Re}(\bar{u}v) = \frac{\bar{u}v + u\bar{v}}{2} \\ \|\mathbf{u}\| &= |u|, \end{aligned} \quad (3.3)$$

where $\bar{u}$ is the complex conjugate of u.

Under the translation (3.3) of elements of the open unit disc $\mathbb{R}^2_{c=1}$ of $\mathbb{R}^2$ into elements of the complex unit disc $\mathbb{D}$, the Möbius addition (3.1) in the open ball $\mathbb{V}_c = \mathbb{R}^2_{c=1}$ of $\mathbb{V} = \mathbb{R}^2$, with $c = 1$ for simplicity, takes the form

$$\begin{aligned} \mathbf{u} \oplus_M \mathbf{v} &= \frac{(1 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2)\mathbf{u} + (1 - \|\mathbf{u}\|^2)\mathbf{v}}{1 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\ &= \frac{(1 + \bar{u}v + u\bar{v} + |v|^2)u + (1 - |u|^2)v}{1 + \bar{u}v + u\bar{v} + |u|^2|v|^2} \\ &= \frac{(1 + u\bar{v})(u + v)}{(1 + \bar{u}v)(1 + u\bar{v})} \\ &= \frac{u + v}{1 + \bar{u}v} \\ &= u \oplus v \end{aligned} \tag{3.4}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2_{c=1}$ and all $u, v \in \mathbb{D}$, as desired. We will now use the notation $\oplus_M = \oplus$.

Since (3.1) forms a most natural extension of the Möbius addition in (2.2), the following theorem is expected [9].

THEOREM 3.2 (Möbius Gyrogroups). *Let $\mathbb{V}_c$ be the c-ball of a real inner product space $\mathbb{V}$, and let $\oplus$ be the Möbius addition* (3.1) *in $\mathbb{V}_c$. Then the groupoid* $(\mathbb{V}_c, \oplus)$ *is a gyrocommutative gyrogroup* (*called a Möbius gyrogroup*).

DEFINITION 3.3 (Möbius Scalar Multiplication). Let $(\mathbb{V}_c, \oplus)$ be a Möbius gyrogroup. The Möbius scalar multiplication $r \otimes \mathbf{v} = \mathbf{v} \otimes r$ is given by the equation

$$\begin{aligned} r \otimes \mathbf{v} &= c\frac{(1 + \|\mathbf{v}\|/c)^r - (1 - \|\mathbf{v}\|/c)^r}{(1 + \|\mathbf{v}\|/c)^r + (1 - \|\mathbf{v}\|/c)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= c\tanh\left(r\tanh^{-1}\frac{\|\mathbf{v}\|}{c}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|}, \end{aligned} \tag{3.5}$$

where $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{V}_c$, $\mathbf{v} \neq \mathbf{0}$; and $r \otimes \mathbf{0} = \mathbf{0}$.

The Möbius addition $\oplus_M = \oplus$ and its associated scalar multiplication $\otimes$ possess the following properties. For all $r, r_1, r_2 \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{V}_c$,

(V1) $1 \otimes \mathbf{v} = \mathbf{v}$

(V2) $(r_1 + r_2) \otimes \mathbf{v} = r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v}$, Scalar Distributive Law

(V3) $(r_1 r_2) \otimes \mathbf{v} = r_1 \otimes (r_2 \otimes \mathbf{v})$, Scalar Associative Law

(V4) $r \otimes (r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v}) = r \otimes (r_1 \otimes \mathbf{v}) \oplus r \otimes (r_2 \otimes \mathbf{v})$, *Monodistributive Law*

(V5) $(|r| \otimes \mathbf{v})/\|r \otimes \mathbf{v}\| = \mathbf{v}/\|\mathbf{v}\|$, Scaling Property

(V6) $\|r \otimes \mathbf{v}\| = |r| \otimes \|\mathbf{v}\|$, Homogeneity Property

(V7) $\|\mathbf{u} \oplus \mathbf{v}\| \leq \|\mathbf{u}\| \oplus \|\mathbf{v}\|$, Gyrotriangle inequality

(V8) $\mathrm{gyr}[\mathbf{a}, \mathbf{b}](r \otimes \mathbf{v}) = r \otimes \mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{v}$, Gyroautomorphism Property.

We may note that, ambiguously, $\oplus$ in (V7) is a binary operation in $\mathbb{V}_c$ and in $\mathbb{R}_c$. The introduction of scalar multiplication satisfying properties (V1)–(V7) suggests the following

DEFINITION 3.4 (Möbius Gyrovector Spaces). Let $(\mathbb{V}_c, \oplus)$ be a Möbius gyrogroup equipped with scalar multiplication $\otimes$. The triple $(\mathbb{V}_c, \oplus, \otimes)$ is called a Möbius gyrovector space.

4. GEODESICS IN MÖBIUS GYROVECTOR SPACES

The unique Euclidean geodesic passing through two given points **a** and **b** of a vector space $\mathbb{R}^n$ can be represented by the expression

$$\mathbf{a} + (-\mathbf{a} + \mathbf{b})t, \tag{4.1}$$

$t \in \mathbb{R}$. It passes through **a** when $t = 0$, and through **b** when $t = 1$. In full analogy, the unique hyperbolic geodesic passing through two given points **a** and **b** of a Möbius gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ can be represented by the expression

$$\mathbf{a} \oplus (\ominus\mathbf{a} \oplus \mathbf{b}) \otimes t, \tag{4.2}$$

$t \in \mathbb{R}$, where $\ominus\mathbf{a} = -\mathbf{a}$. It passes through **a** when $t = 0$, and through **b** when $t = 1$. The latter follows from the left cancellation law in any gyrogroup [9]. The graphs of (4.2) in two dimensions, $\mathbb{V}_c = \mathbb{R}^2_{c=1}$, and in three dimensions, $\mathbb{V}_c = \mathbb{R}^3_{c=1}$, are shown in Figs. 1 and 2. These are circles that intersect the boundary of the ball $\mathbb{V}_c$ orthogonally, and are recognized as the geodesics of the Poincaré ball model of hyperbolic geometry in two and in three dimensions, respectively.

The well known Poincaré distance function d in the ball can be written in terms of Möbius addition as

$$d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} \ominus \mathbf{b}\| \tag{4.3}$$

as explained in [10, 11].

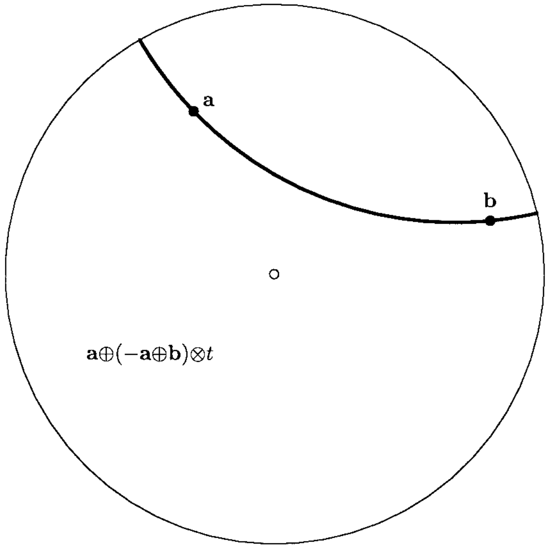

FIG. 1. The unique 2-dimensional Möbius geodesic (4.2) that passes through the two given points **a** and **b**.

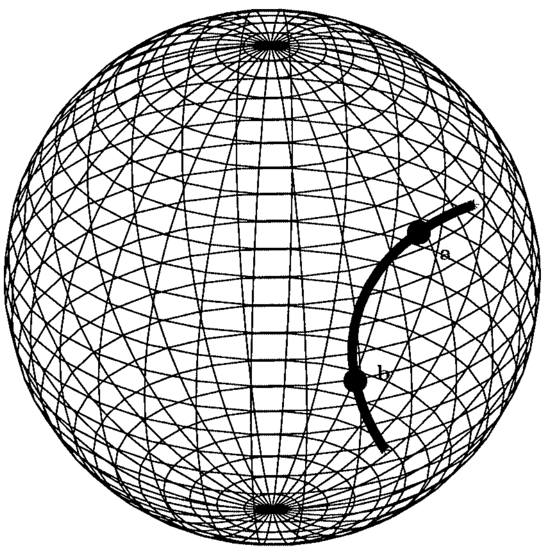

FIG. 2. The unique 3-dimensional Möbius geodesic (4.2) that passes through the two given points **a** and **b**.

To emphasize analogies with Euclidean geometry, the geodesics (4.2) are also called *gyrolines*. The gyrolines (4.2) are geodesics relative to the Poincaré metric (4.3). The Poincaré metric induces a topology relative to which continuity and limits are defined enabling us to define the gyroderivative in a Möbius gyrovector space.

5. GYRODERIVATIVES: THE HYPERBOLIC, GYROVECTOR SPACES DERIVATIVES

Guided by analogies with vector spaces, we define the gyroderivative. The effects of the gyroderivative are then explored by studying its application to parametrized geodesics.

DEFINITION 5.1 (The Gyroderivative). Let $(\mathbb{V}_c, \oplus, \otimes)$ be a Möbius gyrovector space, and let $\mathbf{v}: \mathbb{R} \to \mathbb{V}_c$ be a map from the real line $\mathbb{R}$ into the ball $\mathbb{V}_c$. If the limit

$$\mathbf{v}'(t) = \lim_{h \to 0} \frac{1}{h} \otimes \{\ominus \mathbf{v}(t) \oplus \mathbf{v}(t+h)\} \tag{5.1}$$

exists for any $t \in \mathbb{R}$, we say that the map $\mathbf{v}$ is differentiable on $\mathbb{R}$, and that the gyroderivative (or, hyperbolic derivative) of $\mathbf{v}(t)$ is $\mathbf{v}'(t)$.

The gyroderivative in a Möbius ball gyrovector space $(\mathbb{V}_c, \oplus)$ is closely related to the ordinary derivative in the vector space $\mathbb{V}$ of the ball $\mathbb{V}_c$. Indeed, in the limit of small neighborhood of any point of $\mathbb{V}_c$, hyperbolic geometry reduces to Euclidean geometry. Accordingly, (i) the Möbius addition $\oplus$ in $\mathbb{V}_c$ reduces to the vector addition $+$ in $\mathbb{V}$, and (ii) the Möbius scalar multiplication $\otimes$ reduces to the scalar multiplication in $\mathbb{V}$ near any point of $\mathbb{V}_c$. Hence, the gyroderivative $\mathbf{v}'(t)$ of the map $\mathbf{v}: \mathbb{R} \to \mathbb{V}_c$ given by

$$\mathbf{v}(t) = \mathbf{a} \oplus \mathbf{b} \otimes t \tag{5.2}$$

$t \in \mathbb{R}, \mathbf{a}, \mathbf{b} \in \mathbb{V}_c$ is expected to be a Euclidean vector parallel to the Euclidean tangent line at any point $\mathbf{v}(t_0)$, $t_0 \in \mathbb{R}$, of the geodesic $\mathbf{v}(t)$. We will show in (5.3) and, graphically, in Fig. 3 that this is indeed the case. Despite its close relation to the ordinary derivative in a vector space, the gyroderivative introduces simplicity and elegance when applied to geodesics.

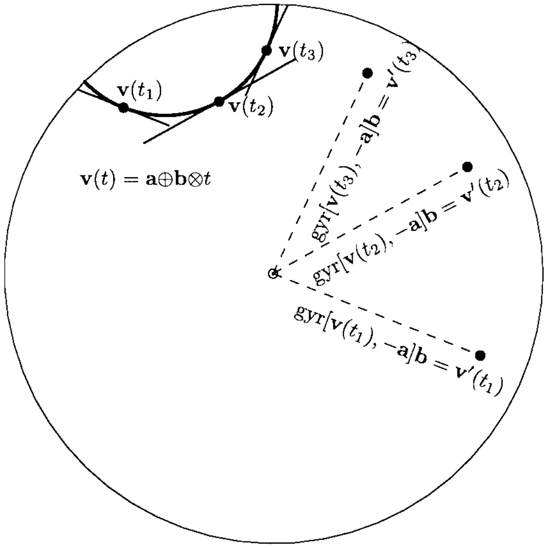

FIG. 3. Euclidean tangent lines at points of Möbius geodesics. $\mathbf{v}(t_n) = \mathbf{a} \oplus \mathbf{b} \otimes t_n$, $t_n \in \mathbb{R}$, $n = 1, 2, 3$, are three points on the geodesic $\mathbf{v}(t) = \mathbf{a} \oplus \mathbf{b} \otimes t$, parametrized by $t \in \mathbb{R}$. The Euclidean tangent line at any point $\mathbf{v}(t)$ of the geodesic is Euclidean parallel to the vector $\mathrm{gyr}[\mathbf{v}(t), -\mathbf{a}]\mathbf{b}$ which, by (5.3), equals $\mathbf{v}'(t)$. Shown are the tangent lines at the three points of the geodesic, $\mathbf{v}(t_1)$, $\mathbf{v}(t_2)$, $\mathbf{v}(t_3)$, and their corresponding Euclidean parallel vectors in the Möbius disc $(\mathbb{R}^2_{c=1}, \oplus, \otimes)$, which is the Poincaré disc model of hyperbolic geometry.

Calculating the gyroderivative of $\mathbf{v}(t)$ in (5.2) and employing gyrogroup manipulations we have the following chain of equations, some of which are numbered for later reference,

$$
\begin{aligned}
\mathbf{v}'(t) &= \lim_{h \to 0} \frac{1}{h} \otimes \{\ominus \mathbf{v}(t) \oplus \mathbf{v}(t+h)\} \\
&= \lim_{h \to 0} \frac{1}{h} \otimes \{\ominus(\mathbf{a} \oplus \mathbf{b} \otimes t) \oplus [\mathbf{a} \oplus \mathbf{b} \otimes (t+h)]\} \\
&\overbrace{=}^{(1)} \lim_{h \to 0} \frac{1}{h} \otimes \{\ominus(\mathbf{a} \oplus \mathbf{b} \otimes t) \oplus [\mathbf{a} \oplus (\mathbf{b} \otimes t \oplus \mathbf{b} \otimes h)]\} \\
&\overbrace{=}^{(2)} \lim_{h \to 0} \frac{1}{h} \otimes \{\ominus(\mathbf{a} \oplus \mathbf{b} \otimes t) \\
&\qquad \oplus [(\mathbf{a} \oplus \mathbf{b} \otimes t) \oplus \mathrm{gyr}[\mathbf{a}, \mathbf{b} \otimes t](\mathbf{b} \otimes h)]\}
\end{aligned}
$$

$$\overbrace{=}^{(3)} \lim_{h\to 0} \frac{1}{h} \otimes \{\mathrm{gyr}[\mathbf{a}, \mathbf{b} \otimes t](\mathbf{b} \otimes h)\}$$

$$\overbrace{=}^{(4)} \lim_{h\to 0} \frac{1}{h} \otimes \{(\mathrm{gyr}[\mathbf{a}, \mathbf{b} \otimes t]\mathbf{b}) \otimes h\}$$

$$\overbrace{=}^{(5)} \lim_{h\to 0} \mathrm{gyr}[\mathbf{a}, \mathbf{b} \otimes t]\mathbf{b}$$

$$= \mathrm{gyr}[\mathbf{a}, \mathbf{b} \otimes t]\mathbf{b}$$

$$\overbrace{=}^{(6)} \mathrm{gyr}[\mathbf{a}, \ominus \mathbf{a} \oplus (\mathbf{a} \oplus \mathbf{b} \otimes t)]\mathbf{b}$$

$$= \mathrm{gyr}[\mathbf{a}, \ominus \mathbf{a} \oplus \mathbf{v}(t)]\mathbf{b}$$

$$\overbrace{=}^{(7)} \mathrm{gyr}[\mathbf{v}(t), \ominus \mathbf{a} \oplus \mathbf{v}(t)]\mathbf{b}$$

$$\overbrace{=}^{(8)} \mathrm{gyr}[\mathbf{v}(t), \ominus \mathbf{a}]\mathbf{b}. \qquad (5.3)$$

The derivation of (5.3) follows.

(1) follows from the scalar distributive law (V2).

(2) follows from the left gyroassociative law (G3).

(3) follows from the left cancellation law (G10).

(4) follows from the gyroautomorphism property (V8).

(5) follows from the scalar associative law (V3).

(6) is verified by applying the left cancellation law (G10).

(7) follows from a left loop and a left cancellation.

(8) follows from a right loop.

In (5.3) we have thus verified the gyrovector space identity

$$\mathbf{v}'(t) = (\mathbf{a} \oplus \mathbf{b} \otimes t)' = \mathrm{gyr}[\mathbf{a} \oplus \mathbf{b} \otimes t, \ominus \mathbf{a}]\mathbf{b} = \mathrm{gyr}[\mathbf{a}, \mathbf{b} \otimes t]\mathbf{b} \qquad (5.4)$$

according to which the gyroderivative of the parametric gyroline

$$\mathbf{v}(t) = \mathbf{a} \oplus \mathbf{b} \otimes t \qquad (5.5)$$

is the gyrovector **b** gyrated by a gyroautomorphism.

The parametric gyroline $\mathbf{v}(t)$ is shown in Fig. 3 where three of its points, $\mathbf{v}(t_1)$, $\mathbf{v}(t_2)$, and $\mathbf{v}(t_3)$ corresponding to the parameter values $t = t_1$, t_2, and t_3 in $\mathbb{R}$, are emphasized. The Euclidean tangent lines of the gyroline at the three selected points $\mathbf{v}(t_1)$, $\mathbf{v}(t_2)$, and $\mathbf{v}(t_3)$ are shown as well as their corresponding Euclidean vectors $\mathrm{gyr}[\mathbf{v}(t_1), \ominus \mathbf{a}]\mathbf{b}$, $\mathrm{gyr}[\mathbf{v}(t_2), \ominus \mathbf{a}]\mathbf{b}$ and $\mathrm{gyr}[\mathbf{v}(t_3), \ominus \mathbf{a}]\mathbf{b}$, to which they are respectively Euclidean parallel.

Hence, the gyroderivative of a gyroline introduces elegance when noticing that the gyroderivative of the parametric gyroline is analogous to the derivative of a parametric line, where the analogy reveals itself in terms of a Thomas gyration.

We now wish to explore another analogy, that gyrolines share with lines, to which the gyroderivative gives rise. Guided by analogies with Euclidean geometry, we define the tangent gyroline of the hyperbolic curve $\mathbf{v}(t)$ in (5.2) at any point $\mathbf{v}(t_0)$, $t_0 \in \mathbb{R}$ being a constant, to be the gyroline

$$\mathbf{y}(t) = \mathbf{v}(t_0) \oplus \mathbf{v}'(t_0) \otimes t, \tag{5.6}$$

$t \in \mathbb{R}$. In Euclidean geometry the tangent line of a line at any of its points is identical with the line itself. We will show that this is the case for gyrolines as well.

Substituting $\mathbf{v}'(t)$ from (5.5) and (5.4) into (5.6) we have the tangent gyroline

$$\mathbf{y}(t) = (\mathbf{a} \oplus \mathbf{b} \otimes t_0) \oplus (\mathrm{gyr}[\mathbf{a} \oplus \mathbf{b} \otimes t_0, \ominus \mathbf{a}]\mathbf{b}) \otimes t \tag{5.7}$$

of the gyroline $\mathbf{v}(t)$, (5.5), at the point t_0. With several gyrovector space algebraic manipulations we simplify it to the point where it is recognized as the gyroline $\mathbf{v}(t)$ itself but with a new parameter. We thus have

$$\begin{aligned}
\mathbf{y}(t) &= (\mathbf{a} \oplus \mathbf{b} \otimes t_0) \oplus (\mathrm{gyr}[\mathbf{a} \oplus \mathbf{b} \otimes t_0, \ominus \mathbf{a}]\mathbf{b}) \otimes t \\
&\overbrace{=}^{(1)} (\mathbf{a} \oplus \mathbf{b} \otimes t_0) \oplus \mathrm{gyr}[\mathbf{a} \oplus \mathbf{b} \otimes t_0, \ominus \mathbf{a}](\mathbf{b} \otimes t) \\
&\overbrace{=}^{(2)} (\mathbf{a} \oplus \mathbf{b} \otimes t_0) \oplus \mathrm{gyr}[\mathbf{a} \oplus \mathbf{b} \otimes t_0, \mathbf{b} \otimes t_0](\mathbf{b} \otimes t) \\
&\overbrace{=}^{(3)} (\mathbf{a} \oplus \mathbf{b} \otimes t_0) \oplus \mathrm{gyr}[\mathbf{a}, \mathbf{b} \otimes t_0](\mathbf{b} \otimes t) \\
&\overbrace{=}^{(4)} \mathbf{a} \oplus (\mathbf{b} \otimes t_0 \oplus \mathbf{b} \otimes t) \\
&\overbrace{=}^{(5)} \mathbf{a} \oplus \mathbf{b} \otimes (t_0 + t)
\end{aligned} \tag{5.8}$$

recovering the gyroline $\mathbf{v}(t)$ parametrized by a new parameter, $t_0 + t$, replacing the old parameter t, (5.5). The derivation of each of the equations in the chain of Eqs. (5.8) follows.

(1) follows from the gyroautomorphism property (V8).

(2) follows from a right loop (G9) followed by a left cancellation (G10).

(3) follows from a left loop (G5).

(4) follows from the left gyroassociative law.

(5) follows from the scalar distributive law (V2).

REFERENCES

1. L. V. Ahlfors, "Conformal Invariants: Topics in Geometric Function Theory," McGraw–Hill, New York, 1973.
2. H. Haruki and T. M. Rassias, A new characteristic of Möbius transformations by use of Apollonius quadrilaterals, *Proc. Amer. Math. Soc.* **126** (1998), 2857–2861.
3. S. G. Krantz, "Complex Analysis: The Geometric Viewpoint," Math. Assoc. of America, Washington, DC, 1990.
4. G. Saad and M. J. Thomsen (Eds.), "Nearrings, Nearfields and K-Loops," Kluwer Academic, Dordrecht, 1997.
5. A. A. Ungar, Thomas rotation and the parametrization of the Lorentz transformation group, *Found. Phys. Lett.* **1** (1988), 57–89.
6. A. A. Ungar, The relativistic noncommutative nonassociative group of velocities and the Thomas rotation, *Resultate Math.* **16** (1989), 168–179.
7. A. A. Ungar, Thomas precession and its associated grouplike structure, *Amer. J. Phys.* **59** (1991), 824–834.
8. A. A. Ungar, Extension of the unit disk gyrogroup into the unit ball of any real inner product space, *J. Math. Anal. Appl.* **202** (1996), 1040–1057.
9. A. A. Ungar, Thomas precession: Its underlying gyrogroup axioms and their use in hyperbolic geometry and relativistic physics, *Found. Phys.* **27** (1997), 881–951.
10. A. A. Ungar, From Pythagoras to Einstein: The hyperbolic Pythagorean theorem, *Found. Phys.* **28** (1998), 1283–1321.
11. A. A. Ungar, The hyperbolic Pythagorean theorem in the Poincaré disc model of hyperbolic geometry, *Amer. Math. Monthly* **106** (1999), 759–763.
12. A. A. Ungar, The bifurcation approach to hyperbolic geometry, *Found. Phys.* **303** (2000), 1253–1281.
13. A. A. Ungar, Gyrovector spaces in the service of hyperbolic geometry, *in* "Mathematical Analysis and Applications" (T. M. Rassias, Ed.), pp. 305–360, Hadronic Press, FL, 2000.
14. A. A. Ungar, Hyperbolic trigonometry and its application in the Poincaré ball model of hyperbolic geometry, *Comput. Math. Appl.*, in press.
15. A. A. Ungar, Hyperbolic trigonometry in the Einstein relativistic velocity model of hyperbolic geometry, *Comput. Math. Appl.* **40** (2000), 313–332.
16. A. A. Ungar, Möbius transformations of the ball, ahlfors' rotation and gyrogroups, *in* "Nonlinear Analysis in Geometry and Topology" (T. M. Rassias, Ed.), pp. 241–287, Hadronic Press, FL, 2000.
17. A. A. Ungar, The relativistic composite-velocity reciprocity principle, *Found. Phys.* **30** (2000), 331–342.