Optimal Computation of Finitely Oriented Convex Hulls

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We define four versions of the "convex hull" of a simple finitely oriented polygon (i.e., a polygon whose edge orientations all belong to some fixed finite set of angles) and give optimal algorithms to find them. Two of these generalize the notions of the orthogonal convex hull of an orthogonal polygon and the traditional "bounding box" of a polygon. Three of the hulls have worst-case time complexity $\Theta(n + f)$ and worst-case space complexity $\Theta(n)$ space, where $n$ is the number of edges of a given polygon and $f (\geq 2)$ is the number of allowed orientations. We also show that testing whether an arbitrary simple polygon is (finitely oriented) convex has worst-case time and space complexity $\Theta(n + f)$ and $\Theta(n)$, respectively. © 1987 Academic Press, Inc.

1. INTRODUCTION AND MOTIVATION

Until very recently the field of computational geometry has mainly concerned itself with only two basic types of polygons, viz. polygons whose edges are restricted to be parallel to either the $x$ or $y$ axis (so called rectilinear, orthogonal, or isothetic polygons) and polygons whose edges are of arbitrary orientation. There seems to be two main reasons for this. The first being that orthogonal polygons are conceptually simple since any interior angle is either 90° or 270° and so algorithms designed for orthogonal polygons are easier to write, understand, and prove correct. The second has more to do with the physical constraints of the usual application areas of computational geometry—most notably VLSI design, geographic databases, computer vision, and computer graphics. These application areas have traditionally used input—output devices and layout schemes based solely on the cartesian coordinate system for almost all graphical computer devices—so, polygons were usually classed as either orthogonal or arbitrary: there was no in between. In a similar way, classical geometry divided the world into either regular polygons or arbitrary polygons: there was nothing in between.

Technology, however, is fast making this an unnatural and burdensome classification; new chip designs use lines with orientations of 45° and 135°.
along with the usual $0^\circ$ and $90^\circ$, and even more choice is expected in the future (see Widmayer et al., 1984). Further, from a theoretical point of view, it is interesting to speculate on what, if anything, makes orthogonal polygons so special: why should there be such a gap in complexity for these two types of polygons? That is, it is frequently the case that if some geometric algorithm is restricted to only orthogonal polygons, then its (worst-case) time bound is linear, whereas if the input is an arbitrary polygon the algorithm tends to be much more expensive. In a sense this is a reflection of the extreme simplicity of orthogonal polygons since a direct case analysis is usually possible, making for very "tight" algorithms.

In an effort to both bridge this complexity gap and provide a theoretical framework for the new tools and techniques of changing technology there have been a number of recent papers in "finite orientation" geometry (see Culberson and Rawlings, 1985; Güting, 1983a, b 1984; Widmayer et al., 1985, 1984). In this paper we define and present optimal algorithms for the natural analogues of one of the first problems studied in computational geometry (see Shamos, 1978), that of finding the convex hull of a simple polygon. Because we feel that this area will come to command much interest in the field as the natural generalization of orthogonal polygons and orthogonal convexity we spend more time in discursive and explanatory material than usual.

In Section 2 we define four versions of the "convex hull" of a polygon, each appropriate under a different set of assumptions. In Section 3 we characterize the various hulls and in Section 4 we demonstrate some elementary relationships between them. We present time and space optimal algorithms to compute them in Section 5 and end in Section 6 with a discussion of further topics of interest in this area.

2. Definitions

All polygons discussed in this paper are simple unless it is explicitly stated otherwise. We assume that polygons are input as a list of vertices in clockwise order where no three consecutive vertices are collinear. We also assume that the plane has a fixed cartesian coordinate system associated to it; thus, the $x$ and $y$ axes are oriented in the positive direction.

First, we define the notion of an orientation and a finite set of orientations quite carefully, since much depends upon these definitions. Given a directed line in the plane its orientation is the angle obtained by rotating the $x$ axis counterclockwise, around their intersection point, until the two lines are collinear and also have the same direction. This is illustrated in Fig. 1. An orientation differs from an angle in that it implicitly includes a
notion of direction with respect to the $x$ axis. However, we will often speak of angle and orientation interchangeably in the remainder of this article.

Let $F$ be a finite set of orientations in the closed-open interval $[0^\circ, 360^\circ)$ such that for all $\theta$ in the range $[0^\circ, 180^\circ)$, $\theta$ is in $F$ if and only if $\theta + 180^\circ$ is in $F$. Clearly, $F$ is symmetric about the horizontal and $|F|$ is always even. We leave for future investigation the case when $F$ is allowed to be asymmetric.

A polygon is said to be finitely oriented with respect to a finite set of angles $F$ if the set of orientations of the edges of the polygon is a subset of $F$. We will treat the set $F$ as fixed for the rest of this paper and speak of “$F$-polygons” when we mean finitely oriented polygons with respect to $F$. Similarly, we shall use the terms “$F$-lines,” “$F$-rays,” and “$F$-segments” to mean lines, rays, and segments whose orientations are in $F$. Note that every undirected $F$-line has two angles associated to it and these differ by $180^\circ$; our convention is that we always choose the smaller of the two.

When $F = \{0^\circ, 90^\circ, 180^\circ, 270^\circ\}$ $F$-polygons are usually called orthogonal (or rectilinear or isothetic) polygons. However, in Appendix I we demonstrate that for all practical purposes when $|F| = 4$, for any $F$, all $F$-polygons may be considered to be orthogonal (Appendix I gives a linear transformation which can be used to convert any such $F$ into $\{0^\circ, 90^\circ, 180^\circ, 270^\circ\}$). To avoid the special cases $|F| = 2$ and $|F| = 0$, we usually assume $|F| \geq 4$ without further comment, although most of the results hold even for these cases.

We say that a polygon is $F$-convex if the intersection of the polygon and any $F$-line is either empty or a line segment. This is a natural generalization of the corresponding notion for orthogonal polygons (see, e.g., Montuno and Fournier, 1982; Nicholl et al., 1983; Ottmann et al., 1984; Sack, 1984; or Wood, 1985). It has already been defined in Widmayer et al. (1985).

In the following we define four versions of the “convex hull” of a (simple)
F-polygon. These variations cover all the possible versions of the “hull”
depending on whether the “hull” is to be convex or F-convex and whether
it is to be an arbitrarily-oriented polygon or an F-polygon.

(1) FCFH(P). Given an F-polygon P we say that an F-polygon Q is
the F-convex F-hull of P if

1. P ⊆ Q,
2. Q is F-convex,
3. Q is the smallest such F-polygon.

(2) FCH(P). Given an F-polygon P we say that a polygon Q is the
F-convex hull of P if

1. P ⊆ Q,
2. Q is F-convex,
3. Q is the smallest such polygon.

(3) CFH(P). Given an F-polygon P we say that an F-polygon Q is
the convex F-hull of P if

1. P ⊆ Q,
2. Q is convex,
3. Q is the smallest such F-polygon.

(4) CH(P). Given an F-polygon P we say that a polygon Q is the
convex hull of P if

1. P ⊆ Q,
2. Q is convex,
3. Q is the smallest such polygon.

In Widmayer et al. (1985), the authors are concerned with the first ver-
sion of the “hull” since, for them, not only must all the polygons under
consideration be F-polygons but also, the only definition of “convexity”
that is applicable to their area of application (VLSI design) is F-convexity;
the reason being that in VLSI design not only are the polygons constrained
to be F-oriented but so are the wires connecting them. The second version
was, initially, included for completeness; however, it turns out to be of
greater importance. This is, simply, because FCH(P) = FCFH(P) when P
is an F-polygon. In other words, we need only require Q to be the smallest
F-convex polygon containing P for it to equal FCFH(P), rather than
requiring Q to be the smallest F-convex F-oriented polygon containing P.
We do not discuss here the effect of dropping the requirement that P be
F-oriented; however, in Rawlins and Wood (1986) we base our discussion
of restricted-oriented convexity (a generalization and unification of F-orien-
ted convexity) on the definition of FCH(P) given here. It is not difficult to
see that $F_{CH}(P)$ "converges" to $CH(P)$ as $|F| \to \infty$; see Rawlins and Wood (1986). As we shall see both $F_{CH}(P)$ and $CF_{CH}(P)$ are easy to compute; therefore, there is no computational nor theoretical reason to prefer one over the other: both are "natural". The third version of the "hull" is, as we shall see, a natural generalization of the (orthogonal) "bounding box" (the smallest orthogonal rectangle which encloses a polygon) which is much used in intersection testing in graphics as an efficient filter. Preparata (private communication) suggested this as a "natural" definition of a convex hull for the orthogonal case, since the bounding box is the smallest convex orthogonal polygon containing the given polygon. Indeed, $CF_{CH}(P)$ also "converges" to $CH(P)$ as $|F| \to \infty$; see Rawlins and Wood (1986). Finally, the fourth version is the usual meaning of the convex hull.

Figure 2 shows the various hulls for an example polygon where $F = \{0^\circ, 45^\circ, 90^\circ, 180^\circ, 225^\circ, 270^\circ\}$.

3. CHARACTERIZING $F$-CONVEX POLYGONS

LEMMA 1. Any simple $F$-polygon with two or more consecutive reflex interior angles cannot be $F$-convex.

Proof. Consider any such $F$-polygon $P$ with three consecutive edges $e_1$, $e_2$, $e_3$ making two reflex interior angles (see Fig. 3).
Clearly there exists an $F$-line whose intersection with $P$ is neither empty nor a line segment, namely any line parallel to $e_2$ and translated a small distance "away from" $P$, since any such line will be an $F$-line and will intersect both $e_1$ and $e_2$.

This means that if any $F$-convex polygon has reflex angles then all are isolated (i.e., surrounded by convex angles). We now show that unlike normal convex polygons, $F$-convex polygons may have reflex angles. Since the set $F$ is fixed we assume that it is kept in sorted order so that given an angle in $F$ we can speak of the "next" angle in $F$ (the successor of the last angle is the first angle).

**Lemma 2.** An $F$-convex polygon $P$ may contain a reflex angle $r$ if and only if the two if the two edges making up $r$ have orientations which are consecutive in $F$. That is, if the edges $e_1$ and $e_2$ have orientations $\theta_1$ and $\theta_2$ then no angle in $F$ lies in the open range $(\theta_1, \theta_2)$.

**Proof.** Lemma 2 established that such a reflex angle must be isolated. Consider such a reflex angle $r$ made by two edges $e_1$ and $e_2$ where without loss of generality $e_1$ is horizontal and $e_2$ has orientation $\theta$ (see Fig. 4).

If there is an orientation in the range $(0, \theta)$ then there exists an $F$-line which intersects both $e_1$ and $e_2$, hence $P$ cannot be $F$-convex.

Conversely, suppose no such orientation exists. If there exists some line which intersects both $e_1$ and $e_2$ then its orientation must lie in the range $(0, \theta)$. Hence it cannot be an $F$-line. Therefore if there is no orientation in the range $(0, \theta)$ then no $F$-line can intersect both $e_1$ and $e_2$. Since all the reflex angles of $P$ are isolated the above argument applies to all reflex angles. Hence $P$ is $F$-convex if and only if each of its reflex angles (if any) are isolated and made by edges with consecutive orientations.

Given two consecutive orientations $\theta_1$ and $\theta_2$ in $F$ we say that the sequence of line segments $l_1, l_2, \ldots, l_m$ is a staircase if

1. $l_i$ meets $l_{i+1}$ and $l_{i-1}$ at its endpoints ($2 \leq i \leq m - 1$),
2. the orientations of $l_1, l_2, \ldots, l_m$ alternate between $\theta_1$ and $\theta_2$,
3. the chain is monotone in both the directions $\theta_1$ and $\theta_2$.

We shall allow $\theta_1$ to equal $\theta_2$ but in that case we restrict $m$ to be 1, so, trivially, any $F$-line segment is a "staircase." Given a staircase made up of

![Figure 4](image-url)
m segments we say that the staircase has length m (see Fig. 5 for examples of staircases of lengths 1, 2, and 4).

Staircases have been previously defined for the special case of orthogonal polygons, see, for example, Nicholl et al. (1983), Sack (1984), and Wood (1985). Using the orthogonal transformation defined in Appendix I we may "lift" any theorems proved for staircases in that special case into theorems for (general) staircases. For example, it is immediate that all of the points on a staircase are visible from a point if and only if the endpoints of the staircase are visible from the point (see Sack, 1984, Sect. 3.2.2, Property 3.3).

Lemma 2 shows that any F-convex polygon may contain many such staircases since it is easy to see that a staircase must consist of alternative reflex and convex angles, and every segment making up the staircase is an F-segment. We now show that an F-convex polygon may be completely characterized by its staircases.

**Lemma 3.** An F-polygon is F-convex if and only if it consists of a sequence of staircases meeting at convex angles.

**Proof.** First, if P is made up only of staircases meeting at convex angles then since all reflex angles are isolated the only way in which P could fail to be F-convex is if some F-line intersects some of the staircases more than once. But this is impossible, since any F-line can only intersect a staircase at most once (unless it is collinear with some part of the staircase), this follows as an easy generalization of the argument used in Lemma 2. Hence such an F-polygon must be F-convex.

Suppose now that the F-polygon P is F-convex. From Lemmas 1 and 2 we know that P may contain reflex angles once they are isolated and made by edges whose orientations are consecutive in F. If P is convex to begin with then it certainly is composed of a sequence of staircases meeting at convex angles since F-segments are by definition staircases. So suppose P contains some reflex angle at the vertex v1 made by two F-line segments with (consecutive) orientations θ1 and θ2 and the vertices of P just before and just after v1 are v0 and v2 (see Fig. 6). We shall show that whenever P contains a length 2 staircase it may contain a staircase of arbitrary length and still be F-convex.

Choose some point in the parallelogram defined by v0, v1, and v2, say x.
Constant the two line segments starting at $x$ with orientations $\theta_1$ and $\theta_2$, respectively, meeting the sides of $P$ at the points $x_1$ and $x_2$ (see Fig. 7).

Clearly, if $P$ was $F$-convex to begin with then if we replace the vertex $v_1$ by the vertices $x_1$, $x$, $x_2$ then $P$ will still be $F$-convex, since any $F$-line intersects a staircase at most once (unless it is collinear). We may repeat this construction on either (or both) of the two new vertices $x_1$ or $x_2$ to place arbitrarily many vertices on a staircase in between the two vertices $v_0$ and $v_2$ (e.g., to replace $x_1$, choose a point in the parallelogram defined by $v_0$, $x_1$, and $x$).

Since we have assumed $P$ to be $F$-convex, we know by Lemma 1 that the reflex angle at $v_1$ is isolated. That is the angles at $v_0$ and $v_2$ are convex. Hence any $F$-convex polygon must consist of a sequence of staircases meeting at convex angles.

Note that (trivially) a convex polygon is always $F$-convex but not conversely.

Lemmas 1, 2, and 3 enable us to prove

**Theorem 1.** If the orientations in $F$ are sorted, then testing a simple $F$-polygon $P$ for $F$-convexity can be accomplished, in the worst case, in $\Theta(n + |F|)$ time and $\Theta(n)$ space, where $n$ is the number of edges of $P$.

**Proof.** Given a simple $F$-polygon $P$ of $n$ edges we first scan it in $O(n)$ time to determine if it has two or more consecutive reflex angles. If it does
then we immediately know that it cannot be $F$-convex by Lemma 1. If we consider an isolated reflex angle made by two edges with orientations $\theta_1$ and $\theta_2$ then we perform a binary search on the (sorted) set $F$ to see whether any orientations lie in the range $(\theta_1, \theta_2)$. If so, then $P$ cannot be $F$-convex. If all the reflex angles in $P$ are isolated and pass this test, then no $F$-line exists which has more than two intersections with $P$, hence $P$ is $F$-convex by Lemma 2. This algorithm takes $O(n \log |F|)$ time in the worst case.

By using Lemma 3 we may reduce the time bound to $O(n + |F|)$ by reconfiguring the algorithm so that it recognizes only $F$-polygons consisting of a sequence of staircases each surrounded by convex angles.

This recognition problem can be solved by first finding one of the staircases (say the one with the first reflex angle in the list of vertices of $P$). Let this be the $i$th vertex in $P$, let the pair of angles making the reflex angle be the $j$th consecutive pair of angles in $F$. If we exhaust $P$ without finding a reflex angle then $P$ is convex and hence $F$-convex. If we find a reflex angle not made by two consecutive angles in $F$ then $P$ cannot be $F$-convex.

Now we scan the vertices of $P$ forward (i.e., clockwise) from the $(i+1)$th vertex looking for another reflex angle. If we find another reflex angle we ask whether it is made by edges whose orientations are consecutive in $F$. To determine this all we need do is examine the current such pair of angles in $F$ (i.e., the $j$th). If not then we ask the same question of the next pair of such angles in $F$ and so on until we exhaust $F$ or find the pair.

If we exhaust $F$ in this way then the current reflex angle was made by two edges whose orientations are not consecutive in $F$, hence, by Lemma 2, $P$ cannot be $F$-convex. If we find such a pair before exhausting $F$ then we continue scanning the vertices of $P$ looking for reflex angles, again if a reflex angle is found then we start scanning the pairs of angles in $F$ starting from the last one found until we either exhaust $F$ or find the pair. If at any time we exhaust $F$ we know that $P$ cannot be $F$-convex, if we exhaust the vertices of $P$ without exhausting the angle pairs of $F$ then we know that $P$ is made up of a sequence of staircases meeting at convex angles, and hence $P$ is $F$-convex by Lemma 3.

Finding the first reflex angle and the orientations of its edges takes $O(n + \log |F|)$ time (worst case), and each new angle can “use up” at most $|F|$ angles in $F$. However, note that at each step of the algorithm we either get rid of an angle in $F$ or a vertex in $P$ in constant time, therefore in total the algorithm requires no more than $O(n + |F|)$ time.

To see that this is optimal consider an $F$-convex $F$-polygon which consists of alternating convex and reflex angles (see Fig. 8). Clearly such a polygon will be non-$F$-convex if any of the pairs of $F$-line segments making up a reflex angle are non-consecutive in $F$, and so any algorithm is forced to examine each possible angle in $F$ at least once. Also
any algorithm must examine each edge of the $F$-polygon at least once thus it must take $\Omega(n + |F|)$ to test any $F$-polygon for $F$-convexity.

Note that the characterization of the $F$-convex $F$-hull as a sequence of staircases is a direct generalization of the case for orthogonal polygons (this result is stated but not proved in Widmayer et al., 1985).

4. RELATIONSHIPS BETWEEN THE HULLS

**Lemma 4.** If $P$ is an $F$-polygon, then

$$P \subseteq \text{FCH}(P) \subseteq \text{FCFH}(P) \subseteq \text{CFH}(P)$$

and

$$P \subseteq \text{FCH}(P) \subseteq \text{CH}(P) \subseteq \text{CFH}(P)$$

**Proof sketch.** To see this it is only necessary to observe that the following statements are equivalent: a polygon is the smallest of some given type containing $P$ and a polygon is the intersection of all polygons, of the same type, containing $P$. Further, a convex polygon is certainly $F$-convex and an $F$-polygon is certainly a polygon hence the above containments follow almost directly from the definition.

It is possible to show that for any $F$-polygon $P$ all the “hulls” exist and that in fact two of the hulls are always equal when $P$ is $F$-oriented. Note that if $P$ is convex, then all of the above containments become equalities.

**Lemma 5.** If $P$ is an $F$-polygon, then $\text{FCH}(P) = \text{FCFH}(P)$.

**Proof sketch.** It is only necessary to observe that if $\text{FCH}(P)$ is not $F$-oriented then it has some non-$F$-oriented edge. This edge cannot be an edge of $P$ since $P$ is an $F$-polygon. Hence there is some gap between the edge and the polygon, and the area of this gap can be decreased by replacing a part of the edge by two edges with orientations nearest the edge's
orientation (i.e., a staircase with edge orientations "surrounding" the orientation of the non-$F$-oriented edge). Clearly a replacement of part of an non-$F$-oriented edge by such a staircase preserves the $F$-convexity of the hull. We have decreased the area of $FCH(P)$ and we still have an $F$-convex polygon which contains $P$, hence we have obtained a contradiction.

Note that the above proof fails if $P$ is not restricted to being $F$-oriented. These relationships hold even when $P$ is not necessarily $F$-oriented except that, as is easy to show, $FCFH(P)$ may not exist.

5. Computation of the Hulls

Using the algorithms already developed for the orthogonal convex hull as our starting point leads to $O(n|F|)$ algorithms to find both the $F$-convex $F$-hull and the convex $F$-hull respectively, since all of the algorithms presented so far (Montuno and Fournier, 1982; Nicholl et al., 1983; Ottmann, 1984; Sack, 1984; Wood, 1985) essentially decomposed the problem into establishing monotonicity in first one then the other orthogonal direction. This leads naturally to a straightforward $O(n|F|)$ algorithm when we have $|F|$ orientations. That this can be bettered is due to the surprisingly high degree of structure possessed by the $F$-convex $F$-hull and the convex $F$-hull, respectively (note that from Bhattacharya and El Gindy, 1984; Graham and Yao, 1983; Lee, 1983; McCallum and Avis, 1979, we know that $CH(P)$ can be found in $\Theta(n)$ time, while the complexity of $FCH(P)$ is, of course, the same as that of $FCFH(P)$).

From the lemmas of the previous section we know that the $F$-convex $F$-hull of a simple $F$-polygon $P$ ($FCFH(P)$) must be the smallest $F$-polygon which both contains $P$ and is composed of a sequence of staircases meeting at convex angles. If we were to find the convex hull of $P$, how would we have to modify it to obtain $FCFH(P)$? Clearly any non-$F$-oriented edge in $CH(P)$ cannot be an edge in $FCFH(P)$, but it is easy to see that the boundary of $FCFH(P)$ must "fit between" the boundaries of $CH(P)$ and $P$. In fact, we can prove

**Lemma 6.** If $P$ is $F$-oriented then all the extremal vertices of $P$ (i.e., all the vertices of $P$ which are also vertices of $CH(P)$) are also vertices of $FCFH(P)$ and the edges of $FCFH(P)$ in between any two consecutive extremal vertices of $P$ form a staircase.

Proof sketch. The proof of this assertion is modelled on the proof that the convex hull of a simple polygon $P$ is identical to the intersection of all halfspaces containing $P$, except in this case halfspaces are delineated by staircases extended to infinity at both ends. The proof is elementary and is
omitted (for a proof outline of the lemma for convex hulls see Grünbaum, 1967).

There is an analogous result for the CFH(P).

We shall adapt the “gift wrapping” method used by Jarvis (1973) to find FCFH(P) in $O(n + |F|)$ time. Consider how to construct the portion of the hull (i.e., FCFH(P)) lying between two consecutive extremal vertices of P (i.e., two vertices of P which are consecutive on CH(P)). Let the two vertices be $v_i$ and $v_j$, where $j \leq i + 1$. Let the symbol $L(v_i, v_j)$ stand for the line segment joining the two vertices. Let $\theta(L)$ be the orientation of the line $L$. Clearly if $\theta(L(v_i, v_j))$ is in $F$ then all we need do to process these two vertices is to add them both to the list of vertices in the FCFH already found. (Note that if $j = i + 1$ then we always add the two vertices to the list since $P$ is an F-polygon.) Otherwise suppose that $\theta_1 < \theta(L(v_i, v_j)) < \theta_2$. Applying the orthogonal transformation defined in Appendix I we transform all the vertices in between $v_i$ and $v_j$ so that $\theta_1$ becomes horizontal and $\theta_2$ becomes vertical. Now imagine walking along the perimeter of $P$ from $v_i$ to $v_j$ (see Fig. 9).

If any vertex is obstructed in the x direction from $v_i$ or in the y direction from $v_j$ then it cannot be on the list of vertices belonging to the FCFH(P). It is now easy to construct an algorithm to step along the vertices in between each pair of consecutive extremal vertices of $P$ and find the vertices of $P$ which belong in the FCFH(P) (see Appendix II) since—as in the proof of Theorem 1—we only need exhaust a particular pair of consecutive orientation at most twice. So at each step we either discard one of the vertices of the polygon or we discard one of the possible orientations in $F$ in constant time per step.

The FCH(P) can be computed in a similar manner.

Note that although both hulls may require the introduction of Steiner vertices, the number of vertices cannot increase. Also it is not necessary to store them (i.e., the “phantom” vertices) since they are reconstructible.
Hence we have proved the following theorem:

**Theorem 2.** If \( P \) is an \( F \)-oriented polygon then each of the four hulls can be found optimally in the following time and space bounds:

1. \( \text{FCFH}(P) \) in \( \Theta(n + |F|) \), by Theorem 1 and the previous argument (see Appendix II for more justification).
2. \( \text{FCH}(P) \) in \( \Theta(n + |F|) \), by Lemma 5 and (1).
3. \( \text{CFH}(P) \) in \( \Theta(n + |F|) \), the lower bound follows from the observation that some input polygons can force us to at least look at all \( |F| \) orientations, the upper bound is a minor variation on the walk idea sketched for the \( \text{FCFH}(P) \).
4. \( \text{CH}(P) \) in \( \Theta(n) \), see Bhattacharya and El Gindy (1984), Graham and Yao (1983), Lee (1983), or McCallum and Avis (1979).

6. **Discussion**

Probably the single-most important observation arising from this work is that there are some problems for which \( F \)-oriented polygons buy you as much as the more restrictive orthogonal polygons. Indeed, in a very real sense problems involving \"\( F \)-convexity\" are basically orthogonal problems, that is, they can be decomposed so that at any point in time we need only consider four orientations. The first open problem then is to somehow classify a priori which class of problems is so decomposable.

This investigation gives rise to many interesting questions. Two of the most obvious ones are What are the purely metric and purely topological properties of the metric space induced by \( F \)?

For example, in the metric space induced by \( F \), staircases are the natural analogues of straight lines in Euclidean space, in that the shortest \( F \)-distance between two points is a staircase and an \( F \)-line meets a staircase at most one point (unless they are collinear). However, the intersection of two staircases is either empty or a set of disconnected points and \( F \)-line segments: unlike the simpler case for straight lines. Further staircases can be non-intersecting without being parallel! Also two points may define exactly one \( F \)-line or an infinity of staircases. We hope to make this geometry more explicit in the future.

Unlike convex sets, \( F \)-convex sets are not closed under affine transformations (unless we also transform \( F \)). But many other properties of convex sets carry over directly; for example, Theorem: The intersection of two \( F \)-convex sets is \( F \)-convex and Theorem: Given an \( F \)-convex set \( P \) and an exterior point \( x \) the set consisting of all line segments from \( x \) to each point in \( P \) is again \( F \)-convex. Clearly not all theorems on convex sets carry over
FINITELY ORIENTED CONVEX HULLS

directly to $F$-convex sets, the open question being to determine which ones do. Some first steps in solving this question are to be found in Rawlins and Wood (1986).

APPENDIX I

In attempting to classify the set of all finitely-oriented polygons over some set of angles $F$ we considered and solved the following problem. For this problem we treat lines, rays, and segments as undirected, and so we always refer to the smaller of the two possible "orientations," thus, $F$ consists solely of orientations in the range $[0^\circ, 180^\circ)$.

**Problem.** Given a collection $C$ of lines and line segments in the plane each of which has one of $f$ possible orientations chosen from the set $F = \{\theta_1, \theta_2, \ldots, \theta_f\}$ where $0 \leq \theta_i < \pi$, $\theta_i \neq \theta_j$, $\forall 1 \leq i \neq j \leq f$. Find an affine transformation$^1$ $T = [\alpha \beta]$ which maps each line of orientation $\theta_i$ in the collection $C$ into a unique line of orientation $\phi_i$ in the collection $C'$ such that $C'$ is a collection of lines and line segments with orientations in the set $F' = \{\phi_1, \phi_2, \ldots, \phi_f\}$, where $0 \leq \phi_i < \pi$, $\phi_i \neq \phi_j$, $\forall 1 \leq i \neq j \leq f$, and all intersections are preserved with none introduced. In essence we want a linear transformation that will send a line (or line segment) with orientation $\theta_i$ into a line (or line segment) with orientation $\phi_i$ while preserving all intersections that the line (line segment) makes.

**Solution.** This problem is solvable for $f = 1, 2, 3$ but is unsolvable for $f > 4$.

**Case 1.** $f = 1$, a simple rotation: send all lines of orientation $\theta$ to lines of orientation $\phi$. Note that if $\theta = \phi$, then $T$ is just the identity transformation.

$$T = \begin{bmatrix} \cos(\theta - \phi) & \sin(\theta - \phi) \\ -\sin(\theta - \phi) & \cos(\theta - \phi) \end{bmatrix}.$$  

**Case 2.** $f = 2$, a skew: send all horizontal lines to horizontal lines but send all lines of orientation $\theta$ to lines of orientation $\phi$ where $\theta \neq 0$ and $\phi \neq 0$. Note that if $\theta = \phi$, then, again, $T$ is just the identity transformation. Using the first transformation to rotate the original collection of lines means that (speaking loosely) any two nonequal angles may be transformed into any other two nonequal angles. If $\phi = \pi/2$ we shall refer to $T$ as the orthogonal transformation. $T$'s existence means that to solve any

$^1$ See Gans (1969) for a thorough treatment of geometric transformations.
geometric problem on 2-oriented lines it suffices to solve the problem only for orthogonals.

$$T = \begin{bmatrix} \sin \theta & 0 \\ \cos \phi - \cos \theta & \sin \phi \end{bmatrix}.$$

Case 3. $f = 3$: this was obtained by construction and solving for the minimal solution to the appropriate system of linear equations for $T$; it proves that any three pairwise nonequal angles are equivalent to any other three pairwise nonequal angles.

$$T = \begin{bmatrix} \sin \phi_1 \sin \phi_2 \sin(\theta_1 - \theta_2) \\ \sin \phi_1 \sin \theta_2 \cos \phi_2 \cos \theta_1 \\ -\sin \phi_2 \sin \theta_1 \cos \phi_1 \cos \theta_2 \sin \theta_1 \sin \theta_2 \sin(\phi_1 - \phi_2) \end{bmatrix}.$$

This transformation maps horizontal lines to horizontal lines, lines of orientation $\theta_1$ to lines of orientation $\phi_1$ and lines of orientation $\theta_2$ to lines of orientation $\phi_2$ simultaneously. Note that if $\theta_1 = \phi_1$ and $\theta_2 = \phi_2$ then $T$ is just the identity transformation.

Case 4. $f \geq 4$: In this case no such linear transformation exists as may be proved by examining the number of constraints. For $f = 4$ (or more) we have an overdetermined system of equations defining $\alpha, \beta, \gamma, \text{and } \delta$ and no solution is possible. This result is a special case of the more general theorem on the maximum number of linearly independent points in an $n$-dimensional vector space. Informally speaking, no linear transformation of the plane can map a four (or more) sided polygon into any other four- (or more) sided polygon.

**APPENDIX II**

Assume that all the vertices have been “orthogonalized” so that the two nearest orientations to the line segment $L(v_i, v_j)$ are $0^\circ$ and $90^\circ$. Further assume that the polygon has been rotated so that the first extremal vertex, $v_i$, is “southwest” of the second extremal vertex, $v_j$, that is, both the $x$ and $y$ component of $v_i$ are less than those $v_j$.

The algorithm is straightforward and makes use of a stack to keep track of all the vertices potentially on the hull. It does this by a simple two-pass, one-stack algorithm which first eliminates all vertices which cannot be seen from the $x$ direction, then all vertices which cannot be seen from the $y$ direction.
Algorithm Walk

Input: a sequence of vertices of a finitely oriented polygon with only the first and last (i.e., $v_i$ and $v_j$) extremal in the polygon (i.e., they lie on the convex hull).

Output: the sequence of all vertices in between the first and last which lie on the $F$-convex $F$-hull of the polygon.

\begin{verbatim}
begin
  MAKENULL(Stack);
  PUSH($v_i$, Stack);
  $i := i + 1$;
  while $i \neq j$
    begin
      /*if the new vertex is visible in y direction*/
      /*then add it to the list of potential vertices*/
      if $v_j \cdot y \geq$ TOP(Stack) \cdot $y$ then PUSH($v_j$, Stack);
      $i := i + 1$;
    end;
  while not EMPTY(Stack)
    begin
      /*output all vertices which are visible in the x direction*/
      /*throw away any vertices not visible in x since they cannot be on the $F$-hull*/
      write(TOP(Stack));
      POP(previous, Stack);
      while previous \cdot $x <$ TOP(Stack) \cdot $x$
        POP(junk, Stack);
    end;
end.
\end{verbatim}

Although we do not do it here it is a simple matter to prove the correctness of this algorithm once it is realized that, because of the way the polygon has been transformed, in going from one vertex to the next it is impossible to have the $x$ coordinate increase (decrease) while the $y$ coordinate decreases (increases). It will always be the case that if the $x$ coordinate is larger (smaller), then so is the $y$ coordinate. Using this precondition it is easy to see that (1) no vertex which should be on the hull is discarded in either pass and (2) no vertex which should be discarded is left in the list at the end of the algorithm.
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