Journal of Pure and Applied Algebra 6 (1975) 313–344

ADJOINT MACHINES, STATE-BEHAVIOR MACHINES, AND DUALITY*

Michael A. ARBIB
Computer and Information Science, University of Massachusetts, Amherst, Ma. 01002, USA

Ernest G. MANES
Mathematics, University of Massachusetts, Amherst, Ma. 01002, USA

Communicated by M. Barr
Received 8 January 1973
Revised 5 November 1974

A process $X: \mathcal{X} \to \mathcal{X}$ is *output* if $\mathrm{Dyn}(X) \to \mathcal{X}$ has a right adjoint; *state-behavior* if $\mathrm{Dyn}(X) \to X$ has both left and right adjoints; and *adjoint* if X has a right adjoint and $\mathcal{X}$ has countable coproducts. Output processes provide the proper setting for a general theory of state observability. We give a minimal realization theory using image factorization of a total response map. We give an adjointness theory for state-behavior machines and a duality theory for adjoint machines which clarifies classical linear system duality and yields an improved duality for nondeterministic automata. Adjoint machines (machines with adjoint input processes) provide the first integration of classical sequential machines (the only state-behavior machines in the category, Set, of sets), metric machines, topological machines, linear systems, nondeterministic automata and Boolean machines. There exist state-behavior machines which are not adjoint (but not in Set).

1. Introduction

We first recall the notation of *reachability*, *observability*, and *realization* [2,11] for sequential machines in a form which will motivate the general theory of this paper. We fix upon an input set X_0 and an output set Y in the following discussion, but will allow different machines to have different state sets Q.

1.1. Definition. *sequential machine* is a sextuple

$$M = (X_0, Q, \delta, q_0, Y, \beta)$$

* The research reported in this paper was supported in part by the National Science Foundation under Grant No. GJ 35759.

where $\delta: Q \times X_0 \to Q$ is the *dynamics* of M;

$q_0 \in Q$ is the *initial state* of M; and

$\beta: Q \to Y$ is the *output map* of M.

From the dynamics and initial state of M we may determine its reachability map (X_0^* is the free monoid on X_0, with empty word Λ):

1.2. Definition. The *reachability map* of (X_0, Q, δ, q_0) is the map

$$r: X_0^* \to Q: w \mapsto (q_0, w)\delta^*$$

[where δ^* is the extension of δ to X_0^*: $(q, \Lambda)\delta^* = q$; $(q, wx)\delta^* = ((q, w)\delta^*, x)\delta$] so that wr is the state reached from the initial state by application of the input string w. We say (X_0, Q, δ, q_0) is *reachable* if r is *onto*; i.e. if every state of Q is reachable from q_0.

From the dynamics and output map of M we may determine its observability map (for sets A and B, B^A is the set of maps from A to B):

1.3. Definition. The *observability map* of $(X_0, Q, \delta, Y, \beta)$ is the map

$$\sigma: Q \to Y^{X_0^*}: q \mapsto {}_qM$$

which assigns to each state q the *response* (or *behavior*) *of M started in q*

$${}_qM: X_0^* \to Y: w \mapsto [(q, w)\delta^*]\beta$$

which tells us, for each input string w, the output that will be emitted by M were it to start in state q and read in w. We say $(X_0, Q, \delta, Y, \beta)$ is *observable* if σ is *one-to-one*; i.e. if distinct states have distinct responses.

To speak of realizations we need all of M:

1.4. Definition. The *total response* of M is the map

$$f^{\blacktriangle} = r \cdot \sigma: X_0^* \to Y^{X_0^*}: w \mapsto {}_{wr}M$$

which assigns to each string w the response of the state $(w)r$ to which w sends M from its initial state q_0. Conversely, we say that M is a *realization* of a given $X_0^* \to Y^{X_0^*}$ if it is the total response of M.

A basic problem of sequential machine theory is to find a realization of a given map which is in some sense *minimal*, and the basic result is the following (we do not give the proof here, since it will be a special case of the general theory of Section 3):

1.5. Theorem. *A map* $f^{\blacktriangle}: X_0^* \to Y^{X_0^*}$ *has a realization iff*

$$(w')[(x)f^{\blacktriangle}] = (w'x)[(w)f^{\blacktriangle}]$$

for all w, w' *in* X_0^* *and* x *in* X_0.

To obtain a minimal realization of $f^{\blacktriangle}$, we take the state-space $Q_f = (X_0^*)f^{\blacktriangle} \subset Y^{X_0^*}$; the dynamics

$$\delta_f : Q_f \times X_0 \to Q_f : ((w)f^{\blacktriangle}, x) \mapsto (wx)f^{\blacktriangle} ;$$

the initial state $(\Lambda)f^{\blacktriangle}$; and the output map

$$\beta_f : Q_f \to Y : (w)f^{\blacktriangle} \to (\Lambda)[(w)f^{\blacktriangle}] .$$

In our introductory paper "Machines in a Category" [3] ‡, we showed that, to build a truly general theory of automata, the input structure of a machine should not be regarded as a *set* of applicable inputs, but rather as a *process* which transforms the state-space Q into a new object QX on which the dynamics can act:

1.6. Definition. Given a functor $X: \mathcal{K} \to \mathcal{K}$, Dyn($X$) denotes the category of *X-dynamics* whose objects are pairs (Q, δ), where Q is a $\mathcal{K}$-object and $\delta : QX \to Q$ is a $\mathcal{K}$-morphism; while *dinamorphisms* g: $(Q, \delta) \to (Q', \delta')$ are $\mathcal{K}$-morphisms $g: Q \to Q'$ for which the following diagram commutes

$$\begin{tikzcd} QX \arrow[r, "\delta"] \arrow[d, "gX"'] & Q \arrow[d, "g"] \\ Q'X \arrow[r, "\delta'"] & Q' \end{tikzcd}$$

Let **Set** = ⟨Sets and Maps⟩, and let $X = -\times X_0$: **Set** → **Set** be the functor $Q \mapsto Q \times X_0; f \mapsto f \times X_0$ where

$$f \times X_0 : Q \times X_0 \to Q' \times X_0 : (q, x) \mapsto (qf, x) .$$

‡ Our basic theory has received a textbook exposition in Chapter 9 of L.S. Bobrow and M.A. Arbib, *Discrete Mathematics: Applied Algebra for Computer and Information Science* (W.B. Saunders, 1974). We have provided a primer on category theory in our *Arrows, Structures, Functors: The Categorical Imperative* (Academic Press, 1975). For further background on any concepts of category theory used in this paper, we recommend S. Mac Lane, *Categories for the Working Mathematician* (Springer-Verlag, 1971) and H. Herrlich and G.E. Strecker, *Category Theory: an Introduction* (Allyn and Bacon, 1973). The proofs herein are unusually complete in deference to our colleagues in computation and control who are not yet fluent with category theory.

Then an X-dynamics is just a map $\delta : Q \times X_0 \to Q$, the next-state function of a sequential machine.

We then showed that if X is to allow the construction of reachability maps it must be an input process in the following sense:

1.7. Definition. X is an *input process* if the forgetful functor $\mathrm{Dyn}(X) \to \mathcal{K} : (Q, \delta) \mapsto Q$ has a left adjoint; i.e. if for each $Q \in \mathcal{K}$ there exists a free dynamics $\mu_0 : (QX^{@})X \to QX^{@}$ with a $\mathcal{K}$-morphism $\eta : Q \to QX^{@}$ such that given any X-dynamics (Q', δ') and any $\mathcal{K}$-morphism $f : Q \to Q'$, there exists a unique dynamorphism $\psi : (QX^{@}, \mu_0) \to (Q', \delta')$ such that $\eta \cdot \psi = f$:

$$\begin{tikzcd} Q \arrow[r, "\eta"] \arrow[dr, "f"'] & QX^{@} \arrow[d, dashed, "\psi"] \\ & Q' \end{tikzcd} \qquad \begin{tikzcd} (QX^{@})X \arrow[r, "\mu_0"] \arrow[d, "\psi X"'] & QX^{@} \arrow[d, "\psi"] \\ Q'X \arrow[r, "\delta'"] & Q' \end{tikzcd} .$$

As is proved in [8], if X is an input process then X generates a free triple and the converse is often true.

In [3], we viewed an initial state q_0 of a sequential machine as a map from a one-element set I to Q to suggest the more general notion of a $\mathcal{K}$-morphism $\tau : I \to Q$ for some *initial state object* I in $\mathcal{K}$. We were then able to generalize Definition 1.2 as follows:

1.8. Definition. Let X be an input process, $\delta : QX \to X$ an X dynamics, and $\tau : I \to Q$ an initial state. Then the *reachability map* of (X, Q, δ, I, τ)

$$r : IX^{@} \to Q$$

is the unique dynamorphism $r : (IX^{@}, \mu_0) \to (Q, \delta)$ such that $\eta \cdot r = \tau$:

$$\begin{tikzcd} I \arrow[r, "\eta"] \arrow[dr, "\tau"'] & IX^{@} \arrow[d, dashed, "r"] \\ & Q \end{tikzcd} .$$

The reader should check that when $X = - \times X_0$, and I has but one element 1, then $IX^{@} \cong X_0^*$, and when (1) $\tau = q_0$, the r's defined by 1.2 and 1.7 are indeed the same.

Using the standard category theory notion of a right adjoint, we have the following variation on the theme of 1.7:

1.9. Definition. A functor $X : \mathcal{K} \to \mathcal{K}$ is an *output process* if the forgetful functor

$\mathrm{Dyn}(X) \to \mathcal{X} : (Q, \delta) \mapsto Q$ has a right adjoint; i.e. if for each $Q \in \mathcal{X}$ there exists a *cofree* dynamics $L : (QX_@)\, X \to QX_@$ with a $\mathcal{X}$-morphism $\Lambda : QX_@ \to Q$ such that given any X-dynamics (Q', δ') and any $\mathcal{X}$-morphism $g : Q' \to Q$ there exists a unique dynamorphism $\phi : (Q', \delta') \to (QX_@, L)$ such that $\phi \cdot \Lambda = g$:

$$\begin{tikzcd} Q & QX_@ \arrow[l, "\Lambda"'] \\ & Q' \arrow[u, dashed, "\phi"'] \arrow[ul, "g"] \end{tikzcd} \qquad \begin{tikzcd} (QX_@)X \arrow[r, "L"] & QX_@ \\ Q'X \arrow[u, "\phi X"] \arrow[r, "\delta'"] & Q' \arrow[u, "\phi"'] \end{tikzcd} .$$

Let us check that $X = -\times X_0$ is indeed an output process:

1.10. Example. Given a set Q we form $Q^{X_0^*}$ as our candidate for $QX_@$ when $X = -\times X_0 : \mathrm{Set} \to \mathrm{Set}$. The dynamics on $QX_@$ is the map $L : Q^{X_0^*} \times X_0 \to Q^{X_0^*}$ defined by $(f, x)\, L = ({}_xL)f : w \mapsto (xw)f$ (${}_xL : X_0^* \to X_0^*$ is the left-translation by x which maps w to xw). Set Λ to be the map $\Lambda : Q^{X_0^*} \to Q$ which evaluates on the empty string Λ. Consider another X-dynamics (Q', δ') and a function $g : Q' \to Q$. Hopefully, the diagrams:

$$\begin{tikzcd} Q & Q^{X_0^*} \arrow[l, "\Lambda"'] \\ & Q' \arrow[u, dashed, "\phi"'] \arrow[ul, "g"] \end{tikzcd} \qquad \begin{tikzcd} Q^{X_0^*} \times X_0 \arrow[r, "L"] & Q^{X_0^*} \\ Q' \times X_0 \arrow[u, dashed, "\phi \times \mathrm{id}"] \arrow[r, "\delta'"] & Q' \arrow[u, dashed, "\phi"'] \end{tikzcd}$$

define a unique map ϕ. This is in fact so, since these tell us that $(\Lambda)[q\phi] = (q)g$ while $(wx)[q\phi] = (w)\, {}_xL\, [q\phi] = (w)[(q, x)\delta']\, \phi$ so that $(q)\, \phi = {}_{(q)g}M'$ on recalling the inductive definition of the response function in 1.2 and 1.3.

Notice in particular that when g is the identity map of Q then $\phi : Q \to Q^{X_0^*}$ is what is normally called the *state-behavior map b* which sends each state q to $(q, \cdot)\delta^*$ the state-response function corresponding to starting in state q. Again, when $Q = Y$ and g is the output map of a machine M with dynamics δ, then $\phi : Q \to Y^{X_0^*}$ is the *observability map* which sends each state q to its behavior ${}_qM$, the response function corresponding to state q. The important point about this derivation of these functions is that it clearly reveals both $b : q \mapsto (q, \cdot)\delta^*$ and $\sigma : q \mapsto {}_qM$ as dynamorphisms of δ into the cofree dynamics on Q and on Y, respectively.

1.11. Definition. Let X be an output process, $\delta : QX \to X$ an X-dynamics and $\beta : Q \to Y$ a $\mathcal{X}$-morphism (the output map). Then $YX_@$ is the *response object of* $M = (X, Q, \delta, Y, \beta)$. The diagrams

$$\begin{tikzcd} Q & QX_@ \arrow[l, "\Lambda"'] \\ & Q \arrow[u, dashed, "b"'] \arrow[ul, "\mathrm{id}_Q"] \end{tikzcd} \qquad \begin{tikzcd} Y & YX_@ \arrow[l, "\Lambda"'] \\ & Q \arrow[u, dashed, "\sigma"'] \arrow[ul, "\beta"] \end{tikzcd}$$

define X-dynamorphisms $b:(Q,\delta)\longrightarrow(QX_@,L)$ and $\sigma:(Q,\delta)\longrightarrow(YX_@,L)$. b is the *state behavior map of* M and σ is the *observability map* of M.

Mimicking 1.4, we must clearly have that X is both an input and output process ‡ in order for a machine $M=(X,Q,\delta,I,\tau,Y,\beta)$ to have a total response:

1.12. Definition. A functor $X:\mathcal{K}\to\mathcal{K}$ is a *state-behavior process* if it is both an input process and an output process.

1.13. Definition. Let $M=(X,Q,\delta,I,\tau,Y,\beta)$ be a machine with X a state-behavior process. Then the total response of M is the dynamorphism

$$f^{\blacktriangle}=r\cdot\sigma:IX^{@}\to YX_@$$

where r is the reachability map, and σ is the observability map, of M. Conversely, we say that M is a *realization* of a given dynamorphism $IX^{@}\to YX_@$ if it is the total response of M.

2. Adjoint processes

The free dynamics $X^{@}$ was invented to play the role of the classical X_0^*. We now study conditions more general than the classical in which we can write "$X^{@}=X^*$"; but first we must explain what we mean by X^*. In the truly classical case of $X=-\times X_0:\mathbf{Set}\to\mathbf{Set}$, $QX^{@}=Q\times X_0^*$. Now regard $Q\times X_0^*$ as the disjoint union

$$Q\times X_0^*=\coprod_{n\geqslant 0}(Q\times X_0^n)$$

indexed by all $n\geqslant 0$, of the sets $Q\times X_0^n$. Note that $Q\times X_0^n$ is QX^n where, for any process $X:\mathcal{K}\longrightarrow\mathcal{K}$, X^n is the n-fold composition. Hence define X^* to be the (pointwise) coproduct

$$X^*=\coprod_{n\geqslant 0}X^n\,.$$

‡ A different approach to a general theory (using hyperdoctrines), which is also based on the requirement that the forgetful functor has left and right adjoints, has been given by Bainbridge [7]. However, he works directly with $X^{@}$ rather than an underlying X, and thus cannot "run" his machines.

Analogously to the situations studied by Ehrig [9] and Goguen [10], we have:

2.1. Theorem. *Let $\mathcal{K}$ have, and let $X: \mathcal{K} \to \mathcal{K}$ preserve, countable coproducts. Then X is an input process and $X^{@} = X^{*}$.*

Proof. If $IX^{@} = IX^{*}$ we must define $\mu_0 : IX^{*}X \to IX^{*}$ and $\eta : I \to IX^{*}$ as in 1.7. Since X preserves the coproduct $IX^{n} \xrightarrow{\mathrm{in}_n} IX^{*}$, we have that

$$IX^{n+1} = IX^{n}X \xrightarrow{\mathrm{in}_n X} IX^{*}X$$

is also a coproduct. Thus we may define μ_0 by the obvious rule

$$\begin{tikzcd}
IX^{*}X \arrow[r, dashed, "\mu_0"] & IX^{*} \\
IX^{n+1} \arrow[u, "\mathrm{in}_n X"] \arrow[ur, "\mathrm{in}_{n+1}"'] &
\end{tikzcd}$$

which certainly reduces to the familiar story in case $X = - \times X_0 : \mathbf{Set} \to \mathbf{Set}$. We define $\eta : I \to IX^{*}$ to be simply in_0. Let us check that this works, i.e. that the diagrams

$$\begin{tikzcd}
I \arrow[r, "\mathrm{in}_0"] \arrow[dr, "f"'] & IX^{*} \arrow[d, "\psi"] \\
& Q
\end{tikzcd}
\qquad
\begin{tikzcd}
& IX^{n+1} \arrow[dl, "\mathrm{in}_n X"'] \arrow[dr, "\mathrm{in}_{n+1}"] & \\
IX^{*}X \arrow[rr, "\mu_0"] \arrow[d, "\psi X"'] & & IX^{*} \arrow[d, "\psi"] \\
QX \arrow[rr, "\delta"] & & Q
\end{tikzcd}$$

define a unique $\psi : IX^{*} \to Q$. But the left-hand diagram says

$$\mathrm{in}_0 \cdot \psi = f$$

while the right-hand diagram asserts that

$$\mathrm{in}_{n+1} \cdot \psi = \mathrm{in}_n X \cdot \psi X \cdot \delta = (\mathrm{in}_n \psi) X \cdot \delta \,, \quad n \geqslant 0$$

and these equations define the unique ψ which satisfies the diagrams. □

2.2. Corollary. *If $\mathcal{K}$ has countable coproducts and $X: \mathcal{K} \to \mathcal{K}$ has a right adjoint, then X is an input process and $X^{@} = X^{*}$. We say such an X is an adjoint process.*

Proof. We simply appeal to the standard category theory result that a functor with a right adjoint preserves *all* coproducts. □

Let $\mathcal{K}$ be category and let X: $\mathcal{K}\to\mathcal{K}$ have a right adjoint $X^{\bullet}$, so that we have the bijection

$$\frac{AX \xrightarrow{f} B}{A \xrightarrow{f^{\bullet}} BX^{\bullet}} .$$

We then have the useful

2.3. Transposition principle. Given maps $f: A_1 \to A, g: AX \to B, h: B \to B_1$, then

$$\frac{A_1X \xrightarrow{fX} AX \xrightarrow{g} B \xrightarrow{h} B_1}{A_1 \xrightarrow[f]{} A \xrightarrow[g^{\bullet}]{} BX^{\bullet} \xrightarrow[hX^{\bullet}]{} B_1X^{\bullet}}$$

that is

$$k^{\bullet} = f \cdot g^{\bullet} \cdot hX^{\bullet} \quad \text{where} \quad k = fX \cdot g \cdot h .$$

Proof. This is immediate from the definitions as is seen by redrawing the diagram

$$\begin{tikzcd}
B_1 & B_1X^{\bullet}X \arrow[l, "\epsilon"'] \\
 & A_1X \arrow[u, "k^{\bullet}X"'] \arrow[ul, "k"]
\end{tikzcd}$$

in the form

$$\begin{tikzcd}
A_1X \arrow[r, "fX"] & AX \arrow[r, "g^{\bullet}X"] \arrow[dr, "g"'] & BX^{\bullet}X \arrow[r, "hX^{\bullet}X"] \arrow[d, "\epsilon"] & B_1X^{\bullet}X \arrow[d, "\epsilon"] \\
 & & B \arrow[r, "h"'] & B_1
\end{tikzcd}$$

where the triangle is the definition of $g^{\bullet}$ and the square (which says "ϵ is natural") is the definition of the way $X^{\bullet}$ is a functor. □

As is usual in category theory, we write $\mathcal{K}^{\mathrm{op}}$ for the opposite category to $\mathcal{K}$, and we write $f: A \prec B$ for the morphism in $\mathcal{K}^{\mathrm{op}}$ (A, B) corresponding to the $\mathcal{K}$ -

morphism $f: B \to A$. Any functor $U: \mathcal{K} \to \mathcal{K}$ defines a functor $U^{\mathrm{op}}: \mathcal{K}^{\mathrm{op}} \to \mathcal{K}^{\mathrm{op}}$ by the rule that $(f: BU \prec AU)U^{\mathrm{op}}$ in $\mathcal{K}^{\mathrm{op}}$ corresponds to $fU: AU \to BU$ in $\mathcal{K}$. Our next theorem, whose proof is clear and whose implications we shall develop in Section 4, then captures the essence of the duality principles of automata and system theory.

2.4. Theorem. *Let $\mathcal{K}$ be a category and let $X: \mathcal{K} \to \mathcal{K}$ be a process in $\mathcal{K}$ which has a right adjoint X^*. Let us also write $X^*: \mathcal{K}^{\mathrm{op}} \to \mathcal{K}^{\mathrm{op}}$ in lieu of the more cumbersome $(X^*)^{\mathrm{op}}$, and let $U: \mathrm{Dyn}(X) \to \mathcal{K}$, $V: \mathrm{Dyn}(X^*) \to \mathcal{K}^{\mathrm{op}}$ denote the forgetful functors. Then the passage*

$$(Q, \delta: QX \to Q) \mapsto (Q, \delta^*: QX^* \prec Q)$$

is an isomorphism of categories

$$(\mathrm{Dyn}(X))^{\mathrm{op}} \cong \mathrm{Dyn}(X^*)$$

rendering commutative the diagram of functors:

$$\begin{tikzcd} (\mathrm{Dyn}(X))^{\mathrm{op}} \arrow[rr, phantom, "\cong"] \arrow[dr, "U^{\mathrm{op}}"'] & & \mathrm{Dyn}(X^*) \arrow[dl, "V"] \\ & \mathcal{K}^{\mathrm{op}} . & \end{tikzcd}$$

It follows that U has a left adjoint if and only if V has a right adjoint and, dually, that U has a right adjoint if and only if V has a left adjoint. Therefore, X is an input process if and only if X^* is an output process, and X is a state-behavior process if and only if X^* is. □

It follows from duality, and the formula $QX^{@} = \amalg QX^n$, that if X is adjoint and $\mathcal{K}$ has countable products, the formula for $QX_{@}$ is $\prod Q(X^*)^n$. This does in fact agree with the formula $Q^{X_0^*} = \prod_{n \geq 0} [Q^{X_0^n}]$ for $X = -\times X_0$. We have

2.5. Theorem. *If X is an adjoint process and $\mathcal{K}$ has countable products and coproducts, X is state-behavior.*

3. Image factorizations and minimal realization

In Section 1, we said that a sequential machine was *reachable* if r was *onto*, and *observable* if σ was one-to-one. To extend these concepts to machines in a category $\mathcal{K}$, we must provide $\mathcal{K}$ with image factorizations: (see, e.g. Herrlich and Strecker, Chap. IX for an exposition or [5, Section 3]). We assume known the standard category theory notions of epimorphism, monomorphism and isomorphism.

3.1. Definition. An *image factorization system* for a category $\mathcal{K}$ consists of a pair $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ and $\mathcal{M}$ are classes of morphisms in $\mathcal{K}$ satisfying the following four axioms:

IFS 1: $\mathcal{E}$ and $\mathcal{M}$ are subcategories of $\mathcal{K}$.

IFS 2: If $e: A \to B \in \mathcal{E}$, e is an epimorphism. Dually, if $m: A \to B \in \mathcal{M}$, m is a monomorphism.

IFS 3: If $f: A \to B$ is an isomorphism then $f \in \mathcal{E}$ and $f \in \mathcal{M}$.

IFS 4: Every $f: A \to B$ in $\mathcal{K}$ has an $\mathcal{E}-\mathcal{M}$ factorization which is unique up to isomorphism. More precisely, there exists an $\mathcal{E}-\mathcal{M}$ *factorization* (e, m) of f, meaning $e \in \mathcal{E}$, $m \in \mathcal{M}$ and $f = em$ (so that there exists an object – call it $(A)f$ – such that e has the form $e: A \to (A)f$ and m has the form $m: (A)f \to B$), and this factorization is unique in the sense that if (e', m')

$$\begin{tikzcd}
 & (A)f \arrow[dr, "m"] \arrow[dd, dashed, "\psi"] & \\
A \arrow[ur, "e"] \arrow[dr, "e'"'] & & B \\
 & {[(A)f]'} \arrow[ur, "m'"'] &
\end{tikzcd}$$

is another such factorization – $f = e'm'$, $e' \in \mathcal{E}$, $m' \in \mathcal{M}$ – then there exists an isomorphism ψ (as shown above) with $e\psi = e'$, $\psi m' = m$.

In the category **Set** of sets,

$$\mathcal{E} = \{\text{onto functions}\} \text{ and } \mathcal{M} = \{\text{one-to-one functions}\}$$

is an image factorization system. The first three axioms are clear. For IFS 4, define $(A)f = \{af : a \in A\} \subset B$, set m to be inclusion function and define $ae = af \in (A)f$. For the uniqueness proof, define $i\psi = ae'$ for any a with $ae = i$. The remaining details are routine. Essentially the same construction demonstrates that

$$\mathcal{E} = \{\text{onto homomorphisms}\} \text{ and } \mathcal{M} = \{\text{one-to-one homomorphisms}\}$$

provide image factorizations in **R-Mod** and **Gp**.

For the balance of this section let $(\mathcal{E}, \mathcal{M})$ be an image factorization system for $\mathcal{K}$ and let X be a state-behavior process in $\mathcal{K}$. We can now generalize two familiar definitions from Section 1.

3.2. Definition. A machine $(X, Q, \delta, I, \tau, Y, \beta)$ is *reachable* just in case its reachability map $r: IX^{@} \to Q$ is in $\mathcal{E}$, and is *observable* providing its observability map $\sigma: Q \to YX_{@}$ is in $\mathcal{M}$.

The following standard lemma (known to control theorists in the form of the Zeiger fill-in lemma [11, Chapter 10]) will be useful for our minimal realization theory:

3.3. Diagonal fill-in Lemma. *Given a commutative square – $eg = fm$ – as shown below*

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "h"'] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

with $e \in \mathcal{E}$ and $m \in \mathcal{M}$ there exists a unique h (as shown) with $eh = f$, $hm = g$.

Proof. Using either that e is an epimorphism or that m is a monomorphism, the uniqueness assertion is clear (and, in fact, either triangle implies the other). To prove existence, consider the diagram below, in which we provide image factorizations $f = e_1 m_1$ and $g = e_2 m_2$:

$$\begin{tikzcd} A \arrow[rr, "e"] \arrow[dr, "e_1"] \arrow[dd, "f"'] & & B \arrow[dl, "e_2"'] \arrow[dd, "g"] \\ & (A)f \arrow[dl, "m_1"] & (B)g \arrow[l, dashed, "\psi"'] \arrow[dr, "m_2"'] & \\ C \arrow[rr, "m"'] & & D \end{tikzcd} .$$

Since $(e_1, m_1 m)$ and (ee_2, m_2) are both $\mathcal{E}$–$\mathcal{M}$ factorizations of $eg = fm$, there exists an isomorphism ψ with $ee_2\psi = e_1$ and $\psi m_1 m = m_2$. Define $h = e_2 \psi m_1$. □

With these tools, we may swiftly develop a theory of minimal realization (similar to that of Bainbridge [7]). Given any dynamorphism $X_0^* \to Y^{X_0^*}$ in the category **Set**, we saw (1.5) that we could define a minimal realization as follows:

$$\text{Factor } f^{\blacktriangle} \text{ into } X_0^* \xrightarrow{e} Q_f \xrightarrow{m} Y^{X_0^*}$$

where $Q_f = (X_0^*)f^{\blacktriangle}$, the image of X_0^* under $f^{\blacktriangle}$, and then e is the *onto* map $w \mapsto (w)f^{\blacktriangle}$ [so that the realization is *reachable*] and m is the *one-to-one* map $q \mapsto q$ which sends $(w)f^{\blacktriangle}$ to itself [so that the realization is *observable*]. We then defined δ_f: $((w)f^{\blacktriangle}, x) \mapsto (wx)f^{\blacktriangle}$; β_f: $(w)f^{\blacktriangle} \mapsto (\Lambda)\,[(w)f^{\blacktriangle}]$; and chose initial state $(\Lambda)f^{\blacktriangle}$.

We now make the appropriate constructions and definitions for any state-behavior process X: $\mathcal{X} \to \mathcal{X}$.

3.4. Definition. Fixing I and Y, but letting the state-space Q vary, consider the category

X-mach whose objects are machines $M = (X, Q, \delta, I, \tau, Y, \beta)$; and whose morphisms are *simulations* $\psi : M \to M'$ (we say M *simulates* M') i.e. dynamorphisms $\psi : (Q, \delta) \to (Q', \delta')$ which commute with the initial state and output:

$$\begin{tikzcd}
 & Q \arrow[dd, "\psi"] \arrow[dr, "\beta"] & \\
I \arrow[ur, "\tau"] \arrow[dr, "\tau'"'] & & Y \\
 & Q' \arrow[ur, "\beta'"'] &
\end{tikzcd}$$

It is an immediate consequence of their definitions, that a simulation commutes with the reachability and observability maps:

$$\begin{tikzcd}
 & Q \arrow[dd, "\psi"] \arrow[dr, "\sigma"] & \\
IX^{@} \arrow[ur, "r"] \arrow[dr, "r'"'] & & YX^{@} \\
 & Q' \arrow[ur, "\sigma'"'] &
\end{tikzcd}$$

Of course, if the dynamorphism ψ satisfies the latter diagram, it is certainly a simulation. In particular, then, the existence of a simulation guarantees that the two machines have the same (total) response:

$$f^{\blacktriangle} = r\sigma = r\psi\sigma' = r'\sigma' = (f')^{\blacktriangle} .$$

3.5. Definition. We say a system M is a *realization* of a (total) response $f^{\blacktriangle}$ if $f^{\blacktriangle}$ is the (total) response of M. We say M is a *minimal realization* if it is a reachable realization of $f^{\blacktriangle}$ with the property that, given any other reachable realization M' of $f^{\blacktriangle}$, there exists a unique simulation $\psi : M' \to M$. In other words, M is a terminal object in the category whose objects are reachable realizations of f, and whose morphisms are simulations (composition and identities being at the level of $\mathcal{X}$), and is thus unique up to isomorphism if it exists (as we shall prove it always does within our $(X, \mathcal{E}, \mathcal{M})$ context). We may speak of *the* minimal realization and denote it by

$$M_f = (X, Q_f, \delta_f, I, \tau_f, Y, \beta_f) .$$

3.6. Definition. M is a *canonical* realization of $f^{\blacktriangle}$ if M is a realization of $f^{\blacktriangle}$ which is reachable and observable.

We will prove that "canonical" and "minimal" are equivalent.

We now set out to show that in the present context of state-behavior X in $(\mathcal{K}, \mathcal{E}, \mathcal{M})$ a straightforward minimal realization theory exists providing either

(a) X preserves $\mathcal{E}$ (i.e. $e \in \mathcal{E}$ implies $eX \in \mathcal{E}$), or

(b) $X^{@}$ preserves $\mathcal{E}$.

The proofs in case (a) are very simple, but there is no compelling reason to suspect (a) holds unless X is adjoint whereas $X^{@}$ always has $X_{@}$ as a right adjoint, i.e.

$$\begin{array}{ll} AX^{@} \xrightarrow{f} B & \mathcal{K}\text{-morphism} \\ \hline AX^{@} \xrightarrow{f^{\blacktriangle}} BX_{@} & X\text{-isomorphism} \\ \hline A \xrightarrow{f^{\bullet}} BX_{@} & \mathcal{K}\text{-morphism}. \end{array}$$

Thus if $\mathcal{E}$ is either all coequalizers or all epimorphisms, (a) holds if X is adjoint and (b) always holds.

The theory below generalizes Kalman's "$\mathcal{K}[z]$-module" approach to the realization theory of linear systems [11, Chapter 10] which is the special case $\mathcal{K}$ = vector spaces and linear maps, X = id. See also [5].

3.7. Dynamorphic image Lemma. *Let $h: (Q, \delta) \to (Q', \delta')$ be a dynamorphism and let $e: Q \to Q''$, $m: Q'' \to Q'$ be an $\mathcal{E}$–$\mathcal{M}$ factorization of h. Then if either $X^{@}$ preserves $\mathcal{E}$ there exists a unique dynamics δ'' on Q'' such that $e: (Q, \delta) \to (Q'', \delta'')$ and $m: (Q'', \delta'') \to (Q', \delta')$ are dynamorphisms.*

Proof. If X preserves $\mathcal{E}$ this is immediate by diagonal fill-in:

$$\begin{tikzcd} QX \arrow[r, "eX"] \arrow[d, "\delta"'] & Q''X \arrow[d, "mX"] \\ Q \arrow[d, "e"'] & QX \arrow[d, "\delta"] \\ Q'' \arrow[r, "m"'] \arrow[ur, dashed, "\delta''"] & Q' \end{tikzcd}.$$

Otherwise, assume $X^{@}$ preserves $\mathcal{E}$. For each A define $A\eta_1 : AX \to AX^{@}$

$$A\eta_1 = AX \xrightarrow{A\eta X} AX^{@}X \xrightarrow{A\mu_0} AX^{@}. \tag{3.8}$$

Then $\eta_1 : X \to X^{@}$ is a natural transformation since η and μ_0 are. Define the *run map* $\delta^{@} : QX^{@} \to Q$ of (Q, δ) as the dynamorphic extension

(3.9)
$$\begin{tikzcd} Q \arrow[r, "Q\eta"] \arrow[dr, "\mathrm{id}"'] & QX^{@} \arrow[d, dashed, "\delta^{@}"] \\ & Q \end{tikzcd} .$$

Then $Q\eta_1 \cdot \delta^{@} = \delta$ as is seen from

(3.10)
$$\begin{tikzcd} QX \arrow[rr, "Q\eta_1"] \arrow[dr, "Q\eta X"] \arrow[ddr, "\mathrm{id}"'] & & QX^{@} \arrow[dd, "\delta^{@}"] \\ & QX^{@}X \arrow[ur, "Q\mu_0"'] \arrow[d, "\delta^{@}X"] & \\ & QX \arrow[r, "\delta"'] & Q \end{tikzcd} .$$

Using the uniqueness of dynamorphic extensions, we have $\delta^{@}h = hX^{@} \cdot (\delta')^{@}$ since this is true preceded by $Q\eta$:

$$\begin{tikzcd} Q \arrow[rr, "h"] \arrow[dr, "Q\eta"] \arrow[ddr, "\mathrm{id}"'] & & Q' \arrow[dl, "Q'\eta"'] \arrow[ddl, "\mathrm{id}"] \\ & QX^{@} \arrow[r, "hX^{@}"] \arrow[d, "\delta^{@}"] & Q'X^{@} \arrow[d, "(\delta')^{@}"] \\ & Q \arrow[r, "h"'] & Q' \end{tikzcd} .$$

The desired dynamics is then $\delta'' = Q''\eta_1 \psi$ where ψ is the diagonal fill-in

$$\begin{tikzcd}
QX \arrow[rr, "eX"] \arrow[dr, "Q\eta_1"] \arrow[dd, "\delta"'] & & Q''X \arrow[dl, "Q''\eta_1"'] \arrow[dd, "mX"] \\
& QX^{@} \arrow[r, "eX^{@}"] \arrow[dl, "\delta^{@}"'] & Q''X^{@} \arrow[dd, "mX^{@}"] \arrow[dddl, dashed, "\psi"'] \\
Q \arrow[dd, "e"'] & & Q'X \arrow[dl, "Q'\eta_1"'] \arrow[dd, "\delta'"] \\
& & Q'X^{@} \arrow[dl, "(\delta')^{@}"'] \\
Q'' \arrow[rr, "m"'] & & Q'
\end{tikzcd}$$

Uniqueness is clear as m is mono. □

3.11. Cancellation Lemma. *Let e: $(Q, \delta) \to (Q', \delta')$ be a dynamorphism with $e \in \mathcal{E}$, let (Q'', δ'') be a dynamics and let $f: Q' \to Q'' \in \mathcal{K}$ be such that $ef: (Q, \delta) \to (Q'', \delta'')$ is a dynamorphism. Then if either X preserves $\mathcal{E}$ or $X^{@}$ preserves $\mathcal{E}$, $f: (Q', \delta') \to (Q'', \delta'')$ is a dynamorphism.*

Proof. If X preserves $\mathcal{E}$ consider

$$\begin{tikzcd}
QX \arrow[r, "eX"] \arrow[d, "\delta"'] & Q'X \arrow[r, "fX"] \arrow[d, "\delta'"] \arrow[dr, phantom, "(?)"] & Q''X \arrow[d, "\delta''"] \\
Q \arrow[r, "e"'] & Q' \arrow[r, "f"'] & Q''
\end{tikzcd}.$$

By hypothesis, the perimeter and left square commute. Therefore, $eX \cdot \delta' \cdot f = eX \cdot fX \cdot \delta''$. As $eX \in \mathcal{E}$ is an epimorphism, (?) commutes as desired.

Now suppose $X^{@}$ preserves $\mathcal{E}$. We observe the general principle that if $g: (Q_1, \delta_1) \to (Q_2, \delta_2)$ is a dynamorphism then $\delta_1^{@} g = gX^{@} \delta_2^{@}$.

(3.12)
$$\begin{tikzcd}
Q_1 \arrow[rr, "g"] \arrow[dr, "Q_1\eta"] \arrow[ddr, "\mathrm{id}"'] & & Q_2 \arrow[dl, "Q_2\eta"'] \arrow[ddl, "\mathrm{id}"] \\
& Q_1X^{@} \arrow[r, "gX^{@}"] \arrow[d, "\delta_1^{@}"'] \arrow[dr, phantom, "(?)"] & Q_2X^{@} \arrow[d, "\delta_2^{@}"] \\
& Q_1 \arrow[r, "g"'] & Q_2
\end{tikzcd}$$

This follows from the uniqueness of dynamorphic extensions, the fact that $\delta_1^{@}, g$, $gX^{@}, \delta_2^{@}$ are dynamorphisms, (3.9) and the naturality of η. Applying this principle to e and ef gives

(3.13)
$$\begin{tikzcd}
QX^{@} \arrow[r, "eX^{@}"] \arrow[d, "\delta^{@}"'] & Q'X^{@} \arrow[r, "fX^{@}"] \arrow[d, "(\delta')^{@}"'] \arrow[dr, phantom, "(?)"] & Q''X^{@} \arrow[d, "(\delta'')^{@}"] \\
Q \arrow[r, "e"'] & Q \arrow[r, "f"'] & Q''
\end{tikzcd}$$

the perimeter and left square of (3.13), and hence (?) of (3.13) since $eX^{@} \in \mathcal{E}$. The proof is completed by using the perimeter of (3.10) and the naturality of η_1:

$$\begin{tikzcd}
Q'X \arrow[rr, "fX"] \arrow[dr, "Q'\eta_1"] \arrow[dd, "\delta'"'] & & Q''X \arrow[dl, "Q''\eta_1"'] \arrow[dd, "\delta''"] \\
& Q'X^{@} \arrow[r, "fX^{@}"] \arrow[dl, "(\delta')^{@}"] & Q''X^{@} \arrow[dr, "(\delta'')^{@}"'] \\
Q' \arrow[rr, "f"'] & & Q''
\end{tikzcd}$$
. □

3.14. Simulation Lemma. *Let $f^{\blacktriangle}: IX^{@} \to YX_{@}$ be a dynamorphism, let M be a reachable realization of $f^{\blacktriangle}$ and let M' be an observable realization of $f^{\blacktriangle}$. Then if either X preserves $\mathcal{E}$ or $X^{@}$ preserves $\mathcal{E}$ there exists a unique simulation $\psi: M \to M'$.*

Proof. Define ψ by diagonal fill-in:

$$\begin{tikzcd} IX^{@} \arrow[r, "r"] \arrow[d, "r'"'] & Q \arrow[d, "\sigma"] \arrow[dl, dashed, "\psi"'] \\ Q' \arrow[r, "\sigma'"'] & YX_{@} \end{tikzcd}$$

(where r, r' are the reachability maps of M, M'; σ, σ' are the observability maps of M, M' and the hypothesis is $r\sigma = f^{\blacktriangle} = r'\sigma'$, $r \in \mathcal{E}$ and $\sigma' \in \mathcal{M}$). ψ is a dynamorphism by 3.11. □

We are now ready for the main theorem of this section. Notice that "$f^{\blacktriangle}$ is a dynamorphism" generalizes the condition of 1.5 for $f^{\blacktriangle}$ to have a realization.

3.15. Minimal realization Theorem. *Let X be a state-behavior process in $\mathcal{K}$ and let $(\mathcal{E}, \mathcal{M})$ be an image factorization system for $\mathcal{K}$ such that either X preserves $\mathcal{E}$ or $X^{@}$ preserves $\mathcal{E}$. Let I, Y be objects of $\mathcal{K}$. Then every dynamorphism $f^{\blacktriangle}$: $(IX^{@}, I\mu_0) \to (YX_{@}, YL)$ has a minimal realization M_f whose state object Q_f is the $\mathcal{E}-\mathcal{M}$ image of $f^{\blacktriangle}$. Further, a system M is a minimal realization of f if and only if it is a canonical realization of f.*

Proof. Let $r_f: IX^{@} \to Q_f$, $\sigma_f: Q_f \to YX_{@}$ be an $\mathcal{E}-\mathcal{M}$ factorization of $f^{\blacktriangle}$. By the dynamorphic image lemma there exists unique δ_f such that $r_f: (IX^{@}, \mu_0) \to (Q_f, \delta_f)$ and $\sigma_f: (Q_f, \delta_f) \to (YX_{@}, YL)$ are dynamorphisms. Define τ_f, β_f by

$$\begin{tikzcd} I \arrow[d, "I\eta"'] \arrow[dr, "\tau_f"] & & Y \arrow[dl, "\beta_f"'] \\ IX^{@} \arrow[r, "r_f"'] & Q_f \arrow[r, "\sigma_f"'] & YX_{@} \arrow[u, "Y\Lambda"'] \end{tikzcd}$$

then $M_f = (X, Q_f, \delta_f, I, \tau_f, Y, \beta_f)$ is a canonical realization of f. It is immediate from the simulation lemma that every canonical relaization is minimal. Since all minimal realizations of $f^{\blacktriangle}$ are isomorphic, the proof is complete. □

For a Nerode equivalence approach [14] to minimal realization see [1] and [3].

4. Duality and adjointness

There are three main definitions of dual machine in the literature of automata and system theory:

4.1. Rabin and Scott [14] looked at *nondeterministic acceptors* $M = (Q, \delta, Q_0, F)$ where $\delta : Q \times X_0 \to P(Q)$, with $P(Q)$ the set of subsets of Q (δ extends to $\delta^* : Q \times X_0^* \to P(Q)$ in the normal way), $Q_0 \subset Q$ and $F \subset Q$; they associated with each M the set

$$T(M) = \{w \in X_0^* | \delta^*(q_0, w) \cap F \neq \emptyset \text{ for some } q_0 \in Q_0\} .$$

They then defined the *dual* of such an M to be the machine $M^R = (Q, \delta^R, F, Q_0)$ where $\delta^R(q, x) = \{q' \in Q | q \in \delta(q', x)\}$, noting that $T(M^R)$ is precisely the set $T(M)^R$ of reversals of strings in $T(M)$. Clearly $M^{RR} = M$.

4.2. In case $\mathcal{X}$ = Vect and X is the identity process, a machine (id, Q, F, I, G, Y, H) simply reduces to a "zero input" dynamics $F : Q \to Q$, an input map (in the formal role of the initial state τ) $G : I \to Q$, and an output map H. Control theorists view (F, G, H) as representing a linear system with behavior described by the equations

$$q(t+1) = Fq(t) + Gu(t)$$

$$y(t) = Hq(t) .$$

The algebraic theory of linear systems owes much to Kalman [11]; and has been placed in a categorical setting in [5].

Kalman introduced $M^* = (F^*, H^*, G^*)$ as the dual of the linear system $M = (F, G, H)$. Again $M = M^{**}$, and Kalman made the crucial observation that a system was reachable iff its dual was observable.

4.3. Arbib and Zeiger [6] tried to modify Rabin and Scott's definition in a way that would extend Kalman's observations to the nonlinear case. Given a sequential machine $M = (X_0, Q, \delta, q_0, Y, \beta)$, they defined a new machine $M^\dagger = (X_0, Y^Q, \delta^\dagger, \beta, Y, \beta^\dagger)$ where, denoting the dynamics of M by $(q, x) = q \cdot x$ and those of $M^\dagger$ by $\delta^\dagger(f, x) = x * f$ we have the basic relation

$$q(x * f) = (q \cdot x) f$$

where on each side we are evaluating an element of Y^Q at an element of Q to yield an element of Y. The initial state of $M^\dagger$ is the output function of M, while the output function $\beta^\dagger$ of $M^\dagger$ is evaluation of an f in Y^Q at the initial state q_0 of M. [Cf. the Rabin–Scott definition when $Y = \{0, 1\}$.] Unfortunately, it is *not* true that $M^{\dagger\dagger} = M$, and while M is reachable iff $M^\dagger$ is observable and M is observable if $M^\dagger$ is reachable, it is not true, in general, that $M^\dagger$ is reachable whenever M is observable.

In this section, we show that state-behavior machines provide the proper setting for an adjointness theory of machines, while adjoint machines yield the proper setting for duality. In particular, we shall recover the duality theory for linear machines as a very special case, and see how to modify the Rabin–Scott definition to yield a full duality for nondeterminate sequential machines (*not* acceptors – their output set of $\{0,1\}$ is too restricted to yield a full duality theory which embraces the duality of reachability and observability).

We start by establishing an adjointness principle: Suppose that $U: \mathcal{B} \to \mathcal{A}$ is a functor which has both a left adjoint $L: \mathcal{A} \to \mathcal{B}$ and a right adjoint $R: \mathcal{A} \to \mathcal{B}$. Then for each pair A, A' of $\mathcal{A}$-objects we have bijective correspondences

$$\frac{\dfrac{f: ALU \to A'}{f^{\blacktriangle}: AL \to A'R}}{f^{\bullet}: A \to A'RU.}$$

Starting on the bottom with $g: A \to A'RU$, the notations going up are $g_{\blacktriangle}: AL \to A'R$ and $g_{\bullet}: ALU \to A'$. Indeed, LU has RU as a right adjoint, $\epsilon: ARULU \to A$ being defined of course by $\epsilon = (\mathrm{id}_{ARU})^{\bullet}$.

Let X be a state-behavior process in a category $\mathcal{K}$. Setting $\mathcal{B} = \mathrm{Dyn}(X)$ and $\mathcal{A} = \mathcal{K}$, we have for U the forgetful functor that $LU = X^{@}$ and $RU = X_{@}$ so that we have established

4.4. Adjointness principle for state-behavior machines. If $X: \mathcal{K} \to \mathcal{K}$ is a state-behavior input process, then $X^{@}$ has $X_{@}$ as a right adjoint. There exist bijective correspondences

$$\frac{\dfrac{g_{\bullet} = f: AX^{@} \to B \qquad \mathcal{K}\text{-morphism}}{g_{\blacktriangle} = f^{\blacktriangle}: AX^{@} \to BX_{@} \qquad X\text{-dynamorphism}}}{g = f^{\bullet}: A \to BG_{@} \qquad \mathcal{K}\text{-morphism}}$$

where A and B are arbitrary $\mathcal{K}$-objects.

Omitting the middle correspondents, we have

4.5. Adjointness table (for state-behavior machines).

concept for M	adjoint concept for M
run map	state-behavior map
$\delta^{@}: QX^{@} \to Q$	$b: Q \to QX_{@}$
full response map	observability map
$\sigma_{\bullet} = \delta^{@}\beta: QX^{@} \to Y$	$\sigma: Q \to YX_{@}$
reachability map	adjoint reachability map
$r: IX^{@} \to Q$	$r^{\bullet}: I \to QX_{@}$
response map	adjoint response map
$f: IX^{@} \to Y$	$f^{\bullet}: I \to YX_{@}$.

In the classical case, where I is the one-element set and $AX_@$ is the set of functions from X_0^* to A, $r^\bullet$ is the "name of" r, that is the label for the function r in the set of all functions from X_0^* to Q; similarly, $f^\bullet$ is the name of f. $\delta^@\beta = \sigma_\bullet$ is called the full response map because in the classical case it is the map $Q \times X_0^* \to Y$ which describes the output given any initial state and input word. In [7], Bainbridge used the intermediate dynamorphism $f^\blacktriangle: IX^@ \to YX_@$ to define "the response of M", just as we did in Section 3.

Let us extend this study by showing how opposite categories "cut the work in half" for adjoint machines:

4.6. Definition. Assume $\mathcal{X}$ has countable coproducts and countable products. Let the machine $M = (X, Q, \delta, I, \tau, Y, \beta)$ in $\mathcal{X}$ be adjoint, i.e., $X: \mathcal{X} \to \mathcal{X}$ has a right adjoint. Then the *dual of M* is the machine M^{op} in $\mathcal{X}^{\mathrm{op}}$ defined by

$$M^{\mathrm{op}} = (X^\bullet, Q, \delta^\bullet, Y, \beta, I, \tau) .$$

It is clear that we have a true duality theory in that $(M^{\mathrm{op}})^{\mathrm{op}} = M$.

From Section 2, we know that an adjoint process X has $QX^@ = \amalg QX^n$, while $QX_@ = \prod Q(X^\bullet)^n$. Thus, as we pass from $\mathcal{X}$ to $\mathcal{X}^{\mathrm{op}}$, $\amalg QX^n$ becomes $\prod QX^n = Q(X^\bullet)_@$; while $\prod Q(X^\bullet)^n$ becomes $\amalg Q(X^\bullet)^n = Q(X^\bullet)^@$, since we interchange products and coproducts in opposed categories.

4.7. Example: Boolean machines. We exemplify this by looking at M^{op} for a "classical" machine M. To do this we first note that $\mathbf{Set}^{\mathrm{op}}$ can be modelled by the category $\mathcal{X}$ of complete atomic Boolean algebras and Boolean homomorphisms which preserve all infima and suprema (including 0 and 1). The crucial fact, due to Stone, is that each object A is canonically isomorphic to the powerset Boolean algebra $(P(\mathrm{At}(A)))$ of the subset $\mathrm{At}(A)$ of its atoms, and each morphism $\psi: A \to B$ is deduced as the inverse image map of a unique function $f: \mathrm{At}(B) \to \mathrm{At}(A)$ [where $f(b) = a$ if $b \in \psi(\{a\})$; this a is unique since $\{a\} \cap \{a'\} = \emptyset$ implies $\psi(\{a\}) \cap \psi(\{a'\}) = \emptyset$]. Thus $\mathbf{Set}^{\mathrm{op}}$ is in fact equivalent to $\mathcal{X}$ under the passage

$$\mathbf{Set}^{\mathrm{op}} \to \mathcal{X}: A \xleftarrow{f} B \longmapsto P(A) \xrightarrow{f^{-1}} P(B) .$$

The standard input process $-\times X_0$: $\mathbf{Set} \to \mathbf{Set}$ is adjoint, and hence so is $(-\times X_0)^\bullet = (-)^{X_0}: \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}^{\mathrm{op}}$. The natural bijection here has the form

$$\frac{A \to P(\mathrm{At}(B)^{X_0})}{P(\mathrm{At}(A) \times X_0) \to B} .$$

Translating our classical machine into its M^{op} in $\mathcal{X}$ we may define a process X by

$$AX = P(\mathrm{At}(A)^{X_0})$$

and

$$(A \xrightarrow{\psi} B)X = (- \cdot f)^{-1} : AX \to BX ,$$

where f: At(B) → At(A) corresponds to ψ, and "$- \cdot f$" means "composing with f".

To chase down the formula for $AX^{@}$ note first that $\mathrm{At}(\amalg A_i) = \prod \mathrm{At}(A_i)$. We also have

$$\begin{aligned}\mathrm{At}(AX^{n+1}) &= \mathrm{At}(AX^n)^{X_0} = (\mathrm{At}(AX^{n-1})^{X_0})^{X_0} \\ &\cong (\mathrm{At}(AX^{n-1}))^{(X_0 \times X_0)} \\ &\cdots \\ &\cong (\mathrm{At}(A))^{X_0^n} .\end{aligned}$$

It follows that $\mathrm{At}(AX^{@}) \cong \prod (\mathrm{At}(A))^{X_0^n} \cong \mathrm{At}(A)^{X_0^*}$.

By our translation process from M to M^{op} we then deduce that

$$AX^{@} = P(\mathrm{At}(A)^{X_0^*})$$

and

$$AX_{@} = A^{X_0^*}$$

the Boolean operations in $AX_{@}$ being pointwise; this is complete and atomic.

If we take morphism symbols relative to $\mathcal{K}$, but take concept names and the product, coproduct, $_{@}$ and $^{@}$ symbols relative to the ambient category, we may translate the machine concept for any M (for an adjoint X) into its dual concept for M^{op} as shown in the following table:

4.8. Duality table for adjoint machines.

M-concept in $\mathcal{K}$	M^{op}-concept in $\mathcal{K}^{\mathrm{op}}$
initial state $\tau : I \to Q$	output map $\tau : Q \prec I$
output map $\beta : Q \to Y$	initial state $\beta : Y \prec Q$
adjoint process X: $\mathcal{K} \to \mathcal{K}$	adjoint process $X^{\bullet}$: $\mathcal{K}^{\mathrm{op}} \to \mathcal{K}^{\mathrm{op}}$
dynamics $\delta : QX \to Q$	dynamics $\delta^{\bullet} : QX^{\bullet} \prec Q$
free dynamics	cofree dynamics
$QX^{@} = \amalg QX^n$	$Q(X^{\bullet})_{@} = \prod QX^n$
$\mu_0 : QX^{@}X \to QX^{@}$	$\mu_0^{\bullet} : Q(X^{\bullet})_{@}X^{\bullet} \prec Q(X^{\bullet})^{@}$
$\eta : Q \to QX^{@}$	$\eta^{\bullet} : Q(X^{\bullet})_{@} \prec Q$

M-concept in $\mathcal{K}$	M^{op}-concept in $\mathcal{K}^{\mathrm{op}}$
cofree dynamics	free dynamics
$QX_{@} = \prod Q(X^{\bullet})^{n}$	$Q(X^{\bullet})^{@} = \coprod Q(X^{\bullet})^{n}$
$L: QX_{@}X \to QX_{@}$	$L^{\bullet}: Q(X^{\bullet})^{@}X^{\bullet} \prec Q(X^{\bullet})^{@}$
$\Lambda: QX_{@} \to Q$	$\Lambda^{\bullet}: Q \prec Q(X^{\bullet})^{@}$
run map	state-behavior map
$\delta^{@}: QX^{@} \to Q$	$\delta^{@}: Q \prec Q(X^{\bullet})_{@}$
state-behavior map	run map
$b: Q \to QX_{@}$	$b: Q(X^{\bullet})^{@} \prec Q$
reachability map	observability map
$r: IX^{@} \to Q$	$r: Q \prec I(X^{\bullet})_{@}$
observability map	reachability map
$\sigma: Q \to YX_{@}$	$\sigma: Y(X^{\bullet})^{@} \prec Q$
response map	adjoint response map
$f: IX^{@} \to Y$	$f: Y \prec I(X^{\bullet})_{@}$
adjoint response map	response map
$f^{\bullet}: I \to YX_{@}$	$f^{\bullet}: Y(X^{\bullet})^{@} \prec I$
full response map	adjoint reachability map
$\sigma_{\bullet}: QX^{@} \to Y$	$\sigma_{\bullet}: Y \prec Q(X^{\bullet})_{@}$
adjoint reachability map	full response map
$r^{\bullet}: I \to QX_{@}$	$r^{\bullet}: Q(X^{\bullet})^{@} \prec I$.

We have established the following principles for adjoint machines: reachability and observability are dual and run and state-behavior are dual. This duality between M and M^{op} is in addition to the fact that certain machine concepts for the same M are "adjoint" to each other as defined in 4.5 for arbitrary state-behavior machines.

We now turn to the problem of recapturing the Rabin–Scott and linear duality theories as applications of the theory of M^{op}. Very special structural properties of $\mathcal{K}$ are needed to model M^{op} in $\mathcal{K}$.

Note that if $\mathcal{K} \xrightarrow{\psi} \mathcal{L}$ is an isomorphism of categories, then $M \to M\psi$, where

$$M\psi = (\psi^{-1}X\psi, Q\psi, \delta\psi, I\psi, \tau\psi, Y\psi, \beta\psi)$$

is an isomorphism of categories of machines in every conceivable way.

4.9. Definition. A category $\mathcal{K}$ is *self-adjoint* for the process X if there exists an isomorphism $\mathcal{K} \xrightarrow{\psi} \mathcal{K}^{\mathrm{op}}$ which is the identity on objects and is *reflexive*; i.e. it provides bijections

$$\frac{A \xrightarrow{f} B}{B \xrightarrow{f^*} A}$$

which satisfy $(\mathrm{id}_A)^* = \mathrm{id}_A$; $(fg)^* = g^* f^*$ and $f^{**} = f$. [Clearly, such a ψ interchanges isomorphisms, epimorphisms, etc. Notice that, as objects, $\amalg X_\alpha = \prod X_\alpha$.] Given such a ψ, we say an adjoint process X is *respectful* if $\psi^{-1} X \psi$ is a right adjoint for X, so that we choose X^* so that $\psi^{-1} X \psi = X^*$ (that is, they are equal as functors, on morphisms as well as on objects).

Given such a respectful functor, we can rework our duality table 4.8 with M^{op} interpreted as a dual machine $M^* = (M^{\mathrm{op}})\psi$ back in $\mathcal{K}$, using $f^\dagger$ as shorthand for $(f^*)^*$.

4.10. Duality table of respectful adjoint machines.

M-concept in $\mathcal{K}$	$M^* = (M^{\mathrm{op}})\ \psi$-concept in $\mathcal{K}$
initial state $\tau: I \to Q$	output map $\tau^*: Q \to I$
output map $\beta: Q \to Y$	initial state $\beta^*: Y \to Q$
adjoint process $X: \mathcal{K} \to \mathcal{K}$	adjoint process $X: \mathcal{K} \to \mathcal{K}$
dynamics $\delta: QX \to Q$	dynamics $\delta^\dagger: QX \to Q$
free dynamics	cofree dynamics
$QX^{@} = \amalg QX^n$	$QX_{@} = \amalg QX^n$
$\mu_0: QX^{@}X \to QX^{@}$	$\mu_0^\dagger: QX_{@}X \to QX_{@}$
$\eta: Q \to QX^{@}$	$\eta^\dagger: QX_{@} \to Q$
cofree dynamics	free dynamics
$QX_{@} = \prod QX^n$	$QX^{@} = \prod QX^n$
$L: QX_{@}X \to QX_{@}$	$L^\dagger: QX^{@}X \to QX^{@}$
$\Lambda: QX_{@} \to Q$	$\Lambda^\dagger: Q \to QX^{@}$
run map	state-behavior map
$\delta^{@}: QX^{@} \to Q$	$(\delta^{@})^*: Q \to QX_{@}$

M-concept in $\mathcal{X}$	$M^* = (M^{\mathrm{op}})$ ψ-concept in $\mathcal{X}$
state-behavior map	run map
$b: Q \to QX_@$	$b^*: QX^@ \to Q$
reachability map	observability map
$r: IX^@ \to Q$	$r^*: Q \to IX_@$
observability map	reachability map
$\sigma: Q \to YX_@$	$\sigma^*: YX^@ \to Q$
response map	adjoint map
$f: IX^@ \to Y$	$f^*: Y \to IX_@$
adjoint response map	response map
$f^{\bullet}: I \to YX_@$	$f^{\dagger}: YX^@ \to I$.

In particular we recapture the classic results:

M reachable $\Longleftrightarrow$ r epimorphism $\Longleftrightarrow$ r^* monomorphism $\Longleftrightarrow$ $M^{\dagger}$ observable
M observable $\Longleftrightarrow$ σ monomorphism $\Longleftrightarrow$ σ^* epimorphism $\Longleftrightarrow$ $M^{\dagger}$ reachable.

4.11. Example: nondeterministic automata. Let A, B be sets. A *relation* $f: A \to B$ *from* A *to* B is a triple (A, f, B) with f a subset of $A \times B$. With the usual composition of relations, we obtain the category **Rel** of sets and relations. The dynamics of nondeterministic automata are usually represented by functions $Q \times X_0 \to P(Q)$ in **Set**; we suggest here that the natural setting is a relation $Q \times X_0 \to Q$ in **Rel**. (But note that $Q \times X_0$ is *not* the categorical product of Q and X_0 in **Rel**!)

Let us fix, then, a set X_0. Given $f: A \to B$ define $fX: A \times X_0 \to B \times X_0$ by $fX = \{(a, x, b, x) | (a, b) \in f \text{ and } x \in X_0\}$.

In this way, $X = - \times X_0$ becomes a process in **Rel**. To verify that X is an input process (and in fact an adjoint process) we shall verify that **Rel** has all countable coproducts, and that X has a right adjoint, and then call on 2.5. The bijection

$$\frac{A \times X_0 \to B}{A \to B \times X_0}$$

is the natural one between subsets of $(A \times X_0) \times B$ and $A \times (B \times X_0)$. Thus X is adjoint with $X = X^{\bullet}$. [This establishes (Section 5) that **Rel** is closed.] Finally, disjoint unions still yield the coproduct in **Rel**, with the usual injection functions being considered as relations.

Given a relation (A, f, B) from A to B, the *inverse* of f is the relation (B, f, A) from B to A, on identifying $f \subset A \times B$ with $\{(b, a) | (a, b) \in f\} \subset B \times A$. Sending f

to its inverse establishes a self-adjointness of **Rel**. For more on this example, and its relation to nondeterministic machines, see [4].

Tying this all back to 4.9, it is clear that $X = -\times X_0$: **Rel** $\to$ **Rel** is respectful of the ψ which sends each (A, f, B) to its inverse (B, f, A) – i.e., $f^{-1} \times \mathrm{id} = (f \times \mathrm{id})^{-1}$ – and so we may apply the table 4.10 to nondeterministic automata, so that the dual of

$$M = (Q \times X_0 \xrightarrow{\delta} Q, I \xrightarrow{\tau} Q, Q \xrightarrow{\beta} Y)$$

is the nondeterministic machine

$$M^\dagger = (Q \times X_0 \xrightarrow{\delta^\dagger} Q, Y \xrightarrow{\beta} Q, Q \xrightarrow{\tau} I).$$

Now, recalling the scheme

$$\frac{Q \times X_0 \xrightarrow{\delta} Q}{Q \xrightarrow{\delta^\bullet} Q \times X_0}$$

we read off that

$$((q, x), q') \in \delta^\dagger \iff (q', (q, x)) \in \delta^\bullet \iff ((q', x), q) \in \delta$$

which is precisely the dynamics given by Rabin and Scott.

We now recapture Kalman's duality theorem [11] for linear systems:

4.12. Example. Let $\mathcal{K}$ be the category whose objects are pairs (V, B) where V is a real vector space and B is a basis for V and whose morphisms $f: (V, B) \to (V', B')$ are linear maps $f: V \to V'$. As $\mathrm{id}_V: (V, B_1) \to (V, B_2)$ is an isomorphism in $\mathcal{K}$, labelling objects with different bases does not affect categorical invariants. In particular, $\mathcal{K}$ has all coproducts and products (form the usual weak direct sum, respectively product of vector spaces and choose any basis). If $\mathcal{E}$ = onto linear maps, $\mathcal{M}$ = one-to-one linear maps, $(\mathcal{E}, \mathcal{M})$ is an image-factorization system for $\mathcal{K}$.

Let F be the class of all (V, B) in $\mathcal{K}$ with B finite. If $f: (V, B) \to (V', B')$, f is determined by the $B \times B'$ matrix, $\mathrm{Mat}(f)$, whose bth column is the B'-tuple of scalars representing bf with respect to B'. In general, the transpose of $\mathrm{Mat}(f)$ is not the matrix of a linear map, but $[\mathrm{Mat}(f)]^{\mathrm{tr}} = \mathrm{Mat}(f^*)$ for unique $f^*: (V', B') \to (V, B)$ if B is finite.

Let X: $\mathcal{K} \to \mathcal{K}$ be the identity functor so that

$$IX^{@} = I^{\S}, \; YX_{@} = Y_{\S}$$

where [5] $I^{\S}$ is the countable copower of I (space of left-infinite finite-support input sequences if $\mathcal{K}$ = **Vect**) and $Y_{\S}$ is the countable power of Y (space of right-infinite output sequences if $\mathcal{K}$ = **Vect**).

If $I, Q \in F, r: I^{\S} \to Q$ induces $r^{\bullet}: Q \to I_{\S}$ by

$$\begin{tikzcd} Q \arrow[rr, dashed, "r^{\bullet}"] \arrow[dr, "(\mathrm{in}_n r)^*"'] & & I_{\S} \arrow[dl, "\pi_n"] \\ & I & \end{tikzcd}$$

In [5,5.10] it is shown that $r: I^{\S} \to Q$ is in $\mathcal{E}$ if and only if $r^{\bullet}: Q \to I_{\S}$ is in $\mathcal{M}$. Moreover, if r is the reachability map of the system (id, Q, F, I, G, Y, H) (with $Y \in F$ as well) then $r^{\bullet}$ is the observability map σ of the Kalman dual $M^* = (\mathrm{id}, Q, F^*, Y, M^*, I, G^*)$ as is seen from

$$r^{\bullet}\pi_0 = (\mathrm{in}_0 r)^* = G^*$$

$$r^{\bullet}\pi_{n+1} = (\mathrm{in}_{n+1} r)^* = (\mathrm{in}_n rF)^*$$

$$= F^*(\mathrm{in}_n r)^* = F^*(r^{\bullet}\pi_n) .$$

Dually, the observability map $\sigma: Q \to Y_{\S}$ induces $\sigma^{\bullet}: Y^{\S} \to Q$ by $\mathrm{in}_n \sigma^{\bullet} = (\sigma\pi_n)^*$, $\sigma \in \mathcal{M}$ if and only if $\sigma^{\bullet} \in \mathcal{E}$ and $\sigma^{\bullet}$ is the reachability map of M^*. Complete details appear in [5].

5. Examples

In this section, we illustrate the concepts of Section 2 by presenting four examples of state-behavior processes. The first three are adjoint, but the fourth example establishes that state-behavior processes need not be adjoint (although in the category of sets, the only example of a state-behavior process is the classical one, $X = -\times X_0$).

5.1. Example: Machines in a closed category. A *closed category* is, essentially, a category $\mathcal{K}$ together with a "tensor product" functor $(A, B) \mapsto A \otimes B$ such that for every B, the functor $- \otimes B: \mathcal{K} \to \mathcal{K}$ has a right adjoint. A fundamental example of a closed category is **Set** with $A \otimes B$ simply the product $A \times B$. Here, the bijection

$$\frac{A \times X_0 \xrightarrow{f} B}{A \xrightarrow[f^{\bullet}]{} B^{X_0}}$$

is defined by $\langle b, af^{\bullet}\rangle = (a, b)f$. It is clear from Theorem 2.5 that a closed category with countable coproducts is a good place to imitate ordinary automata theory, and this is precisely the approach of Goguen [10] and Ehrig et al. [9].

5.2. Example: Metric machines. Let the base category $\mathcal{X}$ have metric spaces (A, d) of diameter $\leqslant 1$ (i.e., $d(x, y) \leqslant 1$ for all x, y) as objects and distance-decreasing functions (i.e., $d'(xf, yf) \leqslant d(x, y)$ for all x, y – but "distance-non-increasing" is too cumbersome a term) as morphisms. The *tensor product* $(A, d) \otimes (A', d')$ of two objects is defined to be $(A \times A', \min(d + d', 1))$. Imitating ordinary automata theory, fix an "input space" (X_0, d_0) and define a process X in $\mathcal{X}$ by $(A, d)X = (A, d) \otimes (X_0, d_0)$. By iterating $\otimes$, the set X_0^n of strings of length n is a metric space.

Any disjoint family (A_i, d_i) induces a metric d on its union A by letting d coincide with d_i on $A_i \times A_i$ and defining all other distances as "infinity", i.e., the disjoint union is made metric by making the injections isometries and defining the distance between points in differently-indexed A_i to be 1. In particular, X_0^* is a metric space (X_0^*, d^*).

The free dynamics $(A, d)X^{@}$ is $(A, d) \otimes (X_0^*, d^*)$. The cofree dynamics $(A, d)X_{@}$ is the set of all distance-decreasing functions from (X_0^*, d^*) to (A, d) in the metric $\sup(d(wf, wg): w \in X_0^*)$.

Notice that the category **Set** of sets and functions sits as a full subcategory of $\mathcal{X}$ by identifying sets A with *discrete* metric spaces (in which all non-zero distances are 1). If X_0 is discrete, so is X_0^*. If Y is also discrete, so is $YX_{@}$. Restricting attention to the discrete case, ordinary automata theory is recaptured.

The category of metric spaces is a closed category. The proof of adjointness for $-\otimes B$ is by restriction of the set case: specifically, one checks that if f is distance-decreasing then $f^{\bullet}$ is well-defined (each $af^{\bullet}$ is distance-decreasing) and $f^{\bullet}$ is itself distance-decreasing; and conversely given that $f^{\bullet}$ is well-defined and distance-decreasing then f is distance-decreasing.

5.3. Example: Topological automata. The constructions of 5.2 work in the category **Top** of topological spaces and continuous maps. The coproduct is the disjoint union with the usual disjoint union topology (the open subsets of the A_i form a basis).

For a fixed space X_0, X_0^n has the cartesian power topology and then, with the disjoint union topology, X_0^* is also a space. $X = -\times X_0$ becomes an input process in **Top** with free dynamics $AX^{@} = A \times X_0^*$. If X_0 is at least locally compact Hausdorff, X is state-behavior with $AX_{@}$ the set of all continuous maps from X_0^* to A with the compact-open topology.

There is also another way to proceed that works for any space X_0. Given spaces A and B there is a unique topology on the set $A \times B$ such that a map out of $A \times B$ to another space is continuous just in case it was already separately continuous; simply provide $A \times B$ with the largest topology making all the functions $x \mapsto (x, y)$ (for all $y \in Y$) and $y \mapsto (x, y)$ (for all $x \in X$) continuous. This defines the *tensor product* $A \otimes B$ (cf. the tensor product of vector spaces which linearizes bilinear maps). $X = -\otimes X_0$ is always state-behavior. $AX^{@} = A \otimes X_0^*$ (the topology on X_0^* being the disjoint union of iterated tensor powers) and $AX_{@}$ is the set of all continuous maps from X_0^* to A in the topology of pointwise convergence.

The category of topological spaces with tensor products thus defined is a closed

category. With cartesian products, topological spaces do not form a closed category and for this reason topological spaces are rejected by Goguen [10]. However, if X_0 is locally compact Hausdorff, then $-\times X_0$ has a right adjoint (continuous functions in the compact-open topology) as is proved, for example, in [12, p. 178].

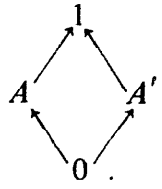
The category of topological spaces has products (the Tychonoff topology!) and the metric space category of 5.2 has products $\Pi(A_i, d_i) = (\Pi A_i, d)$ where $d((x_i), (y_i)) = \sup_i (d(x_i, y_i))$. That the adjoint processes of 5.2 and 5.3 have $QX_@$ formulas $\Pi Q(X^\bullet)^n$ follows from the following closed category truisms which are easy to check directly in these particular cases:

$$(-\times X_0)^{\bullet n} = (-\times X_0)^{n\bullet}$$

$$(-\times X_0)^{\bullet *} = (-\times X_0^*)^\bullet .$$

We can now apply Theorem 2.5 to justify the existence of $X^@$ in these three examples. In the three examples of 5.2 $Q \otimes -$ preserves all coproducts (even if Q is an arbitrary topological space and $\otimes = \times$, as one checks directly) which is why we were able to identify $QX^* = \sqcup(Q \otimes X_0 \otimes \ldots \otimes X_0)$ with $Q \otimes (\sqcup(X_0 \otimes \ldots \otimes X_0)) = Q \otimes X_0^*$. Although we have just seen there can be other reasons, it is clear that in any closed category in which $A \otimes B \cong B \otimes A$ we can write QX^* as $Q \otimes X_0^*$.

5.4. Example (due to Michael Barr). Even if $\mathcal{K}$ has all products and coproducts, not every state-behavior process in $\mathcal{K}$ need be adjoint. Let $\mathcal{K}$ be a partially ordered set which is antisymmetric (as well as reflexive and transitive) considered as a category. Let $X: \mathcal{K} \to \mathcal{K}$ be a process (i.e., an order-preserving map) which also satisfies $AX \leqslant A$ for all A. An element of Dyn(X) is a $\mathcal{K}$-morphism $AX \xrightarrow{\delta} A$, but since $AX \leqslant A$ is guaranteed, the choice of δ is unique for this category $\mathcal{K}$. Moreover, since X is order-preserving, each $\mathcal{K}$-morphism $A \leqslant A'$ is a dynamorphism. Thus U: Dyn(X) $\to \mathcal{K}$ is essentially the identity functor of $\mathcal{K}$, and U^{-1} provides both a left and right adjoint to U. Therefore X is state-behavior. For a particular example, let $\mathcal{K}$ be the set $\{0, A, A', 1\}$ with the partial ordering shown below, and define X by $0X = AX = A'X = 0$, $1X = 1$,



Then $\mathcal{K}$ has all coproducts (suprema) and products (infima). Since 1 is the supremum of A and A' but $1X$ is not the supremum of AX and $A'X$, X does not preserve coproducts. By 2.2, X is not adjoint.

To clarify the import of the above examples, we close this section by showing

that in **Set**, the theory of state-behavior machines is no more general than the "classical" one.

Say that a functor X: **Set** $\to$ **Set** is *classical* if there exists a set X_0 and a natural equivalence $\Gamma: -\times X_0 \to X$; specifically, for each set Q, $Q\Gamma: Q \times X_0 \to QX$ is a bijection and for each function $f: Q \to Q'$ we have

$$\begin{tikzcd} Q \times X_0 \arrow[r, "Q\Gamma"] \arrow[d, "f\times \mathrm{id}"'] & QX \arrow[d, "fX"] \\ Q' \times X_0 \arrow[r, "Q'\Gamma"'] & Q'X \end{tikzcd} .$$

Let 1 denote (any choice of) a one-element set. If X is classical as above, 1Γ identifies X_0 as $1X$; so there is only one candidate for X_0 (up to isomorphism, of course). In fact, if X is any functor and we define $X_0 = 1X$ then there is always a canonical natural transformation $\Gamma: -\times X_0 \to X$ where, thinking of an element of Q as a function $q: 1 \to Q$, $Q\Gamma: Q \times X_0 \to QX$ is defined for any $x \in X_0 = 1X$ by

$$(5.5) \qquad (q, x)Q\Gamma = x(1X \xrightarrow{qX} QX) .$$

Γ is natural precisely because $(q: 1 \to Q)X \cdot (f: Q \to Q')X = (qf: 1 \to Q')X$. Γ enjoys the following universal property. Given an arbitrary set Y_0 and natural transformation $\phi: -\times Y_0 \to X$, there is a unique natural transformation

$$\begin{tikzcd} -\times X_0 \arrow[rr, "\Gamma"] & & X \\ & -\times Y_0 \arrow[ul, dashed, "\psi"] \arrow[ur, "\phi"'] & \end{tikzcd}$$

$\psi: -\times Y_0 \to -\times X_0$ such that $\psi \cdot \Gamma = \phi$. Indeed ψ is constructed as $-\times \phi_0$, where $\phi_0: Y_0 \to X_0$ is the function 1ϕ. The crucial observation is the naturality square

$$\begin{tikzcd} 1 \times Y_0 \arrow[r, "\phi_0"] \arrow[d, "q\times \mathrm{id}"'] & 1X \arrow[d, "qX"] \\ Q \times Y_0 \arrow[r, "Q\phi"'] & QX \end{tikzcd}$$

which proves that ϕ is entirely determined by ϕ_0. This fact is interesting in itself and immediately implies the universal property as advertised. We have also shown:

5.6. Lemma. *Let X: Set → Set be a functor. The following three conditions on X are equivalent.*

(i) *X is classical.*

(ii) *There exists a set Y_0 and a natural transformation $\phi: -\times Y_0 \to X$ such that 1ϕ is bijective.*

(iii) *The canonical natural transformation $\Gamma: -\times X_0 \xrightarrow{\cdot} X$ of 5.6 is a natural equivalence.*

We may now apply this lemma to obtain the promised characterization of adjoint processes in **Set**.

5.7. Theorem. *Let X: Set → Set be a functor. The following three conditions on X are equivalent.*

(i) *X is classical.*

(ii) *X is an adjoint input process.*

(iii) *X preserves coproducts.*

Proof. (i) implies (ii) is the motivating example which was verified in Example 1.10. (ii) implies (iii) is the general principal noted in 2.2.

Turning to the proof that (iii) implies (i), the crucial observation is that for any (even empty) set Q, $(1 \xrightarrow{q} Q : q \in Q)$ is a coproduct diagram (and this is, of course, a very distinguished fact about **Set**). Therefore, by hypothesis, $(X_0 \xrightarrow{qX} QX : q \in Q)$ is a coproduct diagram, where $X_0 = 1X$. This allows us to define the inverse $\tilde{\Gamma}: X \to -\times X_0$ to the canonical $\Gamma: -\times X_0 \to X$ by

$$\begin{tikzcd} X_0 \arrow[r, "qX"] \arrow[dr, "\mathrm{in}_q"'] & QX \arrow[d, dashed, "Q\tilde{\Gamma}"] \\ & Q \times X_0 \end{tikzcd}$$

where in_q sends x to (q, x). The reader can easily verify that $Q\Gamma \cdot Q\tilde{\Gamma} = \mathrm{id}$ and $Q\tilde{\Gamma} \cdot Q\Gamma = \mathrm{id}$. □

A better theorem is

5.8. Theorem. *A functor X: Set → Set is classical if and only if X is a state-behavior input process.*

Proof. As noted preceding 3.7, $X^{@}$ has a right adjoint and so is naturally equivalent to $-\times S$ where $S = 1X^{@}$ by 5.7. Since "free dynamics" is only defined up to isomorphism to begin with, there is no loss of generality in assuming that $X^{@} = -\times S$ as functors. Set $X_0 = 1X$ and define $\Gamma: -\times X_0 \to X$ as in (5.5). By 5.6 it suffices to prove that $Q\Gamma$ is injective and surjective for each Q. Essentially the same square used

following (5.5) proves that an arbitrary natural transformation $\alpha: -\times A \to -\times B$ has the form $-\cdot\psi$ where $\psi: A \to B = 1\alpha$. In particular, if $\eta_1: X \to -\times S$ is as in (3.8), then $\Gamma\eta_1: -\times X_0 \to -\times S = -\times \psi$ where $\psi: X_0 \to S = 1\Gamma\eta_1$. *$Q\Gamma$ is injective*: Let $(q,x) \neq (q',x') \in Q \times X_0$. The diagram

$$\begin{tikzcd}
Q \times X_0 \arrow[r, "Q\Gamma"] \arrow[dr, "\mathrm{id}\times\psi"] \arrow[ddr, bend right, "p'"'] & QX \arrow[d, "Q\eta_1"] \\
& Q \times S \arrow[d, "p"] \\
& Q
\end{tikzcd}$$

(where p, p' are projections) proves that if $q \neq q'$, $(q,x)\,Q\Gamma \neq (q',x')\,Q\Gamma$. Otherwise $q = q'$ but $x \neq x'$. But since

$$\begin{tikzcd}
1 \arrow[r, "q"] \arrow[dr, "\mathrm{id}"'] & Q \arrow[d, "t"] \\
& 1
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
1X \arrow[r, "qX"] \arrow[dr, "\mathrm{id}"'] & QX \arrow[d, "tX"] \\
& 1X
\end{tikzcd}$$

qX is injective and $(q,x)Q\Gamma = \langle x, qX\rangle \neq \langle x', qX\rangle = (q',x')Q\Gamma$.
$Q\Gamma$ is surjective: This is trivial if $Q = \phi$ or $Q = 1$. Otherwise, assume Q possesses two distinct elements q_0, q_1. Fix $r \in QX$. Let $\delta: QX \to Q$ be the characteristic function of r, i.e. $s\delta = q_0$ if $s \neq r$ and $r\delta = q_1$. Recalling (3.10), we have

$$\begin{tikzcd}
Q \times X_0 \arrow[r, "Q\Gamma"] \arrow[rr, bend left, "\mathrm{id}_Q\times\psi"] & QX \arrow[r, "Q\eta_1"] \arrow[dr, "\delta"'] & Q \times S \arrow[d, "\delta^{@}"] \\
& & Q
\end{tikzcd}$$

Define $(q,m) = rQ\eta_1$. Then the naturality diagram

$$\begin{tikzcd}
Q \times X_0 \arrow[r, "Q\Gamma"] \arrow[d, "t \times \mathrm{id}"'] & QX \arrow[r, "Q\eta_1"] \arrow[d, "tX"] & Q \times S \arrow[d, "t \times \mathrm{id}"] \\
X_0 \arrow[r, "1\Gamma = \mathrm{id}"'] \arrow[rr, bend right, "\psi"'] & X_0 = 1X \arrow[r, "1\eta_1"'] & S
\end{tikzcd}$$

(where $t: Q \to 1$) and the observation that $t \times \mathrm{id}$ is a projection establishes that m has the form $x\psi$ for $x = rtX$. As $(q,x)Q\Gamma\delta = (q,x)(\mathrm{id}_Q \times \psi)\delta^{@} = (q, x\psi)\delta^{@}$ $= (q,m)\delta^{@} = rQ\eta_1^{;}\delta^{@} = r\delta = q_1$, $(q,x)Q\Gamma = r$ as desired. □

References

[1] B.D.O. Anderson, M.A. Arbib and E.G. Manes, Foundations of system theory: finitary and infinitary conditions, Spinger Lecture Notes in Economics and Mathematical Systems, to appear.

[2] M.A. Arbib, Theories of Abstract Automata (Prentice-Hall, 1969).

[3] M.A. Arbib and E.G. Manes, Machines in a category: an expository introduction, SIAM Review 16 (1974) 163–192.

[4] M.A. Arbib and E.G. Manes, Fuzzy Machines in a Category, Bull. Australian Math. Soc. 13 (1975) 169–210.

[5] M.A. Arbib and E.G. Manes, Foundations of System Theory: Decomposable Systems, Automatica 10 (1974) 285–302.

[6] M.A. Arbib and H.P. Zeiger, On the Relevance of abstract algebra to control theory, Automatica 5 (1969) 589–606.

[7] E.S. Bainbridge, A Unified minimal realization theory, with duality for machines in a hyperdoctrine (announcement of results), Technical Report, June 1972, Computer and Communication Sciences Department, University of Michigan. (For the full thesis see "A Unified Minimal Realization Theory with Duality", from the same group, November 1972.)

[8] M. Barr, Coequalizers and free triples, Math. Z. 116 (1970) 307–322.

[9] H. Ehrig, K.-D. Kiermeier, M.-J. Kreowski and W. Kühnel, Universal Theory of Automata (Teubner, 1974).

[10] J.A. Goguen, Disrete-time machines in closed monoidal categories, I, J. Comp. Syst. Sci. 10 (1975) 1–43 (condensed version: Minimal realization of machines in closed categories, Bull. Amer. Math. Soc. 78 (1972) 777–783).

[11] R.E. Kalman, P.L. Falb and M.A. Arbib, Topics in Mathematical System Theory (McGraw-Hill, 1969).

[12] G. McCarty, Topology (McGraw-Hill, 1967).

[13] E.G. Manes, Algebraic Theories, Graduate texts in Mathematics 26 (Springer-Verlag) to appear.

[14] M.O. Rabin and D. Scott, Finite automata and their decision problems, IBM. J. Res. and Dev. 3 (1959) 114–125.