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Monochromatic progressions in random colorings

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ABSTRACT

Let $N^+(k) = 2^{k/2}k^{3/2}f(k)$ and $N^-(k) = 2^{k/2}k^{1/2}g(k)$ where $f(k) \to \infty$ and $g(k) \to 0$ arbitrarily slowly as $k \to \infty$. We show that the probability of a random 2-coloring of $\{1, 2, \ldots, N^+(k)\}$ containing a monochromatic k-term arithmetic progression approaches 1, and the probability of a random 2-coloring of $\{1, 2, \ldots, N^-(k)\}$ containing a monochromatic k-term arithmetic progression approaches 0, as $k \to \infty$. This improves an upper bound due to Brown, who had established an analogous result for $N^+(k) = 2^k \log k f(k)$.

1. Introduction

One of the earliest results in Ramsey theory is the theorem of van der Waerden [5], stating that for any positive integer k, there exists an integer $W(k)$ such that any 2-coloring of $\{1, 2, \ldots, W(k)\}$ yields a monochromatic k-term arithmetic progression. The exact values of $W(k)$ are known only for $k \leqslant 6$. Berlekamp [2] showed that $W(p+1) \geqslant p2^p$ whenever p is prime, and Gowers [4] showed that $W(k)$ is bounded above by a tower of finite height, i.e.,

$$W(k) \leqslant 2^{2^{2^{2^{2^{k+9}}}}}.$$

Since the best known upper and lower bounds on $W(k)$ are far apart, a lot of work has been done on variants of the original problem. A natural question from a probabilistic perspective is to obtain upper bounds on the slowest growing function $N^+(k)$ such that the probability of a 2-coloring of $\{1, 2, \ldots, N^+(k)\}$ containing a monochromatic k-term arithmetic progression (hereafter abbreviated as k-AP) approaches 1 as $k \to \infty$. Similarly, one could seek lower bounds on the fastest growing function $N^-(k)$ such that the probability of a 2-coloring of $\{1, 2, \ldots, N^-(k)\}$ containing a monochromatic k-AP approaches 0 as $k \to \infty$. An upper bound for $N^+(k)$ was established by Brown [3] who showed that $N^+(k) = O(2^k \log k f(k))$ where $f(k) \to \infty$ as $k \to \infty$. We improve this bound to $N^+(k) = O(2^{k/2}k^{3/2}f(k))$, and also show that $N^-(k) = \Omega(2^{k/2}k^{1/2}g(k))$ where $g(k) \to 0$ as $k \to \infty$.

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2. Almost disjoint progressions

A family of sets $\mathcal{F} = \{S_1, S_2, \ldots, S_m\}$ is said to be *almost disjoint* if any two distinct elements of $\mathcal{F}$ have at most one element in common, i.e., if $|S_i \cap S_j| \leqslant 1$ whenever $i \neq j$.

Lemma 2.1. *Let $\mathcal{F}_{k,n}$ be the collection of k-APs contained in $\{1, 2, \ldots, n\}$ with common difference d satisfying $n/k \leqslant d < n/(k-1)$. Then $\mathcal{F}_{k,n}$ is an almost disjoint family. Moreover, $|\mathcal{F}_{k,n}| = n^2(1 + o(1))/2k^3$.*

Proof. Let $n/k \leqslant d < n/(k-1)$, and let $A_1 = \{a, a+d, \ldots, a+(k-1)d\}$ be a k-term arithmetic progression in $\mathcal{F}_{k,n}$. For $0 \leqslant \ell \leqslant k-1$, consider the pairwise disjoint half-open intervals $I_\ell = (\ell n/k, (\ell+1)n/k]$. We claim that $a + \ell d \in I_\ell$ for $0 \leqslant \ell \leqslant k-1$. Clearly, $a + \ell d > \ell d \geqslant \ell n/k$. Moreover, $a + \ell d = a + (k-1)d - (k-\ell-1)d \leqslant n - (k-\ell-1)n/k = (\ell+1)n/k$. In particular, $a \leqslant n/k \leqslant d$.

Now suppose that $A_2 = \{a', a' + d', \ldots, a' + (k-1)d'\} \in \mathcal{F}_{k,n}$ with $|A_1 \cap A_2| \geqslant 2$. Let $a + \ell_1 d = a' + \ell_1' d'$ and $a + \ell_2 d = a' + \ell_2' d'$. Since the intervals I_ℓ are pairwise disjoint, it follows that $\ell_1 = \ell_1'$ and $\ell_2 = \ell_2'$. But then we have $a = a'$ and $d = d'$, so that $A_1 = A_2$. Thus $\mathcal{F}_{k,n}$ is an almost disjoint family.

Finally,

$$|\mathcal{F}_{k,n}| = \sum_{\frac{n}{k} \leqslant d < \frac{n}{k-1}} \bigl(n - d(k-1)\bigr) = \frac{n^2(1+o(1))}{2k^3},$$

since there are $n - d(k-1)$ k-term arithmetic progressions of common difference d completely contained in $\{1, 2, \ldots, n\}$. □

For each integer $k \geqslant 3$, let c_k denote the asymptotic constant such that the size of the largest family of almost disjoint k-term arithmetic progressions contained in $[1, n]$ is $c_k n^2/(2k-2)$. It follows from the above lemma that $c_k \geqslant 1/k^2$. Perhaps there is an absolute constant λ such that $c_k \leqslant \lambda/k^2$. Ardal, Brown and Pleasants [1] have shown that $0.476 \leqslant c_3 \leqslant 0.485$.

3. Monochromaticity: Almost surely and almost never

Theorem 3.1. *Let $N^+(k) = 2^{k/2}k^{3/2}f(k)$ where $f(k) \to \infty$ arbitrarily slowly as $k \to \infty$. Then the probability that a 2-coloring of $\{1, 2, \ldots, N^+(k)\}$ chosen randomly and uniformly contains a monochromatic k-term arithmetic progression approaches 1 as $k \to \infty$.*

Proof. Our approach will be similar to that of Brown[3], but rather than work with a family of combinatorial lines in a suitably chosen hypercube, which is an almost disjoint family of size $O(n)$, we work with k-APs of large common difference, which is an almost disjoint family of size $\Omega(n^2/k^3)$, as shown in the previous section.

Let $n = N^+(k) = 2^{k/2}k^{3/2}f(k)$ and $q = \lfloor (f(k))^{4/3} \rfloor$. Let $s = s(k)$ satisfy $n = qs + r$, $0 \leqslant r < s$. We divide the interval $[1, n]$ into q blocks $B_1, B_2, \ldots, B_q$ of length s, and possibly one residual block B_{q+1} of length r. Let $\mathcal{F}_1 = \mathcal{F}_{k,s}$ consist of all k-APs in $B_1 = [1, s]$ with common difference d satisfying $s/k \leqslant d < s/(k-1)$. By Lemma 2.1, the elements of $\mathcal{F}_1$ are almost disjoint, and $s^2/4k^3 \leqslant |\mathcal{F}_1| \leqslant s^2/k^3$ for large k.

For each arithmetic progression $P \in \mathcal{F}_1$, let C_P denote the set of 2-colorings of B_1 in which P is monochromatic. Then $|C_P| = 2^{s-k+1}$. Also, $|C_P \cap C_Q| = 2^{s-2k+2}$, since $|P \cap Q| \in \{0, 1\}$. By Bonferroni's inequality,

$$\left| \bigcup_{P \in \mathcal{F}_1} C_P \right| \geqslant \sum_{P \in \mathcal{F}_1} |C_P| - \sum_{\substack{P, Q \in \mathcal{F}_1 \\ P \neq Q}} |C_P \cap C_Q| = |\mathcal{F}_1| 2^{s-k+1} - \binom{|\mathcal{F}_1|}{2} 2^{s-2k+2}.$$

Since

$$\frac{|\mathcal{F}_1|}{2^k} \leqslant \frac{n^2}{q^2 2^k k^3} \to 0 \quad \text{as } k \to \infty,$$

it follows that

$$\left| \bigcup_{P \in \mathcal{F}_1} C_P \right| > |\mathcal{F}_1| 2^{s-k} \geqslant \frac{2^s s^2}{2^{k+2} k^3}.$$

Similarly, we can consider the blocks $B_2, B_3, \ldots, B_q$ and the corresponding families $\mathcal{F}_2, \mathcal{F}_3, \ldots, \mathcal{F}_q$. Let p_0 be the probability that no arithmetic progression from any of the $\mathcal{F}_i$ is monochromatic under a 2-coloring chosen randomly and uniformly. Then

$$p_0 < \left(1 - \frac{s^2}{2^{k+2} k^3}\right)^q < e^{-s^2 q / 2^{k+2} k^3}.$$

Since

$$\frac{s^2 q}{2^{k+2} k^3} = \Theta\left(\frac{n^2}{2^k k^3 q}\right) \to \infty \quad \text{as } k \to \infty,$$

it follows that p_0 approaches 0 for large k. Thus the probability that some arithmetic progression is monochromatic approaches 1 as $k \to \infty$. □

Theorem 3.2. *Let $N^-(k) = 2^{k/2} k^{1/2} g(k)$ where $g(k) \to 0$ arbitrarily slowly as $k \to \infty$. Then, the probability that a 2-coloring of $\{1, 2, \ldots, N^-(k)\}$ chosen randomly and uniformly contains a monochromatic k-AP approaches 0 as $k \to \infty$.*

Proof. Let $n = N^-(k)$, and let E be the expected number of monochromatic k-APs in a 2-coloring of $\{1, 2, \ldots, n\}$ chosen randomly and uniformly. Note that there are $n^2(1 + o(1))/(2k - 2)$ k-APs contained in $[1, n]$ and each of these is monochromatic with probability 2^{1-k}. By linearity of expectation, $E < k[g(k)]^2/(k-2)$. For $r \geqslant 0$, let p_r be the probability that there are exactly r monochromatic k-APs in a random 2-coloring. Then $E = p_1 + 2p_2 + 3p_3 + \cdots > 1 - p_0$, so that $p_0 > (k - 2 - k[g(k)]^2)/(k-2)$. Thus, the probability that some arithmetic progression is monochromatic approaches 0 as $k \to \infty$. □

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