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Generalized knight's tours on rectangular chessboards

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Abstract

In [Math. Mag. 64 (1991) 325–332], Schwenk has completely determined the set of all integers m and n for which the $m \times n$ chessboard admits a closed knight's tour. In this paper, (i) we consider the corresponding problem with the knight's move generalized to (a, b)-knight's move (defined in the paper, Section 1). (ii) We then generalize a beautiful coloring argument of Pósa and Golomb to show that various $m \times n$ chessboards do not admit closed generalized knight's tour (Section 3). (iii) By focusing on the $(2, 3)$-knight's move, we show that the $m \times n$ chessboard does not have a closed generalized knight's tour if $m = 1, 2, 3, 4, 6, 7, 8$ and 12 and determine almost completely which $5k \times m$ chessboards have a closed generalized knight's tour (Section 4). In addition, (iv) we present a solution to the (standard) open knight's tour problem (Section 2).

Keywords: Generalized knight's tour; Rectangular chessboard; Hamiltonian graph

1. Introduction

An intriguing old puzzle in recreational mathematics is that of finding a closed tour for the knight on the standard 8×8 chessboard. The knight moves one square in a single direction, either horizontally or vertically, and then followed by two squares perpendicular to it. According to [15], this easily understood problem has its history that dates back to the time of Euler and De Moivre. The problem has been extended to any $m \times n$ rectangular

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chessboard but a complete solution was available only recently. It was Schwenk who proved the following:

Theorem 1 (*Schwenk [16]*). *The $m \times n$ chessboard with $m \leqslant n$ admits a closed knight's tour unless one or more of the following conditions holds*:

(i) *m and n are both odd*;
(ii) $m = 1, 2$ *or* 4; *or*
(iii) $m = 3$ *and* $n = 4, 6$ *or* 8.

We observe, in passing, that other problems concerning knight's tour have also been discussed (see [5]). In [21], Watkins and Hoenigman consider knight's tours on the torus. It turns out, unexpectedly, that some of the knight's tours on the torus, when restricted to square chessboards, give rise to magic squares (see [1]). The knight's tour problem has also been considered on cylinders and other surfaces [19] and on chessboards of other shapes, for example the triangular honeycomb [6,18]. In the meantime, a problem concerning the number of knight's tours on the square chessboard has also received due consideration [10]. More about the knight's tour (and other) problems on chessboard are available in the recent book [20] by Watkins.

Knight's moves are amenable to generalization. We consider the following one. Suppose the squares of the $m \times n$ chessboard are (i, j) where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. A move from square (i, j) to square (k, l) is termed an (a, b)-*knight's move* if $\{|k - i|, |l - j|\} = \{a, b\}$. For a given (a, b)-knight's move on an $m \times n$ chessboard, there is associated with it a graph whose vertex set and edge set are $\{(i, j) \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ and $\{(i, j)(k, l) \mid 1 \leqslant i, k \leqslant m, 1 \leqslant j, l \leqslant n, \{|k-i|, |l-j|\} = \{a, b\}\}$, respectively. Let $G((a, b), m, n)$ denote this graph, or just $G(m, n)$ for simplicity if the move (a, b) is understood or not to be emphasized.

A *closed* (a, b)-*knight's tour* is a series of (a, b)-knight's moves that visits every square of the $m \times n$ chessboard exactly once and then returns to the starting square. The *generalized knight's tour problem* asks: which $m \times n$ chessboards admit a closed (a, b)-knight's tour? This amounts to asking: which graph $G((a, b), m, n)$ is Hamiltonian?

We shall make a few easy observations. First, if $a + b$ is even, then no closed (a, b)-knight's tour is possible because only cells of the same color (that is either all black or all white cells) are covered during the moves. Thus $a + b$ is assumed to be odd. Also, we shall assume that $a < b$ since an (a, b)-knight's move and a (b, a)-knight's move are the same.

Next, if m and n are both odd, then no closed (a, b)-knight's tour is possible because $G(m, n)$ is then a bipartite graph with an odd number of vertices mn.

We may further assume that $m \leqslant n$. If $m \leqslant a + b - 1$, then no closed (a, b)-knight's tour on the $m \times n$ chessboard is possible. This is because the vertex $(a, 1)$ in $G(m, n)$ is of degree $\leqslant 1$. Suppose $n < 2b$. Then the vertex (b, b) is of degree 0.

We summarize the above observations in the following:

Theorem 2. *Suppose the $m \times n$ chessboard admit a closed (a, b)-knight's tour, where $a < b$ and $m \leqslant n$. Then*

(i) *$a + b$ is odd*;
(ii) *m or n is even*;

(iii) $m \geqslant a + b$; *and*
(iv) $n \geqslant 2b$.

Perhaps the simplest generalized knight's move is that of the $(0, 1)$-knight's move. In this case, the associated graph $G(m, n)$ is the horizontal grid whose hamiltonicity is easily decided. As for the $(0, b)$-knight's move, where $b \geqslant 3$ is odd, the associated graph $G(m, n)$ is disconnected. Henceforth, we shall assume that $1 \leqslant a < b$.

2. Open knight's tour on rectangular boards

In [16], Schwenk mentioned that the corresponding problem for the open knight's tour can also be solved by the same method he has introduced. The solution was left as a challenge to the interested readers. In this section, we provide a complete solution to the open knight's tour problem. Earlier, Cull and de Curtins [3] proved that every $m \times n$ chessboard with $5 \leqslant m \leqslant n$ admits an open knight's tour.

Theorem 3 (*Cull and de Curtins [3]*). *Every $m \times n$ chessboard with $5 \leqslant m \leqslant n$ admits an open knight's tour*.

The case $m = 3$ was considered in [14] where Van Rees showed that the $3 \times n$ chessboard admits an open knight's tour if and only if $n = 4$ or $n \geqslant 7$. Here, we shall present the solution for the missing case $m = 4$ as well as some constructions for the open knight's tours on the $3 \times n$ chessboard.

We shall make use of the following necessary condition for the existence of a Hamiltonian path in a graph. If H is a graph, we let $\omega(H)$ denote the number of components in H.

Theorem 4. *Let S be a proper subset of the vertex set of a graph G. If G contains a Hamiltonian path, then*

$$\omega(G - S) \leqslant |S| + 1.$$

Theorem 5. *The $m \times n$ chessboard with $m \leqslant n$ admits an open knight's tour unless one or more of the following conditions holds*:

(i) $m = 1$ *or* 2;
(ii) $m = 3$ *and* $n = 3, 5, 6$; *or*
(iii) $m = 4$ *and* $n = 4$.

Proof. Both $G(3, 3)$ and $G(m, n)$ for $m \leqslant 2$ are disconnected and hence do not have Hamiltonian paths.

For the remaining part on the non-existence of Hamiltonian paths, we shall make use of Theorem 4. Fig. 1(a) shows a disconnected graph with seven components. It is the result of removing five vertices $(1, 2)$, $(1, 4)$, $(2, 3)$, $(3, 2)$ and $(3, 4)$ from the graph $G(3, 5)$.

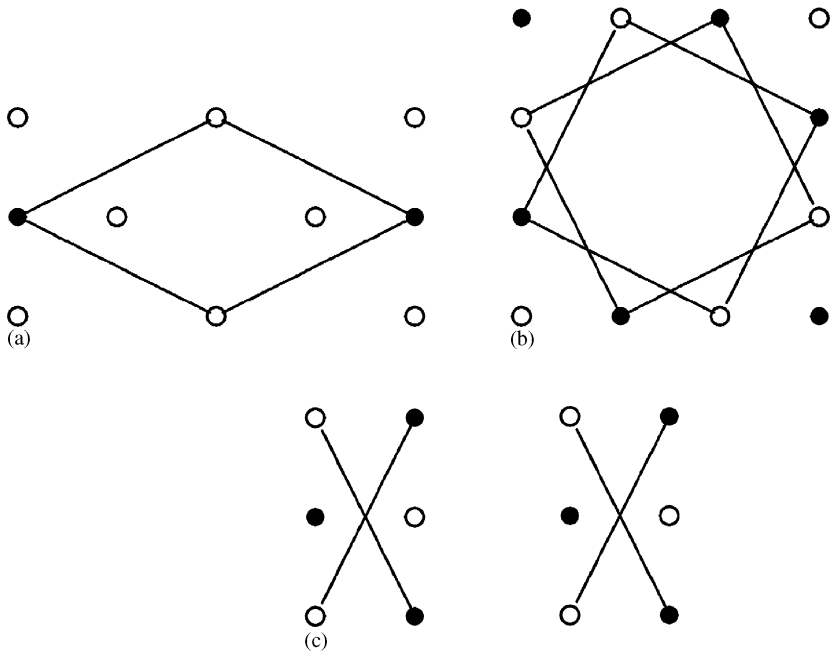

Fig. 1.

Fig. 1(b) is the resulting disconnected graph with six components when the four vertices $(j, 2)$ and $(j, 3)$ for $j=2, 3$ are removed from the graph $G(4, 4)$. Fig. 1(c) shows the resulting disconnected graph with eight components when the six vertices $(i, 3)$ and $(i, 4)$ for $i=1, 2, 3$ are removed from the graph $G(3, 6)$. By Theorem 4, all three graphs $G(3, 5)$, $G(4, 4)$ and $G(3, 6)$ do not contain Hamiltonian paths.

Next, we show that every other board admits an open knight's tour. Fig. 2 depicts a Hamiltonian path in $G(3, n)$ for each $n \in \{4, 7, 8, 9\}$ and in $G(4, k)$ for each $k \in \{5, 6, 7\}$. Let $P(m, n)$ denote a Hamiltonian path in $G(m, n)$. We shall show that each $P(3, n)$, for $n \in \{7, 9\}$, in Fig. 2 is extendable to a $P(3, n + 4)$ and each $P(4, k)$, for $k \in \{5, 6, 7\}$, in Fig. 2 is extendable to a $P(4, k + 3)$. This can be done by placing the graphs $S(3, 4)$ (a subgraph of $G(3, 4)$) and $S(4, 3)$ (a subgraph of $G(4, 3)$) on the right-hand side of $P(3, n)$ and $P(4, k)$, respectively, and joining them by suitable edges as explained below. The graphs $S(3, 4)$ and $S(4, 3)$ are shown in Fig. 3(a) and (b), respectively.

For the case $m = 3$, note that each of the $P(3, n)$, for $n \in \{7, 9\}$, has $(1, n)$ and $(2, n - 1)$ as end vertices. Joining the vertices $(1, n)$ and $(2, n - 1)$ of $P(3, n)$ to the vertices $(3, 1)$ and $(1, 1)$ of $S(3, 4)$, respectively, yields a Hamiltonian path in $G(3, n + 4)$ with $(1, n + 4)$ and $(2, n + 3)$ as end vertices. The extension of a Hamiltonian path in $G(3, 7)$ to a Hamiltonian path in $G(3, 11)$ is shown in Fig. 3(a). Repeat the process, we obtain a Hamiltonian path in $G(3, n)$ for every odd $n \geqslant 7$. For the case where $n \geqslant 10$ is even, Schwenk's result (Theorem 1) implies that $G(3, n)$ contains a Hamiltonian path.

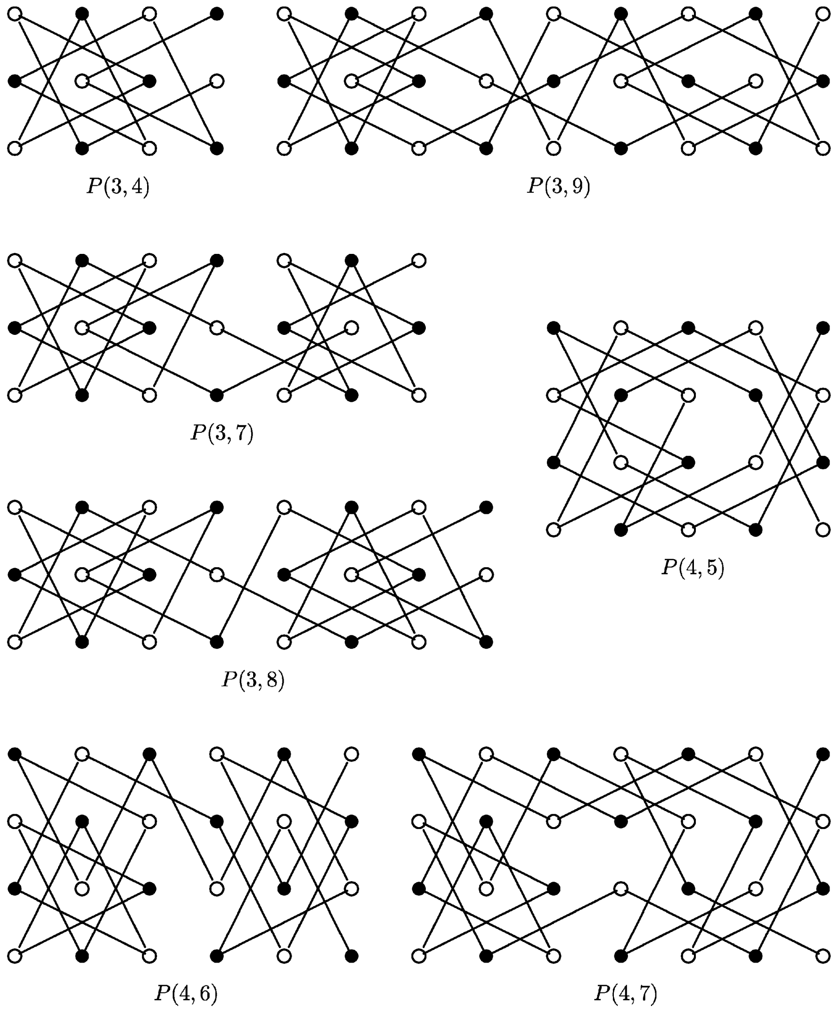

Fig. 2. The Hamiltonian paths $P(3,4)$, $P(3,7)$, $P(3,8)$, $P(3,9)$, $P(4,5)$, $P(4,6)$ and $P(4,7)$.

For the case $m=4$, note that each of the $P(4,k)$, for $k \in \{5,6,7\}$, has $(1,k)$ and $(4,k)$ as end vertices. Joining these two vertices to the vertices $(3,1)$ and $(2,1)$ of $S(4,3)$, respectively, yields a Hamiltonian path in $G(4,k+3)$ with $(1,k+3)$ and $(4,k+3)$ as end vertices. The extension of a Hamiltonian path in $G(4,5)$ to a Hamiltonian path in $G(4,8)$ is shown in Fig. 3(b). Repeat the process, we obtain a Hamiltonian path in $G(4,n)$ for every $n \geqslant 5$.

By Theorem 3, $G(m,n)$ contains a Hamiltonian path for every $m \geqslant 5$. This completes the proof. □

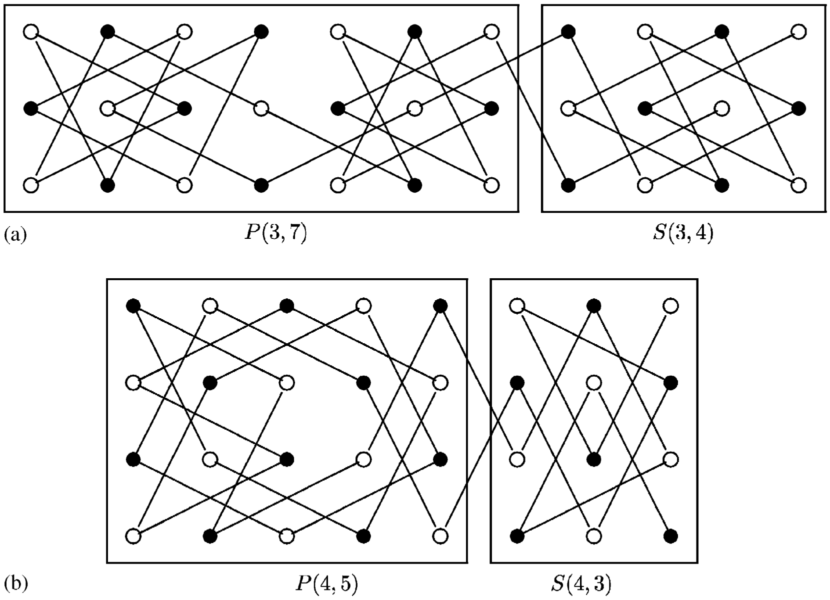

Fig. 3. (a) Extension of $P(3, 7)$ to $P(3, 11)$; (b) Extension of $P(4, 5)$ to $P(4, 8)$.

3. Forbidden rectangular boards

In this section, we show that certain rectangular chessboards do not admit a closed generalized knight's tour. The first two results generalize that of Pósa (see [16]) and Golomb [5] which states that the $4 \times n$ chessboard does not admit a closed $(1, 2)$-knight's tour.

Theorem 6. *Suppose* $m = a + b + 2t + 1$ *where* $0 \leqslant t \leqslant a - 1$. *Then the* $m \times n$ *chessboard admits no closed* (a, b)*-knight's tour*.

Proof. As $a + b$ is odd, we may write $a + b = 2s + 1$. Then $a \leqslant s$ and $b > s$ because $a < b$.

Let $r = \frac{m}{2} = s + t + 1$ and let the vertices of the $m \times n$ chessboard B be colored using r distinct colors $c_1, c_2, \ldots, c_r$ in the following manner.

If $1 \leqslant i \leqslant s+t+1$, then vertices in the ith row of B are colored with c_i. If $s+t+2 \leqslant i \leqslant m$, then vertices in the ith row of B are colored with c_{m+1-i}.

Since the case $a + b = 3$ (where $a = 1$ and $b = 2$) has been settled by Pósa (and also Golomb [5]) and discussed in [16], we may assume that $a + b \geqslant 5$ (so that $s \geqslant 2$).

Consider vertices in the $(t + 1)$th row. They are all colored with c_{t+1}. Moreover these vertices are adjacent only to the vertices in the $(a + t + 1)$th and $(b + t + 1)$th rows because $0 \leqslant t \leqslant a - 1$.

Since $a + t + 1 \leqslant s + t + 1$ and $b + t + 1 > s + t + 1$, vertices in these two rows are colored with c_{a+t+1}.

Now, look at those vertices in the $(m - t)$th row. They are colored with c_{t+1}. Moreover these vertices are adjacent only to the vertices in the $(a + t + 1)$th and $(b + t + 1)$th rows which are colored c_{a+t+1} (as explained earlier). This means that vertices which are colored c_{t+1} together with their neighbors force a proper subcycle and the proof is complete. □

Lemma 1. *Suppose the vertices of an $m \times n$ chessboard B are colored in equal amount with two colors, red and blue. Suppose further that every red vertex is adjacent only to the blue vertices and that at least one blue vertex is adjacent to a blue vertex. Then B admits no closed (a, b)-knight's tour.*

Proof. Suppose that there is a closed (a, b)-knight's tour $C = v_1 v_2 \ldots v_{mn} v_1$ of B. Since B contains an equal amount of vertices of each color and a red vertex must always be sandwiched by two blue vertices, the red and blue vertices must alternate around C. Let all the odd-labelled vertices v_{2r+1} be colored in red and all the even-labelled vertices v_{2r} be colored in blue. But from the original coloring of the chessboard B with black and white, we may conclude that all the vertices v_{2r+1} are also white. Thus all red vertices are white vertices, but this contradicts the different pattern chosen for the two colorings. We conclude that no closed (a, b)-knight's tour is possible. □

Pósa's and Golomb's theorem can also be generalized to the following:

Theorem 7. *Suppose $m = a(k + 2l)$ where $1 \leqslant l \leqslant \frac{k}{2}$. Then the $m \times n$ chessboard admits no closed (a, ak)-knight's tour, where a is odd and k is even.*

Proof. The proof is reminiscent of that of Pósa.

First note that, as $a + ak = a(1 + k)$ is odd (by Theorem 2), a is odd and k is even.

Next, let B be an $m \times n$ chessboard. For each $i = 1, 2, \ldots, k + 2l$, let A_i denote the $a \times n$ chessboard which consists of the $((i - 1)a + 1)$th, $((i - 1)a + 2)$th, $\ldots$, iath rows of B. In other words, B is partitioned into $k + 2l$ sub-chessboards $A_1, A_2, \ldots, A_{k+2l}$ each of size $a \times n$.

Now, let the vertices of B be colored with two colors in the following manner:

For $1 \leqslant i \leqslant k$, let the vertices in A_i be colored with red if i is odd and with blue otherwise.

For $k + 1 \leqslant i \leqslant k + 2l$, let the vertices in A_i be colored with blue if i is odd and with red otherwise.

Consider the vertices in the jth row. They are adjacent only to the vertices in the $(j \pm a)$th and the $(j \pm ak)$th rows. Note that not all the four rows are always possible. For example, if $j \leqslant a$, then the $(j - a)$th and the $(j - ak)$th rows do not exist.

Suppose the jth row belongs to A_i. Then the $(j + a)$th and the $(j - a)$th rows belong to A_{i+1} and A_{i-1}, respectively. Also, the $(j + ak)$th and the $(j - ak)$th rows belong to A_{i+k} and A_{i-k}, respectively.

Suppose $1 \leqslant i \leqslant k$. Then a vertex in the jth row is not adjacent to a vertex in the $(j - ak)$th row (since there is no A_{i-k} sub-chessboard).

If i is odd, then the vertices in A_i are colored with red whereas the vertices in A_{i+1} and A_{i-1} are colored with blue. Since $k + i$ is odd and $k + i \geqslant k + 1$, the vertices in A_{i+k} are colored with blue.

If i is even, then the vertices in A_i are colored with blue. Clearly, the vertices in A_{i-1} are colored with red. Since $k + i$ is even and $k + i \geqslant k + 1$, the vertices in the A_{i+k} are colored with red. The vertices in A_{i+1} are colored with red when $i < k$, but they are colored with blue when $i = k$.

Now, suppose $k+1 \leqslant i \leqslant k+2l$. Then, a vertex in the jth row is not adjacent to a vertex in the $(j+ak)$th row (since there is no A_{i+k} sub-chessboard).

If i is even and $i < k+2l$ then the vertices in A_i are colored with red and the vertices in A_{i+1} and A_{i-1} are colored with blue. Since $i-k$ is even and $i-k \leqslant k$, the vertices in A_{i-k} are colored with blue. If $i = k+2l$, then the vertices in A_{k+2l} are adjacent only to the vertices in A_{k+2l-1} and A_{2l} which are both colored with blue.

If i is odd, then the vertices in A_i are colored with blue. Clearly, the vertices in A_{i+1} and A_{i-k} are colored with red. The vertices in A_{i-1} are colored with red when $i > k+1$, but they are colored with blue when $i = k+1$.

Thus, we may make the conclusion that every red vertex in B is adjacent only to the blue vertices; however there is a blue vertex that is adjacent to a blue vertex. By Lemma 1, no closed (a, ak)-knight's tour is possible. □

Theorem 8. *Suppose* $m = 2(ak+l)$ *where* $1 \leqslant k \leqslant l \leqslant a$. *Then the* $m \times n$ *chessboard admits no closed* $(a, a+1)$*-knight's tour*.

Proof. Let B be an $m \times n$ chessboard. As $k \leqslant l$, we have $m > k(2a+1)$. Partition the first $k(2a+1)$ rows of vertices into k sub-chessboards $A_1, A_2, \ldots, A_k$, each of size $(2a+1) \times n$. For each A_i, $i = 1, 2, \ldots, k$, we shall color the first a rows of vertices with red and the next $a+1$ rows of vertices that follow with blue. Note that in the chessboard B, we have ak rows of vertices colored with red, $k(a+1)$ rows of vertices colored with blue and $2l-k$ rows uncolored.

Let D denote the $(2l-k) \times n$ sub-board that contains all the uncolored vertices of B. As $l \geqslant k$, we have $2l - k = k + s$ for some $s \geqslant 0$. Clearly, s is even. We shall color the first $k + \frac{s}{2}$ rows of vertices in D with red and the remaining $\frac{s}{2}$ rows of vertices with blue. The number of vertices colored with red in B is now equal to the number of vertices colored with blue.

Consider the vertices in the jth row. They are adjacent only to the vertices in the $(j \pm a)$th and the $(j \pm (a+1))$th rows. Note that not all the four rows are always possible. For example, if $j \leqslant a$, then the $(j-a)$th and the $(j-a-1)$th rows do not exist.

Suppose the jth row belongs to A_i, for some $i = 1, 2, \ldots, k$. If the jth row is colored red, then the $(j \pm a)$th and the $(j \pm (a+1))$th rows are colored blue. So, every vertex colored with red in A_i is adjacent only to vertices colored with blue.

Suppose the jth row belongs to D. Since $k + \frac{s}{2} \leqslant a$, every vertex colored with red in D can only be adjacent to vertices colored in blue.

Consider a vertex in the $(a+1)$th row. It is colored with blue and is adjacent to a vertex in the $(2a+1)$th row which is also colored with blue.

Thus, we may make the conclusion that every red vertex in B is adjacent only to the blue vertices; however there is a blue vertex that is adjacent to another blue vertex. By Lemma 1, B does not admit a closed $(a, a+1)$-knight's tour. □

The previous three results deal with forbidden boards of size $m \times n$ with m even. The next result considers a case where the move is $(a, a+1)$ and m is odd. However, the result is not enjoyed by the $(1, 2)$-knight's move.

Theorem 9. *Suppose* $m = 2a + 2t + 1$ *where* $1 \leqslant t \leqslant a - 1$. *Then the* $m \times n$ *chessboard admits no closed* $(a, a + 1)$*-knight's tour*.

Proof. Let A_u (respectively, A_l) denote the $a \times a$ sub-board located at the upper (respectively, lower) left corner of the $m \times n$ chessboard. It is easy to see that vertices in A_u or A_l are of degree 2 in $G(m, n)$.

Consider the vertex $(a + t + 1, a + 2)$. It is adjacent to the vertices $(t + 1, 1)$, $(t, 2)$ and $(2a + t + 1, 1)$. Clearly, $(t + 1, 1)$ and $(t, 2)$ belong to A_u. Since $1 \leqslant t \leqslant a - 1$, it is easy to see that $(2a + t + 1, 1)$ belongs to A_l. Hence $(a + t + 1, a + 2)$ is adjacent to three vertices of degree 2 and thus $G(m, n)$ is non-Hamiltonian. □

4. (2, 3)-knight's move

In this section, we shall confine our attention to the (2, 3)-knight's move. Clearly, if $m \leqslant 4$, then the $m \times n$ chessboard admits no closed (2, 3)-knight's tour (by Theorem 2). By Theorem 8, no closed (2, 3)-knight's tour is possible if m is 6, 8 or 12. By Theorem 9, there is no closed (2, 3)-knight's tour on the $7 \times n$ chessboard.

Corollary 1. *If* $m \leqslant 4$ *or* $m = 6, 7, 8, 12$, *then the* $m \times n$ *chessboard does not admit a closed* (2, 3)*-knight's tour*.

It is thus natural to look at the smallest undecided case which is the $5 \times n$ chessboard. In fact, in the rest of the paper, we determine the values of n for which the $5k \times n$ chessboard, except for the 5×18, admits a closed (2, 3)-knight's tour. The result is summarized in Theorem 10. It is very likely that the 5×18 chessboard admits no closed (2, 3)-knight's tour but we are unable to show it.

Similar question could also be asked for the $9k \times n$ and $11k \times n$ cases, but a full account (if available) may have to appear elsewhere.

Proposition 1. *Suppose* $n \neq 18$. *Then the* $5 \times n$ *chessboard admits a closed* (2, 3)*-knight's tour if and only if* $n \geqslant 16$ *is even*.

Proof. Since $G(5, n)$ is a bipartite graph, n must be even in order that $G(5, n)$ is hamiltonian.

If $n \leqslant 4$, then clearly $G(5, n)$ is non-Hamiltonian because the board is not wide enough to permit a closed (2, 3)-knight's tour.

If n is 6, 8 or 12, Corollary 1 shows that $G(5, n)$ is non-Hamiltonian.

If $n = 10$, the fact that $G(5, 10)$ is non-Hamiltonian is easily seen. The two vertices (3, 2) and (3, 8) are both of degree 2 and they force a 4-cycle $(3, 2)(5, 5)(3, 8)(1, 5)(3, 2)$ in $G(5, 10)$.

For $n = 14$, suppose $G(5, 14)$ contains a Hamiltonian cycle $C(5, 14)$. Then the path $(2, 13)(5, 11)(3, 14)(1, 11)(4, 13)$ must be part of $C(5, 14)$ because $(3, 14)$, $(2, 13)$ and $(4, 13)$ are vertices of degree 2. This implies that the path $P_1 = (1, 5)(3, 8)(5, 5)$ must also be part of $C(5, 14)$ because the neighbors of $(3, 8)$ are $(1, 11)$, $(5, 11)$, $(1, 5)$ and $(5, 5)$.

1	36	31	42	57	18	47	6	29	12	53	22	75	70	65	24
38	45	4	33	40	79	8	15	50	59	10	77	68	27	72	63
43	56	17	2	35	30	13	54	19	48	61	66	25	52	21	74
32	41	80	37	46	5	58	11	78	7	28	71	64	23	76	69
3	34	39	44	55	16	49	60	9	14	51	20	73	62	67	26

$C(5, 16)$

1	88	83	78	11	52	67	6	95	42	15	38	57	72	17	46	25	36	31	62
90	81	4	85	76	99	8	55	74	97	40	69	20	49	64	27	34	59	22	29
79	10	53	2	87	92	43	12	51	66	71	94	45	14	37	32	61	18	47	24
84	77	100	89	82	5	96	41	68	7	56	73	16	39	58	21	30	63	26	35
3	86	91	80	9	54	75	98	93	44	13	50	65	70	19	48	23	28	33	60

$C(5, 20)$

1	102	97	108	73	12	113	6	95	18	69	78	23	48	93	44	59	64	83	52	29	34	39	54
104	111	4	99	106	119	8	15	88	75	10	115	90	25	46	67	80	21	62	27	36	57	32	41
109	72	13	2	101	96	17	70	77	86	117	94	19	60	65	84	49	92	43	38	55	82	51	30
98	107	120	103	112	5	74	11	114	7	24	47	68	79	22	63	26	45	58	33	40	53	28	35
3	100	105	110	71	14	87	118	9	16	89	76	85	116	91	20	61	66	81	50	31	42	37	56

$C(5, 24)$

1	98	93	104	7	114	83	88	13	112	81	120	49	126	17	36	77	24	47	30	19	40	69	64	59	42
100	107	4	95	102	129	90	109	86	15	122	75	10	51	124	73	26	33	22	53	28	71	62	45	66	57
105	6	115	2	97	92	111	8	117	84	127	12	35	80	119	48	31	18	37	78	55	60	43	20	39	68
94	103	130	99	108	87	14	113	82	89	50	125	16	121	76	23	52	29	72	25	46	65	58	41	70	63
3	96	101	106	5	116	85	128	91	110	9	118	123	74	11	34	79	54	27	32	21	38	67	56	61	44

$C(5, 26)$

Fig. 4. Hamiltonian cycles $C(5, 16)$, $C(5, 20)$, $C(5, 24)$ and $C(5, 26)$.

Since $(3, 2)$ is of degree 2, the path $P_2 = (1, 5)(3, 2)(5, 5)$ must also be part of $C(5, 14)$. But then $P_1 \cup P_2$ is a 4-cycle in $C(5, 14)$, a contradiction.

We now show that every other board admits a closed (2, 3)-knight's tour. This is done by first showing that some smaller boards contain Hamiltonian cycles and then use these to build up Hamiltonian cycles in bigger boards.

Fig. 4 depicts a Hamiltonian cycle each in $G(5, n)$ for $n \in \{16, 20, 24, 26\}$. These Hamiltonian cycles are indicated by the sequences of consecutive integers from 1 to $5n$. Let these Hamiltonian cycles be denoted $C(5, n)$, $n \in \{16, 20, 24, 26\}$.

For each $t \in \{11, 19, 21\}$, let R_t denote the subgraph of $G(5, t)$ depicted in Fig. 5. Note that each R_t consists of three disjoint paths whose union includes all the vertices in $G(5, t)$, $t \in \{11, 19, 21\}$. Let $u - v$ denote a path whose end vertices are u and v. We further note that the three disjoint paths in R_t are $x_1 - x_2$, $y_1 - y_2$ and $z_1 - z_2$ where $x_1 = (1, t)$, $x_2 = (4, t - 2)$, $y_1 = (3, t - 2)$, $y_2 = (4, t - 1)$, $z_1 = (4, t)$ and $z_2 = (5, t)$.

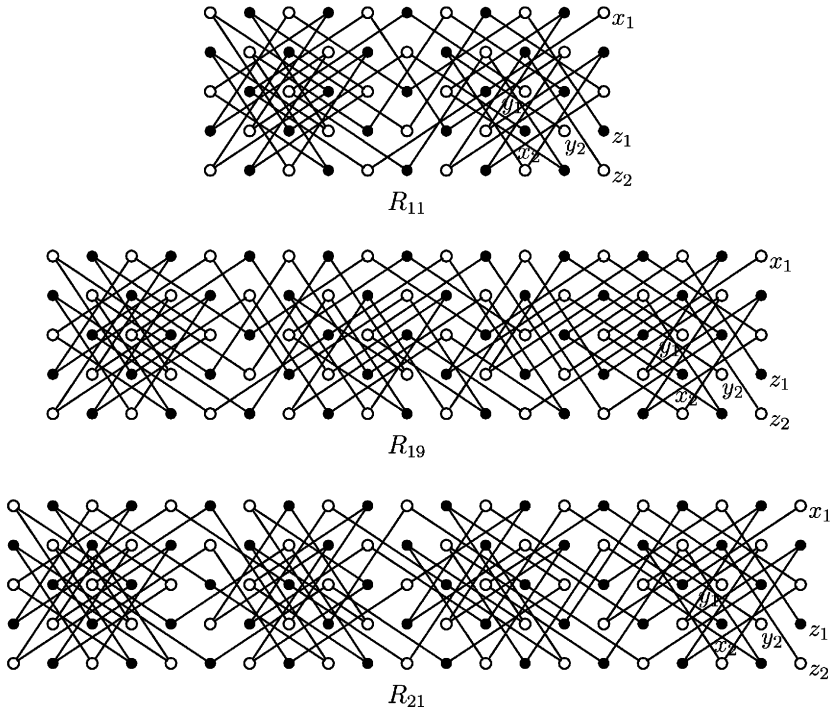

Fig. 5. The graphs R_{11}, R_{19} and R_{21}.

Now suppose there is a subgraph of $G(5, s)$, denoted L_s, which consists of three disjoint paths whose union includes all the vertices in $G(5, s)$. Suppose further that the end vertices of these paths are $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1$ and γ_2. Moreover, these end vertices are such that, when R_t is placed on the left hand side of L_s, there is a $(2, 3)$-knight's move from x_i to α_i, from y_i to β_i and from z_i to γ_i, $i = 1, 2$. It is easy to see that if the three paths in L_s are

(i) $\alpha_1 - \gamma_1, \gamma_2 - \beta_2, \beta_1 - \alpha_2$,
(ii) $\alpha_1 - \gamma_1, \gamma_2 - \beta_1, \beta_2 - \alpha_2$ or
(iii) $\alpha_1 - \beta_2, \beta_1 - \gamma_1, \gamma_2 - \alpha_2$,

then we have a Hamiltonian cycle, denoted $R_t + L_s$, in $G(5, t + s)$. This is illustrated in Fig. 6.

We now show the existence of the graphs L_s which meet the above conditions for every $s = 11 + 6k$ where $k \geqslant 0$. Note that $R_{11} + L_s$ takes care of $n = 22, 28, 34, \ldots$; $R_{19} + L_s$ takes care of $n = 30, 36, 42, \ldots$; and $R_{21} + L_s$ takes care of $n = 32, 38, 44, \ldots$.

The graphs L_{17} and L_{23} are depicted in Fig. 7. They satisfy conditions (i) and (ii) above, respectively. We shall use these two graphs to build up L_{11+6k}. For this purpose, let B_{12} denote the spanning subgraph of $G(5, 12)$ which is depicted in Fig. 7. Note that B_{12} consists

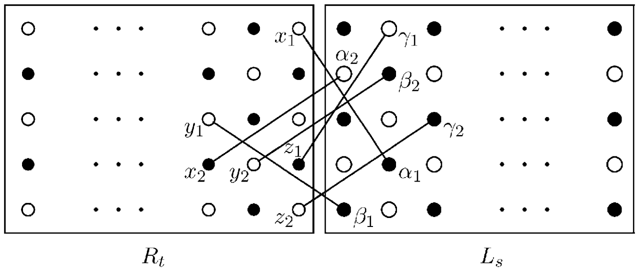

Fig. 6. $R_t + L_s$.

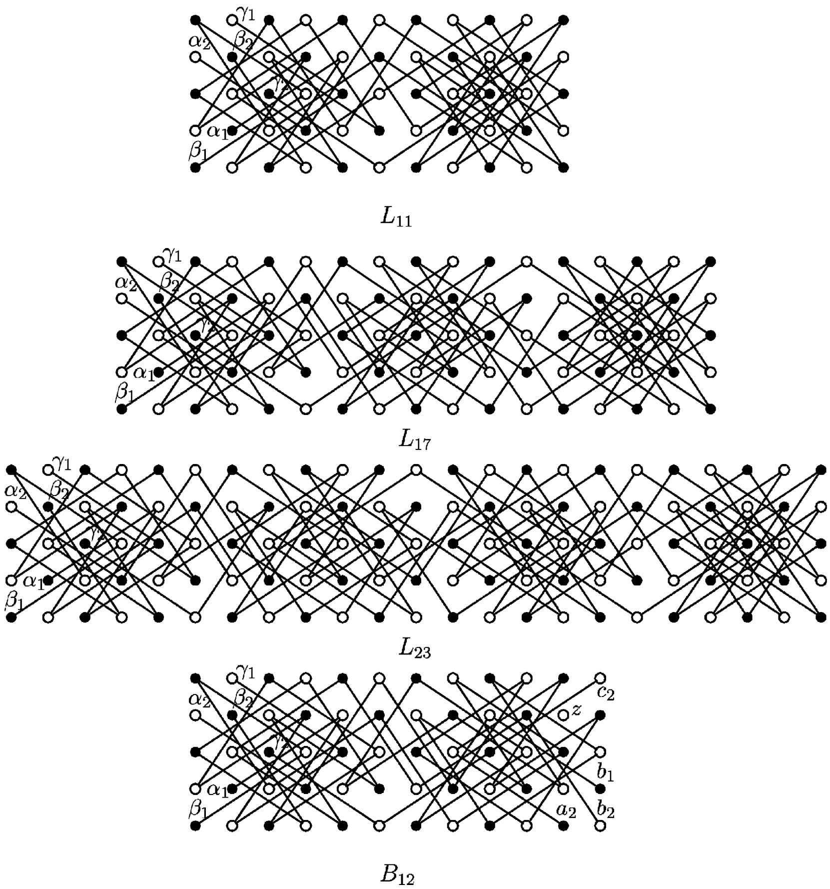

Fig. 7. The graphs L_{11}, L_{17}, L_{23} and B_{12}.

of five disjoint paths $\alpha_1 - \gamma_1$, $\alpha_2 - a_2$, $\beta_1 - b_1$, $\beta_2 - b_2$ and $\gamma_2 - c_2$ together with the isolated vertex $z = (2, 11)$. Here, $\alpha_1 = (4, 2)$, $\alpha_2 = (2, 1)$, $\beta_1 = (5, 1)$, $\beta_2 = (2, 2)$, $\gamma_1 = (1, 2)$, $\gamma_2 = (3, 3)$, $a_2 = (5, 11)$, $b_1 = (4, 12)$, $b_2 = (5, 12)$ and $c_2 = (1, 12)$.

To obtain L_{29}, place B_{12} on the left-hand side of L_{17}. Then add six new edges $z\beta_1$, $z\alpha_1$, $a_2\alpha_2$, $b_1\gamma_1$, $b_2\beta_2$ and $c_2\gamma_2$. Note that L_{29} satisfies condition (i) above. Continue the process, we obtain L_{17+12k} which satisfies condition (i) above for any $k \geqslant 0$.

Similarly, we obtain L_{23+12k} which satisfies condition (ii) above for any $k \geqslant 0$.

To complete the proof, we need to construct L_{11}. This graph is depicted in Fig. 7. Note that L_{11} satisfies condition (iii) above. □

Proposition 2. *The* $10 \times n$ *chessboard admits a closed* $(2, 3)$*-knight's tour if and only if* $n \geqslant 10$ *and* $n \neq 12$.

Proof. By Proposition 1 and Corollary 1, the graph $G(10, n)$ is non-Hamiltonian for $n \leqslant 8$ or $n = 12$.

For $n = 9$, suppose $G(10, 9)$ contains a Hamiltonian cycle $C(10, 9)$. Then the paths $(2, 2)(4, 5)(2, 8)$, $(10, 2)(8, 5)(10, 8)$ and the edge $(1, 9)(3, 6)$ must be a part of $C(10, 9)$ because $(2, 2)$, $(2, 8)$, $(10, 2)$, $(10, 8)$ and $(1, 9)$ are vertices of degree 2 in $G(10, 9)$. This implies that the edge $(1, 3)(3, 6)$ must also be included in $C(10, 9)$, but then the vertex $(6, 8)$ cannot be included since it has only one available edge $(9, 6)(6, 8)$, a contradiction.

Next, we shall show that $G(10, n)$ is Hamiltonian for every other value of n. Fig. 8 depicts a Hamilton cycle $C(10, n)$ in $G(10, n)$ for $n \in \{10, 11, 13, 14, 17\}$. Note that each $C(10, n)$ in Fig. 8 contains the edges $e_1 = (1, n)(4, n-2)$, $e_2 = (1, n-2)(4, n)$ and $e_3 = (3, n-2)(6, n)$.

Fig. 9 shows a subgraph of $G(10, 5)$, denoted $S(10, 5)$, which consists of three disjoint paths $P_1 = a_1 - a_2$, $P_2 = b_1 - b_2$ and $P_3 = c_1 - c_2$ whose end vertices are $a_1 = (1, 1)$, $a_2 = (8, 3)$, $b_1 = (2, 1)$, $b_2 = (3, 3)$, $c_1 = (3, 1)$ and $c_2 = (2, 3)$. Note that $V(P_1) \cup V(P_2) \cup V(P_3) = V(G(10, 5))$.

The process of extension is to replace each edge e_i, $i = 1, 2, 3$, in $C(10, n)$ by a path P_j for some j such that $1 \leqslant j \leqslant 3$, and obtain an extension of a Hamiltonian cycle in $G(10, n+5)$ for $n \in \{10, 11, 13, 14, 17\}$.

Place $S(10, 5)$ on the right-hand side of a $C(10, n)$. Remove the edge $e_1 = (1, n)(4, n-2)$ from $C(10, n)$ and join $(1, n)$ and $(4, n-2)$ to the vertices b_2 and b_1 of $S(10, 5)$, respectively. Next, remove the edge $e_2 = (1, n-2)(4, n)$ from $C(10, n)$ and join $(1, n-2)$ and $(4, n)$ to the vertices c_1 and c_2 of $S(10, 5)$, respectively. Finally, remove the edge $e_3 = (3, n-2)(6, n)$ from $C(10, n)$ and join $(3, n-2)$ and $(6, n)$ to the vertices a_1 and a_2 of $S(10, 5)$, respectively. Thus, we obtain a Hamiltonian cycle $C(10, n+5)$ which also includes the edges $(1, n+5)(4, n+3)$, $(1, n+3)(4, n+5)$ and $(3, n+3)(6, n+5)$. The extension of a $C(10, 10)$ to a $C(10, 15)$ is shown in Fig. 10.

Repeating the above construction, we obtain a Hamiltonian cycle in $G(10, n)$ for each $n \geqslant 10$ and $n \neq 12$. □

Proposition 3. *Suppose* $k \geqslant 3$ *is an integer. Then the* $5k \times n$ *chessboard admits a closed* $(2, 3)$*-knight's tour if and only if*

(i) $n \geqslant 10$ *is even and* $n \neq 12$ *when* k *is odd, or*
(ii) $n = 5, 9, 10, 11$ *or* $n \geqslant 13$ *when* k *is even.*

1	10	67	46	95	56	29	8	23	50
12	65	4	69	48	27	6	53	86	25
45	96	57	2	9	22	51	94	55	30
68	47	100	11	66	85	24	49	28	7
3	70	13	64	5	54	87	26	21	52
58	39	44	97	60	91	78	31	84	93
37	16	73	42	99	80	33	18	75	82
14	63	90	71	40	35	20	61	88	77
43	98	59	38	17	74	83	92	79	32
72	41	36	15	62	89	76	81	34	19

$C(10,10)$

1	130	39	70	31	114	75	132	51	102	55	94	89	136
128	73	28	37	68	139	20	105	134	53	92	77	18	87
71	32	113	2	131	40	101	30	95	90	137	50	103	56
38	69	140	129	74	133	52	115	76	19	88	135	54	93
27	36	127	72	29	106	67	138	21	104	17	86	91	78
112	3	8	33	110	13	122	41	100	15	96	57	62	49
11	124	43	6	35	116	81	24	47	98	59	120	83	64
126	109	26	45	4	9	22	107	66	61	118	79	16	85
7	34	111	12	123	42	99	14	121	82	63	48	97	58
44	5	10	125	108	25	46	117	80	23	84	65	60	119

$C(10,14)$

1	42	25	78	7	58	73	12	23	18	107
40	71	4	27	76	109	14	21	100	9	16
79	6	59	2	43	24	19	106	57	74	11
26	77	110	41	72	99	8	17	108	13	22
3	44	39	70	5	28	75	10	15	20	101
52	85	80	47	60	105	34	65	98	93	56
87	36	49	82	45	54	67	96	31	62	91
38	69	104	51	84	89	94	29	102	33	64
81	46	53	86	35	48	61	92	55	66	97
50	83	88	37	68	103	32	63	90	95	30

$C(10,11)$

1	116	75	46	101	58	3	118	73	18	97	88	27
114	15	120	77	44	129	104	29	42	99	60	5	86
47	102	57	2	117	74	17	96	89	26	127	72	19
76	45	130	115	14	119	100	59	4	87	28	41	98
121	78	113	16	103	30	43	128	105	6	85	90	61
56	13	48	81	38	95	108	63	40	71	20	25	126
11	110	65	50	79	54	35	124	69	22	93	106	33
112	37	122	67	52	9	82	31	24	91	62	7	84
49	80	55	12	109	64	39	94	107	34	125	70	21
66	51	10	111	36	123	68	53	8	83	32	23	92

$C(10,13)$

$C(10,17)$

1	78	83	10	123	70	135	36	85	130	117	38	145	160	87	48	139
76	133	4	81	8	119	126	143	6	121	128	141	114	89	162	157	46
11	124	71	2	79	84	131	116	69	136	35	86	49	138	39	146	159
82	9	170	77	134	5	122	129	118	37	144	161	156	47	140	113	88
3	80	75	132	125	68	7	120	127	142	115	90	147	158	45	50	163
72	17	12	105	58	169	100	63	34	29	98	137	112	53	150	155	40
19	102	65	14	107	74	61	32	95	56	27	42	51	152	93	148	25
104	59	168	67	16	21	30	97	166	109	54	23	154	91	164	111	44
13	106	73	18	101	64	57	28	99	62	33	94	149	26	41	52	151
66	15	20	103	60	167	108	55	22	31	96	165	110	43	24	153	92

Fig. 8. Hamiltonian cycles $C(10, n)$, $n = 10, 11, 13, 14, 17$.

Proof. First, we note that, by Corollary 1, the $5k \times n$ chessboard does not admit a closed (2, 3)-knight's tour if $n \leqslant 4$ or $n = 6, 7, 8, 12$. Further, if k is odd, then the $5k \times n$ chessboard does not admit a closed (2, 3)-knight's tour if $n \leqslant 9$ or if n is odd (by Theorem 2).

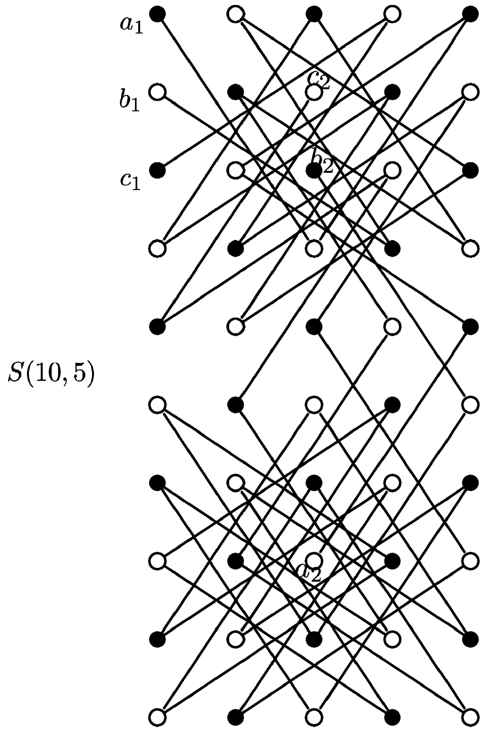

Fig. 9. The graph $S(10, 5)$.

Next, we show that every other $5k \times n$ chessboard admits a closed (2, 3)-knight's tour. The following construction shall be used throughout.

Construction (*): Suppose $G(m, n)$ has a Hamiltonian cycle $C(m, n)$ which contains the edges $(1, 1)(3, 4)$ and $(m - 2, 3)(m, 6)$. Take a copy of $C_t = C(m_t, n)$ and a copy of $C_b = C(m_b, n)$. Place C_b below C_t. Delete the edge $(m_t - 2, 3)(m_t, 6)$ (respectively, $(1, 1)(3, 4)$) from C_t (respectively, C_b). Joining the vertex $(m_t - 2, 3)$ (respectively, $(m_t, 6)$) of C_t to the vertex $(1, 1)$ (respectively, $(3, 4)$) of C_b, we obtain a Hamiltonian cycle $C(m_t + m_b, n)$ in $G(m_t + m_b, n)$ which contains the edges $(1, 1)(3, 4)$ and $(m_t + m_b - 2, 3)(m_t + m_b, 6)$.

Case (1): *k is odd*

Suppose $n \geqslant 16$ is even and $n \neq 18$. Note that every Hamiltonian cycle $C(5, n)$ constructed in Proposition 1 contains the edges $(1, 1)(3, 4)$ and $(3, 3)(5, 6)$. Take two copies of $C(5, n)$ and place one above the other. By the construction (*), we obtain a Hamiltonian cycle in $G(10, n)$ which contains the edges $(1, 1)(3, 4)$ and $(8, 3)(10, 6)$. Repeating the construction (*) by taking $C_t = C(10, n)$ and $C_b = C(5, n)$, we have a Hamiltonian cycle $G(5k, n)$ which contains the edges $(1, 1)(3, 4)$ and $(5k - 2, 3)(5k, 6)$ for $k \geqslant 3$ and $n \geqslant 16$ is even except $n = 18$.

Suppose $n \in \{10, 14, 18\}$. The required Hamiltonian cycles $C(10, 10)$, $C(10, 14)$ and $C(15, 14)$, $C(15, 18)$ are shown in Figs. 8 and 11, respectively. Now, $C(10, 18)$ can be

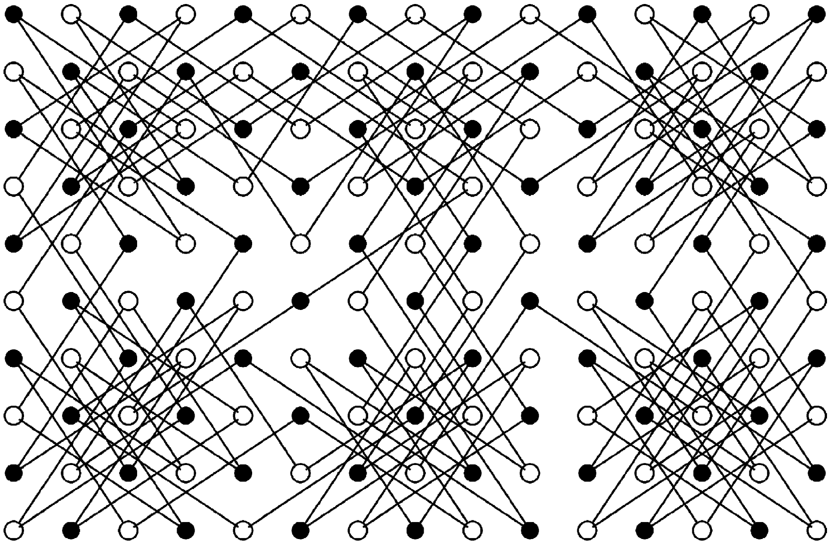

Fig. 10. Extension of a closed (2, 3)-knight's tour in the 10 × 10 chessboard to one in the 10 × 15 chessboard.

constructed by using the method described in the proof of Proposition 2 while $C(15, 10)$ can be obtained by taking a 90° clockwise rotation on the Hamiltonian cycle $C(10, 15)$ of Fig. 10. Note that, all these Hamiltonian cycles $C(5s, n)$ contain the edges $(1, 1)(3, 4)$ and $(5s - 2, 3)(5s, 6)$ for $s = 2, 3$ and $n \in \{10, 14, 18\}$. Now, by taking $C_t = C(15, n)$ and $C_b = C(10, n)$ and applying the construction $(^*)$, we obtain a Hamiltonian cycle in $G(5k, n)$ for all odd $k \geqslant 3$ and $n = 10, 14, 18$.

Case (2): *k is even*

In this case, $5k \equiv 0 \pmod{10}$.

For $n = 5$, $C(10i, 5)$ can be obtained by a 90° clockwise rotation on the Hamiltonian cycle $C(5, 10i)$ (constructed in Proposition 1), where $i \geqslant 2$.

For $n = 9$, note that the Hamiltonian cycles $C(20, 9)$ and $C(30, 9)$ in Fig. 12 both contain the edges $(1, 1)(3, 4)$ and $(10i - 2, 3)(10i, 6)$ where $i = 2, 3$. As such, these two Hamiltonian cycles can be used to obtain a Hamiltonian cycle in $G(10i, 9)$ for $i \geqslant 2$ by the construction $(^*)$.

For $n \geqslant 10$ and $n \neq 12$, note that all the Hamiltonian cycles obtained in the proof of Proposition 2 contain the edges $(1, 1)(3, 4)$ and $(10i - 2, 3)(10i, 6)$. So, by the construction $(^*)$, we have a Hamiltonian cycle in $G(10i, n)$ for $i \geqslant 1$, $n \geqslant 10$ and $n \neq 12$.

This completes the proof. □

Putting all the above propositions together, we have the following result.

Theorem 10. *The* $5k \times n$ *chessboard where* $(5k, n) \neq (5, 18)$ *admits a closed* (2, 3)*-knight's tour if and only if*

1	154	47	172	209	32	3	156	59	22	55	34	187	158
152	27	182	45	170	5	50	29	184	57	36	19	52	189
173	206	31	2	155	48	23	208	33	186	157	60	21	54
46	171	210	153	28	183	58	169	4	51	188	159	56	35
181	44	151	26	207	30	185	6	49	20	53	190	37	18
42	67	174	205	24	97	134	65	160	109	168	117	122	61
69	132	85	180	203	40	75	106	63	162	119	136	77	124
150	25	98	43	66	7	110	167	116	121	38	17	108	191
175	204	41	68	133	64	161	96	135	76	123	62	163	118
84	179	70	131	86	105	202	39	74	107	78	125	120	137
99	144	149	176	101	166	89	8	111	164	115	192	197	16
142	87	10	147	178	95	130	81	14	113	194	91	128	199
71	102	83	12	145	140	73	104	201	196	93	138	79	126
148	177	100	143	88	9	112	165	90	129	198	15	114	193
11	146	141	72	103	82	13	94	139	80	127	200	195	92

$C(15,14)$

1	254	259	248	137	58	267	6	135	114	149	46	265	116	51	96	91	144
252	245	4	257	250	269	140	55	152	9	262	111	142	53	94	147	48	89
247	138	57	2	255	260	113	150	45	266	7	134	97	92	145	264	117	50
258	249	270	253	244	5	136	59	268	141	54	115	148	47	90	143	52	95
3	256	251	246	139	56	153	8	261	112	151	10	263	110	49	88	93	146
62	243	158	15	32	23	44	77	30	133	98	83	28	39	118	129	124	103
241	18	75	156	13	60	107	20	37	80	25	42	105	120	127	100	85	122
16	33	22	63	154	159	132	35	82	11	78	109	130	125	102	27	40	87
157	14	61	242	19	76	31	24	43	106	29	38	99	84	123	104	119	128
74	155	240	17	34	21	12	79	108	131	36	81	26	41	86	121	126	101
175	230	235	160	177	64	165	170	223	200	189	218	69	194	213	202	207	184
228	237	172	233	162	167	180	71	192	221	66	197	182	211	204	187	216	209
239	178	73	174	231	226	199	190	219	164	169	224	201	206	185	68	195	214
234	161	176	229	236	171	222	65	166	181	70	193	188	217	208	183	212	203
173	232	227	238	179	72	163	168	225	198	191	220	67	196	215	210	205	186

$C(15,18)$

Fig. 11. Hamiltonian cycles $C(15, 14)$ and $C(15, 18)$.

(i) $k = 1$ *and* $n \geqslant 16$ *is even*; *or*
(ii) $k = 2$ *and* $n \geqslant 10$ *and* $n \neq 12$; *or*
(iii) $k \geqslant 3$ *is odd and* $n \geqslant 10$ *is even and* $n \neq 12$; *or*
(iv) $k \geqslant 4$ *is even and* $n = 5, 9, 10, 11$ *or* $n \geqslant 13$.

1	120	21	66	113	82	179	122	19
118	45	138	23	162	43	140	47	124
65	114	81	180	121	20	67	112	83
22	161	2	119	46	123	18	163	178
137	24	117	44	139	48	125	42	141
80	99	64	115	164	111	84	101	68
3	34	167	160	41	86	177	36	17
116	165	136	25	100	69	142	49	126
63	40	79	98	35	102	55	110	85
168	159	4	33	166	37	16	87	176
135	26	103	54	39	50	127	70	143
78	97	62	105	52	109	28	91	56
5	32	169	158	107	88	175	38	15
104	53	134	27	90	71	144	51	128
61	106	77	96	31	92	57	108	29
170	157	6	151	172	155	14	89	174
133	74	149	8	153	12	129	72	145
76	95	60	147	10	131	30	93	58
7	152	171	156	73	150	173	154	13
148	9	132	75	94	59	146	11	130

$C(20, 9)$

116	159	94	191	144	165	114	157	96
161	68	221	98	163	112	223	70	155
192	145	62	115	158	95	66	143	166
93	190	117	160	69	156	97	164	113
220	99	162	67	222	71	154	111	224
61	106	193	146	63	142	167	104	65
118	149	92	189	140	169	108	151	102
147	74	219	100	105	110	225	72	153
194	139	60	107	150	103	64	141	168
91	188	119	148	73	152	101	170	109
218	9	172	75	138	59	186	11	226
7	214	195	210	77	216	263	212	13
120	261	90	187	10	171	30	137	58
173	76	217	8	213	12	227	78	185
196	209	6	215	262	211	14	133	264
89	250	121	260	31	136	57	248	29
252	53	174	27	134	79	184	51	228
5	32	197	208	249	132	265	38	15
122	259	88	251	52	247	28	135	56
175	26	253	54	39	50	229	80	183
198	207	4	33	200	37	16	131	266
87	40	123	258	35	254	55	246	129
240	201	176	25	256	81	182	49	230
3	34	199	206	41	130	267	36	17
124	257	86	241	202	245	128	255	82
177	24	239	44	179	48	231	42	181
22	205	2	237	46	233	18	203	268
85	242	125	270	235	20	83	244	127
238	45	178	23	204	43	180	47	232
1	236	21	84	243	126	269	234	19

$C(30, 9)$

Fig. 12. Hamiltonian cycles $C(20, 9)$ and $C(30, 9)$.

Acknowledgements

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