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Note

Embedding graphs as isometric medians

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Abstract

We show that every connected graph can be isometrically (i.e., as a distance preserving subgraph) embedded in some connected graph as its median. As an auxiliary result we also show that every connected graph is an isometric subgraph of some Cayley graph.

Keywords: Distance; Median; Isometric; Embedding

1. Introduction

Given a finite connected graph G, we denote the geodesic (i.e., shortest path) distance of two vertices x, y by $\rho_G(x, y)$. The *distance sum* of a vertex x of G is $\sigma_G(x) := \sum_{y \in V(G)} \rho_G(x, y)$. The *median subgraph* (or simply *median*) of G, denoted by M_G, is the subgraph of G induced by the vertices of minimum distance sum. An old result of Slater [7] states that any graph H is the median of some connected graph G. Slater's paper was followed by several others [2,3,5] that were concerned with finding bounds (in terms of various parameters of H) for the order of a smallest graph G containing H as median.

In view of the fact that the median is defined in terms of the metric it is natural to ask, in the case where the given graph H is connected, whether the embedding of H as a median can be so arranged that the metrics of H and the ambient graph G coincide, i.e. that H is an isometric subgraph of G. (Recall that a connected subgraph H of a connected graph G is *isometric* in G if $\rho_H(x, y) = \rho_G(x, y)$ for any $x, y \in V(H)$.) The embeddings constructed in all the papers mentioned above are such that in the target graph G the vertices of H are at distance ≤ 2 from each other, destroying the metric structure of H as soon as the diameter of H exceeds 2.

In the present note we show that any connected graph H can be *isometrically* embedded in some connected graph G as its median. Our construction yields a graph G of order $O((2r)^n)$, where n is the order of H and r its diameter, leaving open the question of an ambient graph G whose order is polynomial in n. This is in sharp contrast to the case for graphs constructed in the earlier papers, where H is not required to be the isometric median of G (in fact, as shown in [2], G can be chosen so as to have order $< 2n$).

All graphs considered in this paper are finite and simple.

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2. Isometric embeddings in vertex-transitive graphs

In the proof of our main result (Theorem 3.1) we make use of the fact that any connected graph can be isometrically embedded in a graph all of whose vertices have the same distance sum. One way of achieving this is to make sure that the target graph of the embedding is vertex-transitive. That such an embedding always exists is probably folklore; we present here a proof which is a slight variation of the Nowakowski–Rival embedding theorem which says that every connected graph can be isometrically embedded in a strong product of paths ([6], and for strong products in general see Imrich and Klavžar [4], Chapter 5).

Theorem 2.1. *Any connected graph can be isometrically embedded in a strong torus (i.e., a strong product of cycles).*

Strong products of vertex-transitive graphs being vertex-transitive, it follows that strong tori are vertex-transitive. In fact, they are Cayley graphs based on direct products of cyclic groups.

Proof. Let H be a connected graph with vertices $x_1, \ldots, x_n$, and denote the diameter of H by r. Consider the cycle C whose vertex-set is the cyclic group $\mathbf{Z}_s$, where $s \geq 2r$, two vertices i, j being adjacent if and only if $i - j = \pm 1$, and let G be the strong product of n copies of C. Note that $V(G) = \mathbf{Z}_s^n$. Define a map $\varphi : V(H) \longrightarrow \mathbf{Z}_s^n$ as follows. Given $x_i \in V(H)$, define the j-th coordinate of $\varphi(x_i)$ to be the distance $\rho_H(x_i, x_j)$, $i, j = 1, \ldots, n$. As in the proof of the Nowakowski–Rival theorem it is then a matter of straightforward checking that φ is an isometric embedding of H in G. □

Remark 2.2. The map φ used in the preceding proof can be thought of as first mapping H isometrically into the strong product P of n paths of length r (invoking the theorem of Nowakowski–Rival in its original form), and then embedding P into a strong torus G which is the strong product of sufficiently long cycles so that the embedding $P \longrightarrow G$ can be made isometric (whence the condition that $s \geq 2r$). The number of factors in P could be reduced from n to d, the strong isometric dimension of H (as defined by Fitzpatrick and Nowakowski in [1]). Taking s as small as possible the strong isometric dimension of the torus then is rd. It would be interesting to know whether any H can always be isometrically embedded in a vertex-transitive graph whose strong isometric dimension only depends on that of H.

3. Isometrically embedded medians

The following is the main result of this paper.

Theorem 3.1. *Given any connected graph H, there exists a connected graph G whose median is an isometric subgraph which is isomorphic to H.*

We first establish a result concerning the median subgraph of a partial cartesian product (Proposition 3.4). From this, Theorem 3.1 follows as an easy corollary.

Definition 3.2 (*Partial Cartesian Product*). Let G, H be graphs, A a subset of $V(G)$. By $G \Box_A H$ we denote the graph P defined by

$$V(P) = V(G) \times V(H),$$
$$(x, y)(x', y') \in E(P) \iff xx' \in E(G),\ y = y'; \quad \text{or } x = x' \in A,\ yy' \in E(H).$$

Remark 3.3. (1) The subgraph P_A of P induced by $A \times V(H)$ is $G_A \Box H$, where G_A is the subgraph of G induced by A. In particular, for $A = V(G)$, P is the full cartesian product $G \Box H$. If $A \neq \emptyset$ and G and H are connected, then so also is P.

(2) If G, G_A, and H are connected, and G_A is isometric in G, then P_A is isometric in P. This follows from the fact that if $(u, v), (u', v') \in A \times V(H)$, then $\rho_{P_A}((u, v), (u', v')) = \rho_{G_A}(u, u') + \rho_H(v, v')$.

Proposition 3.4. *Given:* (1) *a connected graph G which is its own median;*

(2) *a non-empty isometric subgraph F of G;*

(3) *a connected graph H of order ≥ 2.*

Let $P = G \Box_A H$, where $A = V(F)$. Then $M_P = F \Box M_H$. Moreover, if M_H is isometric in H, then M_P is isometric in P.

Proof. We use the following notation:

- $n_G := |V(G)|, n_H := |V(H)|$;
- given $u \in A$, the H-fiber H_u is the copy of H in P with $V(H_u) = \{u\} \times V(H)$; for $v \in V(H)$, the G-fiber G_v is defined analogously.

By hypothesis, all vertices of G have the same distance sum, say s. Put $\min_{v \in V(H)} \sigma_H(v) =: s_H$.

Let $u \in A, u' \in V(G)$, and $v, v' \in V(H)$. Then P contains a $(u, v)(u', v')$-geodesic which consists of a segment in the fiber H_u from (u, v) to (u, v'), followed by a segment in the fiber $G_{v'}$ from (u, v') to (u', v'). Hence $\rho_P((u, v)(u', v')) = \rho_G(u, u') + \rho_H(v, v')$ as in the full product $G \Box H$, and therefore

$$\sigma_P(u, v) = n_H \sigma_G(u) + n_G \sigma_H(v) \geq n_H s + n_G s_H, \tag{1}$$

with equality if and only if $v \in V(M_H)$.

Now let $u \notin A, u' \in V(G)$, and $v, v' \in V(H), v \neq v'$. Because F is isometric in G, $F \Box H$ is isometric in P; hence P contains a $(u, v)(u', v')$-geodesic which consists of a segment in the fiber G_v from (u, v) to some vertex (w, v), $w \in A$, then a segment in H_w from (w, v) to (w, v'), and finally a segment in $G_{v'}$ from (w, v') to (u', v'). Hence

$$\begin{aligned} \rho_P((u, v), (u', v')) &= \min_{w \in A} \left(\rho_G(u, w) + \rho_H(v, v') + \rho_G(w, u') \right) \\ &\geq \rho_G(u, u') + \rho_H(v, v'), \end{aligned}$$

and for $u' = u$ this inequality is strict. Therefore, since $n_H \geq 2$,

$$\sigma_P(u, v) > n_H \sigma_G(u) + n_G \sigma_H(v) \geq n_H s + n_G s_H. \tag{2}$$

(1) and (2) together imply that $\sigma_P(u, v)$ attains its minimum if and only if $(u, v) \in A \times V(M_H)$. Hence $M_P = F \Box M_H$.

The statement that M_P is isometric in P if M_H is isometric in H follows from Remark 3.3(2). □

Proof of Theorem 3.1. At variance with the notation in the statement of the theorem we continue to use the notation of Proposition 3.4. Let a connected graph F be given. By Theorem 2.1 there is a connected vertex-transitive graph G which contains F as an isometric subgraph. Being vertex-transitive, G coincides with its median. Applying Proposition 3.4 to G and any connected graph H of order ≥ 3 whose median is a single vertex v (e.g. a star $K_{1,r}, r \geq 2$) it follows that the median of $P = G \Box_A H$ is the cartesian product of F with the single vertex v, and hence isomorphic to F. M_P is isometric in P because v, considered as a subgraph of H, is isometric. □

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