A NON-IMMERSION THEOREM FOR REAL PROJECTIVE SPACE

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§1. INTRODUCTION

Let $P_n$ denote real projective $n$-space, with its usual differential structure. The purpose of this note is to prove

**Theorem (1.1).** $P_n$ cannot be immersed in $(2^{r+1} - 1)$-space where

\[
\begin{align*}
n &= 2^r + r + 2 & r &\neq 1 \mod 4 \\
n &= 2^r + r + 3 & r &\equiv 1 \mod 4.
\end{align*}
\]

The method of Stiefel-Whitney classes shows that if $n = 2^r$ then $P_n$, and consequently $P_m$ for $m > n$, cannot be immersed in $(2^{r+1} - 2)$-space. In [1] Atiyah has proved, by the method of exterior powers, that $P_n$ cannot be immersed in $(2n - p)$-space where $p$ is approximately $n/2$. Theorem (1.1) first gives a new result when $n = 22$. Levine [3] and Mahowald [4] proved the analogous non-embedding theorem; $P_n$ cannot be embedded in $2^{r+1}$-space when $n = 2^r + 1$. Our theorem may be true when $n = 2^r + 2$ (if $n = 2^r + 1$ the immersion is possible (5.3) of [6], or see (5.7) of [5]). Improvements on the Stiefel-Whitney class result for complex and quaternion projective space can be obtained from (1.1) by using the fibrations of projective spaces (see (5.2) of [6]). However better results are available in this case [7].

Our methods rely heavily on the work of James, and I am indebted to him for sending me a preprint of [2].

§2. IMMERSIONS AND AXIAL MAPS

Suppose that $P_q$ has its standard cell structure with base point $e$. When $r > q$ regard $P_r$ in the usual way, as a subspace of $P_q$. An axial map of $P_s \times P_t$ into $P_q$ is a map

\[f : P_s \times P_t \to P_q\]

such that if $x \in P_s$ and $y \in P_t$, then

\[f(x, e) = x, \; f(e, y) = y.\]

In this section we shall prove

**Theorem (2.1).** If $P_n$ is immersible in $(n + k)$-space then there exists an axial map of $P_n \times P_n$ into $P_{n+k}$. The converse is true if $n < 2k$. 

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Let $K\tilde{O}(P_n)$ denote the reduced Grothendieck group of classes of real vector bundles over $P_n$. A generator of this group is the class $x$ of the non-trivial line bundle $\xi$ over $P_n$. The group is cyclic of order a power of 2 and multiplicative structure is given by $x^2 = -2x$.

Let $g(rx)$ denote the geometrical dimension, as defined in §1 of [1], of the element $rx \in K\tilde{O}(P_n)$. The proof of (2.1) rests on the following lemma and the work of James [2].

**Lemma (2.2).** $g(rx) \leq k$ if and only if $g((k - r)x) \leq k$.

**Proof.** Suppose $g(rx) \leq k$, then there exists a real vector bundle $\eta$ such that $\theta(\eta) = rx + k$, $\theta$ being the homomorphism defined in [1]. Now $\theta(\eta \otimes \xi) = (rx + k)(x + 1) = (k - r)x + k$. Conversely if there exists a bundle $\zeta$ such that $\theta(\zeta) = (k - r)x + k$ then $\theta(\zeta \otimes \xi) = rx + k$.

The class of the tangent bundle of $P_n$ is $(n + 1)x,$ hence from (2.1) of [6], or see §3 of [1], $P_n$ is immersible in $(n + k)$-space if and only if $g(-(n + 1)x) \leq k$. Applying the lemma, with $r = -(n + 1)$, to this result we have

**Theorem (2.3).** $P_n$ is immersible in $(n + k)$-space if and only if $g((n + k + 1)x) \leq k$.

Recall from [2] that a $t$-field on $P_n$ of tangents to $P_n$ is a continuous function which assigns to each point of $P_n$ a set of $t$ linearly independent tangent vectors to $P_n$ at that point.

**Lemma (2.4).** Suppose $s < q$, then $P_n$ admits a $t$-field of tangents to $P_q$ if and only if $g((q + 1)x) \leq q - t$ on $P_s$.

**Proof.** Suppose $g((q + 1)x) \leq q - t$ on $P_s$. Then there exists a vector bundle $\eta$ such that $\theta(\eta) = (q + 1)x + q - t$, and since $q > s$ the bundle sum of $\eta$ with a trivial vector bundle of dimension $t$ is isomorphic with the restriction of the tangent bundle of $P_q$ to $P_s$. Hence $P_s$ admits a $t$-field of tangents to $P_q$. The converse follows directly from the definition of geometrical dimension.

Combining (2.4) with Theorem (4.1) of [2] we have

**Theorem (2.5).** There exists an axial map of $P_s \times P_t$ into $P_{s+t}$ if $g((q + 1)x) \leq q - t$ on $P_s$. The converse is true if $s < 2(q - t)$.

Setting $s = t = n$ and $q = n + k$ Theorem (2.1) now follows from (2.3) and (2.5).

§3. PROOF OF THEOREM (1.1)

Let $\phi(k, n)$ denote the number of values of $m$ in the range $k \leq m \leq n$ which are congruent to 0, 1, 2 or 4 mod 8. The following theorem is a special case of a theorem proved by James using the Adams $\psi$ operations, (6.2) of [2].

**Theorem (3.1).** Let $C_{n+k,n}$ be odd. If there exists an axial map of $P_n \times P_n$ into $P_{n+k}$ then

\[
\begin{align*}
n + k + 1 & \text{ is a multiple of } 2^{\phi(k,n) - 1} \text{ if } n \not\equiv 3 \mod 4 \\
\text{and} \quad n + k + 1 & \text{ is a multiple of } 2^{\phi(k,n) - 2} \text{ if } n \equiv 3 \mod 4.
\end{align*}
\]
Suppose now that $P_1$ is immersible in $(2^r+3-1)$-space where $n$ is as in (1.1) and $r > 2$ to avoid triviality. It follows from (2.1) and (3.1), since by the dyadic rule $C_{n+k,n}$ is odd, that

$$r + 1 \geq \phi(2^r - r - 3, 2^r + r + 2) - 1 \quad \text{if } r \neq 1 \mod 4$$

and

$$r + 1 \geq \phi(2^r - r - 4, 2^r + r + 3) - 2 \quad \text{if } r \equiv 1 \mod 4.$$

Now $\phi(2^r - r - 3, 2^r + r + 2) = r + 3$ and $\phi(2^r - r - 4, 2^r + r + 3) = r + 4$. This provides a contradiction proving (1.1).

REFERENCES


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