Abstract

In [Math. Mag. 64 (1991) 325–332], Schwenk has completely determined the set of all integers \( m \) and \( n \) for which the \( m \times n \) chessboard admits a closed knight’s tour. In this paper, (i) we consider the corresponding problem with the knight’s move generalized to \((a, b)\)-knight’s move (defined in the paper, Section 1). (ii) We then generalize a beautiful coloring argument of Pósa and Golomb to show that various \( m \times n \) chessboards do not admit closed generalized knight’s tour (Section 3). (iii) By focusing on the \((2, 3)\)-knight’s move, we show that the \( m \times n \) chessboard does not have a closed generalized knight’s tour if \( m = 1, 2, 3, 4, 6, 7, 8 \) and 12 and determine almost completely which \( 5k \times m \) chessboards have a closed generalized knight’s tour (Section 4). In addition, (iv) we present a solution to the (standard) open knight’s tour problem (Section 2).

© 2005 Elsevier B.V. All rights reserved.

Keywords: Generalized knight’s tour; Rectangular chessboard; Hamiltonian graph

1. Introduction

An intriguing old puzzle in recreational mathematics is that of finding a closed tour for the knight on the standard \( 8 \times 8 \) chessboard. The knight moves one square in a single direction, either horizontally or vertically, and then followed by two squares perpendicular to it. According to [15], this easily understood problem has its history that dates back to the time of Euler and De Moivre. The problem has been extended to any \( m \times n \) rectangular
chessboard but a complete solution was available only recently. It was Schwenk who proved the following:

**Theorem 1** (Schwenk [16]). The $m \times n$ chessboard with $m \leq n$ admits a closed knight's tour unless one or more of the following conditions holds:

(i) $m$ and $n$ are both odd;
(ii) $m = 1, 2$ or $4$; or
(iii) $m = 3$ and $n = 4, 6$ or $8$.

We observe, in passing, that other problems concerning knight’s tour have also been discussed (see [5]). In [21], Watkins and Hoenigman consider knight’s tours on the torus. It turns out, unexpectedly, that some of the knight’s tours on the torus, when restricted to square chessboards, give rise to magic squares (see [1]). The knight’s tour problem has also been considered on cylinders and other surfaces [19] and on chessboards of other shapes, for example the triangular honeycomb [6,18]. In the meantime, a problem concerning the number of knight’s tours on the square chessboard has also received due consideration [10]. More about the knight’s tour (and other) problems on chessboard are available in the recent book [20] by Watkins.

Knight’s moves are amenable to generalization. We consider the following one. Suppose the squares of the $m \times n$ chessboard are $(i, j)$ where $1 \leq i \leq m$ and $1 \leq j \leq n$. A move from square $(i, j)$ to square $(k, l)$ is termed an $(a, b)$-knight’s move if $|k-i|, |l-j| = a, b$.

For a given $(a, b)$-knight’s move on an $m \times n$ chessboard, there is associated with it a graph whose vertex set and edge set are \{(i, j) | 1 \leq i \leq m, 1 \leq j \leq n\} and \{(i, j)(k, l) | 1 \leq i, k \leq m, 1 \leq j, l \leq n, |k-i|, |l-j| = a, b\}, respectively. Let $G((a, b), m, n)$ denote this graph, or just $G(m, n)$ for simplicity if the move $(a, b)$ is understood or not to be emphasized.

A closed $(a, b)$-knight’s tour is a series of $(a, b)$-knight’s moves that visits every square of the $m \times n$ chessboard exactly once and then returns to the starting square. The **generalized knight’s tour problem** asks: which $m \times n$ chessboards admit a closed $(a, b)$-knight’s tour? This amounts to asking: which graph $G((a, b), m, n)$ is Hamiltonian?

We shall make a few easy observations. First, if $a + b$ is even, then no closed $(a, b)$-knight’s tour is possible because only cells of the same color (that is either all black or all white cells) are covered during the moves. Thus $a + b$ is assumed to be odd. Also, we shall assume that $a < b$ since an $(a, b)$-knight’s move and a $(b, a)$-knight’s move are the same.

Next, if $m$ and $n$ are both odd, then no closed $(a, b)$-knight’s tour is possible because $G(m, n)$ is then a bipartite graph with an odd number of vertices $mn$.

We may further assume that $m \leq n$. If $m \leq a + b - 1$, then no closed $(a, b)$-knight’s tour on the $m \times n$ chessboard is possible. This is because the vertex $(a, 1)$ in $G(m, n)$ is of degree $\leq 1$. Suppose $n < 2b$. Then the vertex $(b, b)$ is of degree $0$.

We summarize the above observations in the following:

**Theorem 2.** Suppose the $m \times n$ chessboard admit a closed $(a, b)$-knight’s tour, where $a < b$ and $m \leq n$. Then

(i) $a + b$ is odd;
(ii) $m$ or $n$ is even;
(iii) \( m \geq a + b \); and
(iv) \( n \geq 2b \).

Perhaps the simplest generalized knight’s move is that of the \((0, 1)\)-knight’s move. In this case, the associated graph \( G(m, n) \) is the horizontal grid whose hamiltonicity is easily decided. As for the \((0, b)\)-knight’s move, where \( b \geq 3 \) is odd, the associated graph \( G(m, n) \) is disconnected. Henceforth, we shall assume that \( 1 \leq a < b \).

2. Open knight’s tour on rectangular boards

In [16], Schwenk mentioned that the corresponding problem for the open knight’s tour can also be solved by the same method he has introduced. The solution was left as a challenge to the interested readers. In this section, we provide a complete solution to the open knight’s tour problem. Earlier, Cull and de Curtins [3] proved that every \( m \times n \) chessboard with \( 5 \leq m \leq n \) admits an open knight’s tour.

**Theorem 3** (Cull and de Curtins [3]). Every \( m \times n \) chessboard with \( 5 \leq m \leq n \) admits an open knight’s tour.

The case \( m = 3 \) was considered in [14] where Van Rees showed that the \( 3 \times n \) chessboard admits an open knight’s tour if and only if \( n = 4 \) or \( n \geq 7 \). Here, we shall present the solution for the missing case \( m = 4 \) as well as some constructions for the open knight’s tours on the \( 3 \times n \) chessboard.

We shall make use of the following necessary condition for the existence of a Hamiltonian path in a graph. If \( H \) is a graph, we let \( \omega(H) \) denote the number of components in \( H \).

**Theorem 4.** Let \( S \) be a proper subset of the vertex set of a graph \( G \). If \( G \) contains a Hamiltonian path, then

\[
\omega(G - S) \leq |S| + 1.
\]

**Theorem 5.** The \( m \times n \) chessboard with \( m \leq n \) admits an open knight’s tour unless one or more of the following conditions holds:

(i) \( m = 1 \) or \( 2 \);
(ii) \( m = 3 \) and \( n = 3, 5, 6 \); or
(iii) \( m = 4 \) and \( n = 4 \).

**Proof.** Both \( G(3, 3) \) and \( G(m, n) \) for \( m \leq 2 \) are disconnected and hence do not have Hamiltonian paths.

For the remaining part on the non-existence of Hamiltonian paths, we shall make use of Theorem 4. Fig. 1(a) shows a disconnected graph with seven components. It is the result of removing five vertices \((1, 2), (1, 4), (2, 3), (3, 2), \) and \((3, 4)\) from the graph \( G(3, 5) \).
Fig. 1 (b) is the resulting disconnected graph with six components when the four vertices \((j, 2)\) and \((j, 3)\) for \(j=2, 3\) are removed from the graph \(G(4, 4)\). Fig. 1 (c) shows the resulting disconnected graph with eight components when the six vertices \((i, 3)\) and \((i, 4)\) for \(i=1, 2, 3\) are removed from the graph \(G(3, 6)\). By Theorem 4, all three graphs \(G(3, 5), G(4, 4)\) and \(G(3, 6)\) do not contain Hamiltonian paths.

Next, we show that every other board admits an open knight’s tour. Fig. 2 depicts a Hamiltonian path in \(G(m, n)\) for each \(n \in \{4, 7, 8, 9\}\) and in \(G(4, k)\) for each \(k \in \{5, 6, 7\}\).

Let \(P(m, n)\) denote a Hamiltonian path in \(G(m, n)\). We shall show that each \(P(3, n)\), for \(n \in \{7, 9\}\), in Fig. 2 is extendable to a \(P(3, n+4)\) and each \(P(4, k)\), for \(k \in \{5, 6, 7\}\), in Fig. 2 is extendable to a \(P(4, k+3)\). This can be done by placing the graphs \(S(3, 4)\) (a subgraph of \(G(3, 4)\)) and \(S(4, 3)\) (a subgraph of \(G(4, 3)\)) on the right-hand side of \(P(3, n)\) and \(P(4, k)\), respectively, and joining them by suitable edges as explained below. The graphs \(S(3, 4)\) and \(S(4, 3)\) are shown in Fig. 3(a) and (b), respectively.

For the case \(m = 3\), note that each of the \(P(3, n)\), for \(n \in \{7, 9\}\), has \((1, n)\) and \((2, n-1)\) as end vertices. Joining the vertices \((1, n)\) and \((2, n-1)\) of \(P(3, n)\) to the vertices \((3, 1)\) and \((1, 1)\) of \(S(3, 4)\), respectively, yields a Hamiltonian path in \(G(3, n+4)\) with \((1, n+4)\) and \((2, n+3)\) as end vertices. The extension of a Hamiltonian path in \(G(3, 7)\) to a Hamiltonian path in \(G(3, 11)\) is shown in Fig. 3(a). Repeat the process, we obtain a Hamiltonian path in \(G(3, n)\) for every odd \(n \geq 7\). For the case where \(n \geq 10\) is even, Schwenk’s result (Theorem 1) implies that \(G(3, n)\) contains a Hamiltonian path.
For the case $m = 4$, note that each of the $P(4, k)$, for $k \in \{5, 6, 7\}$, has $(1, k)$ and $(4, k)$ as end vertices. Joining these two vertices to the vertices $(3, 1)$ and $(2, 1)$ of $S(4, 3)$, respectively, yields a Hamiltonian path in $G(4, k + 3)$ with $(1, k + 3)$ and $(4, k + 3)$ as end vertices. The extension of a Hamiltonian path in $G(4, 5)$ to a Hamiltonian path in $G(4, 8)$ is shown in Fig. 3(b). Repeat the process, we obtain a Hamiltonian path in $G(4, n)$ for every $n \geq 5$.

By Theorem 3, $G(m, n)$ contains a Hamiltonian path for every $m \geq 5$. This completes the proof. \qed
3. Forbidden rectangular boards

In this section, we show that certain rectangular chessboards do not admit a closed generalized knight’s tour. The first two results generalize that of Pósa (see [16]) and Golomb [5] which states that the $4 \times n$ chessboard does not admit a closed $(1, 2)$-knight’s tour.

**Theorem 6.** Suppose $m = a + b + 2t + 1$ where $0 \leq t \leq a - 1$. Then the $m \times n$ chessboard admits no closed $(a, b)$-knight’s tour.

**Proof.** As $a + b$ is odd, we may write $a + b = 2s + 1$. Then $a \leq s$ and $b > s$ because $a < b$.

Let $r = \frac{m}{2} = s + t + 1$ and let the vertices of the $m \times n$ chessboard $B$ be colored using $r$ distinct colors $c_1, c_2, \ldots, c_r$ in the following manner.

If $1 \leq i \leq s + t + 1$, then vertices in the $i$th row of $B$ are colored with $c_i$. If $s + t + 2 \leq i \leq m$, then vertices in the $i$th row of $B$ are colored with $c_{m+1-i}$.

Since the case $a + b = 3$ (where $a = 1$ and $b = 2$) has been settled by Pósa (and also Golomb [5]) and discussed in [16], we may assume that $a + b \geq 5$ (so that $s \geq 2$).

Consider vertices in the $(t+1)$th row. They are all colored with $c_{t+1}$. Moreover these vertices are adjacent only to the vertices in the $(a+t+1)$th and $(b+t+1)$th rows because $0 \leq t \leq a - 1$.

Since $a + t + 1 \leq s + t + 1$ and $b + t + 1 > s + t + 1$, vertices in these two rows are colored with $c_{a+t+1}$.

Now, look at those vertices in the $(m-t)$th row. They are colored with $c_{t+1}$. Moreover these vertices are adjacent only to the vertices in the $(a + t + 1)$th and $(b + t + 1)$th rows
which are colored $c_{a+t+1}$ (as explained earlier). This means that vertices which are colored $c_{t+1}$ together with their neighbors force a proper subcycle and the proof is complete. □

**Lemma 1.** Suppose the vertices of an $m \times n$ chessboard $B$ are colored in equal amount with two colors, red and blue. Suppose further that every red vertex is adjacent only to the blue vertices and that at least one blue vertex is adjacent to a blue vertex. Then $B$ admits no closed $(a, b)$-knight’s tour.

**Proof.** Suppose that there is a closed $(a, b)$-knight’s tour $C = v_1v_2\ldots v_{mn}v_1$ of $B$. Since $B$ contains an equal amount of vertices of each color and a red vertex must always be sandwiched by two blue vertices, the red and blue vertices must alternate around $C$. Let all the odd-labelled vertices $v_{2r+1}$ be colored in red and all the even-labelled vertices $v_{2r}$ be colored in blue. But from the original coloring of the chessboard $B$ with black and white, we may conclude that all the vertices $v_{2r+1}$ are also white. Thus all red vertices are white vertices, but this contradicts the different pattern chosen for the two colorings. We conclude that no closed $(a, b)$-knight’s tour is possible. □

Pósa’s and Golomb’s theorem can also be generalized to the following:

**Theorem 7.** Suppose $m = a(k + 2l)$ where $1 \leq l \leq \frac{k}{2}$. Then the $m \times n$ chessboard admits no closed $(a, ak)$-knight’s tour, where $a$ is odd and $k$ is even.

**Proof.** The proof is reminiscent of that of Pósa.

First note that, as $a + ak = a(1 + k)$ is odd (by Theorem 2), $a$ is odd and $k$ is even.

Next, let $B$ be an $m \times n$ chessboard. For each $i = 1, 2, \ldots, k + 2l$, let $A_i$ denote the $a \times n$ chessboard which consists of the $((i-1)a+1)$th, $((i-1)a+2)$th, $\ldots$, $i$th rows of $B$. In other words, $B$ is partitioned into $k + 2l$ sub-chessboards $A_1, A_2, \ldots, A_{k+2l}$ each of size $a \times n$.

Now, let the vertices of $B$ be colored with two colors in the following manner:

- For $1 \leq i \leq k$, let the vertices in $A_i$ be colored with red if $i$ is odd and with blue otherwise.
- For $k + 1 \leq i \leq k + 2l$, let the vertices in $A_i$ be colored with blue if $i$ is odd and with red otherwise.

Consider the vertices in the $j$th row. They are adjacent only to the vertices in the $(j \pm a)$th and the $(j \pm ak)$th rows. Note that not all the four rows are always possible. For example, if $j \leq a$, then the $(j - a)$th and the $(j - ak)$th rows do not exist.

Suppose the $j$th row belongs to $A_i$. Then the $(j + a)$th and the $(j - a)$th rows belong to $A_{i+1}$ and $A_{i-1}$, respectively. Also, the $(j + ak)$th and the $(j - ak)$th rows belong to $A_{i+k}$ and $A_{i-k}$, respectively.

Suppose $1 \leq i \leq k$. Then a vertex in the $j$th row is not adjacent to a vertex in the $(j - ak)$th row (since there is no $A_{i-k}$ sub-chessboard).

If $i$ is odd, then the vertices in $A_i$ are colored with red whereas the vertices in $A_{i+1}$ and $A_{i-1}$ are colored with blue. Since $k + i$ is odd and $k + i \geq k + 1$, the vertices in $A_{i+k}$ are colored with blue.

If $i$ is even, then the vertices in $A_i$ are colored with blue. Clearly, the vertices in $A_{i+1}$ are colored with red. Since $k + i$ is even and $k + i \geq k + 1$, the vertices in the $A_{i+k}$ are colored
with red. The vertices in $A_{i+1}$ are colored with red when $i < k$, but they are colored with blue when $i = k$.

Now, suppose $k + 1 \leq i \leq k + 2l$. Then, a vertex in the $j$th row is not adjacent to a vertex in the $(j + ak)$th row (since there is no $A_{i+k}$ sub-chessboard).

If $i$ is even and $i < k + 2l$ then the vertices in $A_i$ are colored with red and the vertices in $A_{i+1}$ and $A_{i-1}$ are colored with blue. Since $i - k$ is even and $i - k \leq k$, the vertices in $A_{i-k}$ are colored with blue. If $i = k + 2l$, then the vertices in $A_{k+2l}$ are adjacent only to the vertices in $A_{k+2l-1}$ and $A_{2l}$ which are both colored with blue.

If $i$ is odd, then the vertices in $A_i$ are colored with blue. Clearly, the vertices in $A_{i+1}$ and $A_{i-k}$ are colored with red. The vertices in $A_{i-1}$ are colored with red when $i > k + 1$, but they are colored with blue when $i = k + 1$.

Thus, we may make the conclusion that every red vertex in $B$ is adjacent only to the blue vertices; however there is a blue vertex that is adjacent to a blue vertex. By Lemma 1, no closed $(a, ak)$-knight’s tour is possible. \hfill \Box

**Theorem 8.** Suppose $m = 2(ak + l)$ where $1 \leq k \leq l \leq a$. Then the $m \times n$ chessboard admits no closed $(a, a + 1)$-knight’s tour.

**Proof.** Let $B$ be an $m \times n$ chessboard. As $k \leq l$, we have $m > k(2a + 1)$. Partition the first $k(2a + 1)$ rows of vertices into $k$ sub-chessboards $A_1, A_2, \ldots, A_k$, each of size $(2a + 1) \times n$. For each $A_i, i = 1, 2, \ldots, k$, we shall color the first $a$ rows of vertices with red and the next $a + 1$ rows of vertices colored with blue. Note that in the chessboard $B$, we have $ak$ rows of vertices colored with red, $k(a + 1)$ rows of vertices colored with blue and $2l - k$ rows uncolored.

Let $D$ denote the $(2l - k) \times n$ sub-board that contains all the uncolored vertices of $B$. As $l \geq k$, we have $2l - k = k + s$ for some $s \geq 0$. Clearly, $s$ is even. We shall color the first $k + \frac{s}{2}$ rows of vertices in $D$ with red and the remaining $\frac{s}{2}$ rows of vertices with blue. The number of vertices colored with red in $B$ is now equal to the number of vertices colored with blue.

Consider the vertices in the $j$th row. They are adjacent only to the vertices in the $(j \pm a)$th and the $(j \pm (a + 1))$th rows. Note that not all the four rows are always possible. For example, if $j \leq a$, then the $(j - a)$th and the $(j - a - 1)$th rows do not exist.

Suppose the $j$th row belongs to $A_i$, for some $i = 1, 2, \ldots, k$. If the $j$th row is colored red, then the $(j \pm a)$th and the $(j \pm (a + 1))$th rows are colored blue. So, every vertex colored with red in $A_i$ is adjacent only to vertices colored with blue.

Suppose the $j$th row belongs to $D$. Since $k + \frac{s}{2} \leq a$, every vertex colored with red in $D$ can only be adjacent to vertices colored in blue.

Consider a vertex in the $(a + 1)$th row. It is colored with blue and is adjacent to a vertex in the $(2a + 1)$th row which is also colored with blue.

Thus, we may make the conclusion that every red vertex in $B$ is adjacent only to the blue vertices; however there is a blue vertex that is adjacent to another blue vertex. By Lemma 1, $B$ does not admit a closed $(a, a + 1)$-knight’s tour. \hfill \Box

The previous three results deal with forbidden boards of size $m \times n$ with $m$ even. The next result considers a case where the move is $(a, a + 1)$ and $m$ is odd. However, the result is not enjoyed by the $(1, 2)$-knight’s move.
Theorem 9. Suppose \( m = 2a + 2t + 1 \) where \( 1 \leq t \leq a - 1 \). Then the \( m \times n \) chessboard admits no closed \((a, a + 1)\)-knight’s tour.

Proof. Let \( A_u \) (respectively, \( A_l \)) denote the \( a \times a \) sub-board located at the upper (respectively, lower) left corner of the \( m \times n \) chessboard. It is easy to see that vertices in \( A_u \) or \( A_l \) are of degree 2 in \( G(m, n) \).

Consider the vertex \((a + t + 1, a + 2)\). It is adjacent to the vertices \((t + 1, 1), (t, 2)\) and \((2a + t + 1, 1)\). Clearly, \((t + 1, 1)\) and \((t, 2)\) belong to \( A_u \). Since \( 1 \leq t \leq a - 1 \), it is easy to see that \((2a + t + 1, 1)\) belongs to \( A_l \). Hence \((a + t + 1, a + 2)\) is adjacent to three vertices of degree 2 and thus \( G(m, n) \) is non-Hamiltonian. \( \square \)

4. (2, 3)-knight’s move

In this section, we shall confine our attention to the \((2, 3)\)-knight’s move. Clearly, if \( m \leq 4 \), then the \( m \times n \) chessboard admits no closed \((2, 3)\)-knight’s tour (by Theorem 2). By Theorem 8, no closed \((2, 3)\)-knight’s tour is possible if \( m \) is 6, 8 or 12. By Theorem 9, there is no closed \((2, 3)\)-knight’s tour on the \( 7 \times n \) chessboard.

Corollary 1. If \( m \leq 4 \) or \( m = 6, 7, 8, 12 \), then the \( m \times n \) chessboard does not admit a closed \((2, 3)\)-knight’s tour.

It is thus natural to look at the smallest undecided case which is the \( 5 \times n \) chessboard. In fact, in the rest of the paper, we determine the values of \( n \) for which the \( 5k \times n \) chessboard, except for the \( 5 \times 18 \), admits a closed \((2, 3)\)-knight’s tour. The result is summarized in Theorem 10. It is very likely that the \( 5 \times 18 \) chessboard admits no closed \((2, 3)\)-knight’s tour but we are unable to show it.

Similar question could also be asked for the \( 9k \times n \) and \( 11k \times n \) cases, but a full account (if available) may have to appear elsewhere.

Proposition 1. Suppose \( n \neq 18 \). Then the \( 5 \times n \) chessboard admits a closed \((2, 3)\)-knight’s tour if and only if \( n \geq 16 \) is even.

Proof. Since \( G(5, n) \) is a bipartite graph, \( n \) must be even in order that \( G(5, n) \) is hamiltonian.

If \( n \leq 4 \), then clearly \( G(5, n) \) is non-Hamiltonian because the board is not wide enough to permit a closed \((2, 3)\)-knight’s tour.

If \( n = 6, 8 \) or 12, Corollary 1 shows that \( G(5, n) \) is non-Hamiltonian.

If \( n = 10 \), the fact that \( G(5, 10) \) is non-Hamiltonian is easily seen. The two vertices \((3, 2)\) and \((3, 8)\) are both of degree 2 and they force a 4-cycle \((3, 2)(5, 5)(3, 8)(1, 5)(3, 2)\) in \( G(5, 10) \).

For \( n = 14 \), suppose \( G(5, 14) \) contains a Hamiltonian cycle \( C(5, 14) \). Then the path \((2, 13)(5, 11)(3, 14)(1, 11)(4, 13)\) must be part of \( C(5, 14) \) because \((3, 14), (2, 13)\) and \((4, 13)\) are vertices of degree 2. This implies that the path \( P_1 = (1, 5)(3, 8)(5, 3)\) must also be part of \( C(5, 14) \) because the neighbors of \((3, 8)\) are \((1, 11), (5, 11), (1, 5)\) and \((5, 5)\).
Since \((3, 2)\) is of degree 2, the path \(P_2 = (1, 5)(3, 2)(5, 5)\) must also be part of \(C(5, 14)\).

But then \(P_1 \cup P_2\) is a 4-cycle in \(C(5, 14)\), a contradiction.

We now show that every other board admits a closed \((2, 3)\)-knight’s tour. This is done by first showing that some smaller boards contain Hamiltonian cycles and then use these to build up Hamiltonian cycles in bigger boards.

Fig. 4 depicts a Hamiltonian cycle each in \(G(5, n)\) for \(n \in \{16, 20, 24, 26\}\). These Hamiltonian cycles are indicated by the sequences of consecutive integers from 1 to 5n. Let these Hamiltonian cycles be denoted \(C(5, n)\), \(n \in \{16, 20, 24, 26\}\).

For each \(t \in \{11, 19, 21\}\), let \(R_t\) denote the subgraph of \(G(5, t)\) depicted in Fig. 5. Note that each \(R_t\) consists of three disjoint paths whose union includes all the vertices in \(G(5, t)\), \(t \in \{11, 19, 21\}\). Let \(u \rightarrow v\) denote a path whose end vertices are \(u\) and \(v\). We further note that the three disjoint paths in \(R_t\) are \(x_1 \rightarrow x_2\), \(y_1 \rightarrow y_2\) and \(z_1 \rightarrow z_2\) where \(x_1 = (1, t)\), \(x_2 = (4, t - 2)\), \(y_1 = (3, t - 2)\), \(y_2 = (4, t - 1)\), \(z_1 = (4, t)\) and \(z_2 = (5, t)\).
Now suppose there is a subgraph of $G(5, s)$, denoted $L_s$, which consists of three disjoint paths whose union includes all the vertices in $G(5, s)$. Suppose further that the end vertices of these paths are $x_1, x_2, \beta_1, \beta_2, \gamma_1$ and $\gamma_2$. Moreover, these end vertices are such that, when $R_t$ is placed on the left hand side of $L_s$, there is a $(2, 3)$-knight’s move from $x_i$ to $z_i$, from $y_i$ to $\beta_i$ and from $z_i$ to $\gamma_i$, $i = 1, 2$. It is easy to see that if the three paths in $L_s$ are

(i) $x_1 - \gamma_1, \gamma_2 - \beta_2, \beta_1 - x_2$,
(ii) $x_1 - \gamma_1, \gamma_2 - \beta_1, \beta_2 - x_2$ or
(iii) $x_1 - \beta_2, \beta_1 - \gamma_1, \gamma_2 - x_2$,

then we have a Hamiltonian cycle, denoted $R_t + L_s$, in $G(5, t + s)$. This is illustrated in Fig. 6.

We now show the existence of the graphs $L_s$ which meet the above conditions for every $s = 11 + 6k$ where $k \geq 0$. Note that $R_{11} + L_s$ takes care of $n = 22, 28, 34, \ldots$; $R_{19} + L_s$ takes care of $n = 30, 36, 42, \ldots$; and $R_{21} + L_s$ takes care of $n = 32, 38, 44, \ldots$.

The graphs $L_{17}$ and $L_{23}$ are depicted in Fig. 7. They satisfy conditions (i) and (ii) above, respectively. We shall use these two graphs to build up $L_{11+6k}$. For this purpose, let $B_{12}$ denote the spanning subgraph of $G(5, 12)$ which is depicted in Fig. 7. Note that $B_{12}$ consists
Fig. 6. $R_t + L_s$.

Fig. 7. The graphs $L_{11}$, $L_{17}$, $L_{23}$ and $B_{12}$.

of five disjoint paths $x_1 - \gamma_1$, $x_2 - a_2$, $\beta_1 - b_1$, $\beta_2 - b_2$ and $\gamma_2 - c_2$ together with the isolated vertex $z = (2, 11)$. Here, $x_1 = (4, 2)$, $x_2 = (2, 1)$, $\beta_1 = (5, 1)$, $\beta_2 = (2, 2)$, $\gamma_1 = (1, 2)$, $\gamma_2 = (3, 3)$, $a_2 = (5, 11)$, $b_1 = (4, 12)$, $b_2 = (5, 12)$ and $c_2 = (1, 12)$. 
To obtain $L_{29}$, place $B_{12}$ on the left-hand side of $L_{17}$. Then add six new edges $z\beta_1, z\alpha_1, a_2\alpha_2, b_1\gamma_1, b_2\beta_2$ and $c_2\gamma_2$. Note that $L_{29}$ satisfies condition (i) above. Continue the process, we obtain $L_{17+12k}$ which satisfies condition (i) above for any $k \geq 0$.

Similarly, we obtain $L_{23+12k}$ which satisfies condition (ii) above for any $k \geq 0$.

To complete the proof, we need to construct $L_{11}$. This graph is depicted in Fig. 7. Note that $L_{11}$ satisfies condition (iii) above. $\square$

**Proposition 2.** The $10 \times n$ chessboard admits a closed $(2, 3)$-knight’s tour if and only if $n \geq 10$ and $n \neq 12$.

**Proof.** By Proposition 1 and Corollary 1, the graph $G(10, n)$ is non-Hamiltonian for $n \leq 8$ or $n = 12$.

For $n = 9$, suppose $G(10, 9)$ contains a Hamiltonian cycle $C(10, 9)$. Then the paths $(2, 2)(4, 5)(2, 8), (10, 2)(8, 5)(10, 8)$ and the edge $(1, 9)(3, 6)$ must be a part of $C(10, 9)$ because $(2, 2), (2, 8), (10, 2), (10, 8)$ and $(1, 9)$ are vertices of degree 2 in $G(10, 9)$. This implies that the edge $(1, 3)(3, 6)$ must also be included in $C(10, 9)$, but then the vertex $(6, 8)$ cannot be included since it has only one available edge $(9, 6)(6, 8)$, a contradiction.

Next, we shall show that $G(10, n)$ is Hamiltonian for every other value of $n$. Fig. 8 depicts a Hamilton cycle $C(10, n)$ in $G(10, n)$ for $n \in \{10, 11, 13, 14, 17\}$. Note that each $C(10, n)$ in Fig. 8 contains the edges $e_1 = (1, n)(4, n-2), e_2 = (1, n-2)(4, n)$ and $e_3 = (3, n-2)(6, n)$.

Fig. 9 shows a subgraph of $G(10, 5)$, denoted $S(10, 5)$, which consists of three disjoint paths $P_1 = a_1 - a_2$, $P_2 = b_1 - b_2$ and $P_3 = c_1 - c_2$ whose end vertices are $a_1 = (1, 1), a_2 = (8, 3), b_1 = (2, 1), b_2 = (3, 3), c_1 = (3, 1)$ and $c_2 = (2, 3)$. Note that $V(P_1) \cup V(P_2) \cup V(P_3) = V(G(10, 5))$.

The process of extension is to replace each edge $e_i, i = 1, 2, 3$, in $C(10, n)$ by a path $P_j$ for some $j$ such that $1 \leq j \leq 3$, and obtain an extension of a Hamiltonian cycle in $G(10, n + 5)$ for $n \in \{10, 11, 13, 14, 17\}$.

Place $S(10, 5)$ on the right-hand side of a $C(10, n)$. Remove the edge $e_1 = (1, n)(4, n-2)$ from $C(10, n)$ and join $(1, n)$ and $(4, n-2)$ to the vertices $b_2$ and $b_1$ of $S(10, 5)$, respectively. Next, remove the edge $e_2 = (1, n-2)(4, n)$ from $C(10, n)$ and join $(1, n-2)$ and $(4, n)$ to the vertices $c_1$ and $c_2$ of $S(10, 5)$, respectively. Finally, remove the edge $e_3 = (3, n-2)(6, n)$ from $C(10, n)$ and join $(3, n-2)$ and $(6, n)$ to the vertices $a_1$ and $a_2$ of $S(10, 5)$, respectively. Thus, we obtain a Hamiltonian cycle $C(10, n + 5)$ which also includes the edges $(1, n+5)(4, n+3), (1, n+3)(4, n+5)$ and $(3, n+3)(6, n+5)$. The extension of a $C(10, 10)$ to a $C(10, 15)$ is shown in Fig. 10.

Repeating the above construction, we obtain a Hamiltonian cycle in $G(10, n)$ for each $n \geq 10$ and $n \neq 12$. $\square$

**Proposition 3.** Suppose $k \geq 3$ is an integer. Then the $5k \times n$ chessboard admits a closed $(2, 3)$-knight’s tour if and only if

(i) $n \geq 10$ is even and $n \neq 12$ when $k$ is odd, or
(ii) $n = 5, 9, 10, 11$ or $n \geq 13$ when $k$ is even.
Proof. First, we note that, by Corollary 1, the $5k \times n$ chessboard does not admit a closed $(2, 3)$-knight’s tour if $n \leq 4$ or $n = 6, 7, 8, 12$. Further, if $k$ is odd, then the $5k \times n$ chessboard does not admit a closed $(2, 3)$-knight’s tour if $n \leq 9$ or if $n$ is odd (by Theorem 2).
Next, we show that every other $5k \times n$ chessboard admits a closed $(2, 3)$-knight’s tour. The following construction shall be used throughout.

Construction $(\ast)$: Suppose $G(m, n)$ has a Hamiltonian cycle $C(m, n)$ which contains the edges $(1, 1)(3, 4)$ and $(m - 2, 3)(m, 6)$. Take a copy of $C_t = C(m_t, n)$ and a copy of $C_b = C(m_b, n)$. Place $C_b$ below $C_t$. Delete the edge $(m_t - 2, 3)(m_t, 6)$ (respectively, $(1, 1)(3, 4)$) from $C_t$ (respectively, $C_b$). Joining the vertex $(m_t - 2, 3)$ (respectively, $(m_t, 6)$) of $C_t$ to the vertex $(1, 1)$ (respectively, $(3, 4)$) of $C_b$, we obtain a Hamiltonian cycle $C(m_t + m_b, n)$ in $G(m_t + m_b, n)$ which contains the edges $(1, 1)(3, 4)$ and $(m_t + m_b - 2, 3)(m_t + m_b, 6)$.

Case (1): $k$ is odd

Suppose $n \geq 16$ is even and $n \neq 18$. Note that every Hamiltonian cycle $C(5, n)$ constructed in Proposition 1 contains the edges $(1, 1)(3, 4)$ and $(3, 3)(5, 6)$. Take two copies of $C(5, n)$ and place one above the other. By the construction $(\ast)$, we obtain a Hamiltonian cycle in $G(10, n)$ which contains the edges $(1, 1)(3, 4)$ and $(8, 3)(10, 6)$. Repeating the construction $(\ast)$ by taking $C_t = C(10, n)$ and $C_b = C(5, n)$, we have a Hamiltonian cycle $G(5k, n)$ which contains the edges $(1, 1)(3, 4)$ and $(5k - 2, 3)(5k, 6)$ for $k \geq 3$ and $n \geq 16$ is even except $n = 18$.

Suppose $n \in \{10, 14, 18\}$. The required Hamiltonian cycles $C(10, 10)$, $C(10, 14)$ and $C(15, 14)$, $C(15, 18)$ are shown in Figs. 8 and 11, respectively. Now, $C(10, 18)$ can be
constructed by using the method described in the proof of Proposition 2 while $C(15, 10)$ can be obtained by taking a 90° clockwise rotation on the Hamiltonian cycle $C(10, 15)$ of Fig. 10. Note that, all these Hamiltonian cycles $C(5s, n)$ contain the edges $(1, 1)(3, 4)$ and $(5s - 2, 3)(5s, 6)$ for $s = 2, 3$ and $n \in \{10, 14, 18\}$. Now, by taking $C_i = C(15, n)$ and $C_{b} = C(10, n)$ and applying the construction (*), we obtain a Hamiltonian cycle in $G(5k, n)$ for all odd $k \geq 3$ and $n = 10, 14, 18$.

Case (2): $k$ is even

In this case, $5k \equiv 0 \pmod{10}$.

For $n = 5$, $C(10i, 5)$ can be obtained by a 90° clockwise rotation on the Hamiltonian cycle $C(5, 10i)$ (constructed in Proposition 1), where $i \geq 2$.

For $n = 9$, note that the Hamiltonian cycles $C(20, 9)$ and $C(30, 9)$ in Fig. 12 both contain the edges $(1, 1)(3, 4)$ and $(10i - 2, 3)(10i, 6)$ where $i = 2, 3$. As such, these two Hamiltonian cycles can be used to obtain a Hamiltonian cycle in $G(10i, 9)$ for $i \geq 2$ by the construction (*).

For $n \geq 10$ and $n \neq 12$, note that all the Hamiltonian cycles obtained in the proof of Proposition 2 contain the edges $(1, 1)(3, 4)$ and $(10i - 2, 3)(10i, 6)$. So, by the construction (*), we have a Hamiltonian cycle in $G(10i, n)$ for $i \geq 1, n \geq 10$ and $n \neq 12$.

This completes the proof. □

Putting all the above propositions together, we have the following result.

**Theorem 10.** The $5k \times n$ chessboard where $(5k, n) \neq (5, 18)$ admits a closed $(2, 3)$-knight’s tour if and only if
Fig. 11. Hamiltonian cycles $C(15, 14)$ and $C(15, 18)$.

(i) $k = 1$ and $n \geq 16$ is even; or
(ii) $k = 2$ and $n \geq 10$ and $n \neq 12$; or
(iii) $k \geq 3$ is odd and $n \geq 10$ is even and $n \neq 12$; or
(iv) $k \geq 4$ is even and $n = 5, 9, 10, 11$ or $n \geq 13$. 

\begin{align*}
&\begin{array}{cccccccccccccccccccc}
1 & 154 & 47 & 172 & 209 & 32 & 3 & 156 & 59 & 22 & 55 & 34 & 187 & 58 \\
152 & 27 & 182 & 45 & 170 & 5 & 50 & 29 & 184 & 57 & 36 & 19 & 52 & 189 \\
173 & 206 & 31 & 2 & 155 & 48 & 23 & 208 & 33 & 186 & 157 & 60 & 21 & 54 \\
181 & 44 & 151 & 26 & 207 & 30 & 185 & 6 & 49 & 20 & 53 & 100 & 37 & 18 \\
69 & 132 & 85 & 180 & 203 & 40 & 75 & 106 & 63 & 162 & 119 & 136 & 77 & 124 \\
175 & 204 & 41 & 68 & 133 & 64 & 161 & 96 & 135 & 76 & 123 & 62 & 163 & 118 \\
84 & 179 & 70 & 131 & 86 & 105 & 202 & 39 & 74 & 107 & 78 & 125 & 120 & 137 \\
99 & 144 & 149 & 176 & 101 & 166 & 89 & 8 & 111 & 164 & 115 & 192 & 197 & 16 \\
142 & 87 & 10 & 147 & 178 & 95 & 130 & 81 & 14 & 113 & 194 & 91 & 128 & 199 \\
71 & 102 & 83 & 12 & 145 & 140 & 73 & 104 & 201 & 196 & 93 & 138 & 79 & 126 \\
148 & 177 & 100 & 143 & 88 & 9 & 112 & 165 & 90 & 129 & 198 & 15 & 114 & 193 \\
11 & 146 & 141 & 72 & 103 & 82 & 13 & 94 & 139 & 80 & 127 & 200 & 195 & 92 \\
\end{array} \\
C(15,14) \\
\begin{array}{cccccccccccccccccccc}
252 & 245 & 4 & 257 & 250 & 269 & 140 & 55 & 152 & 9 & 262 & 111 & 142 & 53 & 94 & 147 & 48 & 89 \\
258 & 249 & 270 & 253 & 244 & 5 & 136 & 59 & 288 & 141 & 54 & 115 & 148 & 47 & 90 & 143 & 52 & 95 \\
3 & 256 & 251 & 246 & 139 & 56 & 153 & 8 & 261 & 112 & 151 & 10 & 263 & 110 & 49 & 88 & 93 & 146 \\
241 & 18 & 75 & 156 & 13 & 60 & 107 & 20 & 37 & 80 & 25 & 42 & 105 & 127 & 100 & 85 & 122 \\
16 & 33 & 22 & 63 & 154 & 159 & 132 & 35 & 82 & 11 & 78 & 109 & 130 & 125 & 102 & 27 & 40 & 87 \\
74 & 155 & 240 & 17 & 34 & 21 & 12 & 79 & 108 & 131 & 36 & 81 & 26 & 41 & 86 & 121 & 126 & 101 \\
175 & 230 & 235 & 160 & 177 & 64 & 165 & 270 & 223 & 200 & 189 & 218 & 69 & 194 & 213 & 202 & 207 & 184 \\
\end{array} \\
C(15,18)
Fig. 12. Hamiltonian cycles $C(20, 9)$ and $C(30, 9)$.

Acknowledgements

The authors wish to thank the referees for the invaluable suggestions.
References


Further reading