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A note on (k, n)-arcs

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Abstract

We construct (k, n)-arcs in $\mathrm{PG}(2, q)$ with k approximately q^2/d and n approximately q/d for each divisor d of $q - 1$.

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We denote by $\mathrm{PG}(2, q)$ the projective plane over the finite field $\mathrm{GF}(q)$ of q elements. A (k, n)-arc in $\mathrm{PG}(2, q)$ is a subset of k points with at most n on a line. In the First Irsee Conference on Finite Geometry, Hill talked about (k, n)-arcs with $n=(q+1)/2$ and remarked that very little is known about (k, n)-arcs except in the extremal cases of $n = 2, 3, q - 1, q$ (see [1,2]). The purpose of this note is to improve the knowledge of the intermediate cases by proving the following result.

Theorem 1. *Let q be a power of the prime p. For any divisor d of $q - 1$ with $2 < d < p$, there exists a (k, n)-arc in $\mathrm{PG}(2, q)$ with*

$$k \geqslant (q^2 + q + 1)/d - (d - 2)\sqrt{q} - 11dq$$

and $n \leqslant (q + 1)/d + (d - 2)\sqrt{q}$.

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Proof. Let d be a divisor of $q-1$ and let $f(x,y,z)$ be a homogeneous polynomial of degree d such that the curve $X: f(x,y,z)=0$ is smooth. For any coset C of the dth powers in $\mathrm{GF}(q)^*$, the set $\{(x,y,z)\in \mathrm{PG}(2,q) \mid f(x,y,z)\in C\}$ is well-defined and the union of those sets is the complement of X in $\mathrm{PG}(2,q)$. Therefore, one of these sets has at least $(q^2+q+1-\#X(\mathrm{GF}(q)))/d$ points. We select this set K, minus any lines it might contain, as our arc and the bound on k follows from the Weil bound applied to X once we show that K contains at most $11d-24$ lines.

Let L be a line in $\mathrm{PG}(2,q)$, which we may assume by a change of coordinates to be $z=0$. The number of points of the intersection of our set with L is the number of points (x,y) in $\mathrm{PG}(1,q)$ with $f(x,y,0)$ in the chosen coset. By choosing a coset representative c we want to count the number of solutions to $f(x,y)=cw^d$ and note that a given (x,y) will lead to d values of w. The equation $f(x,y,0)=cw^d$ determines an algebraic curve and it is easy to check that it is either smooth or a union of d lines. If the curve is smooth, then the bound on the number of points (x,y) follows from the Weil bound applied to this curve. If it is a union of lines, then L is contained in K but then it was removed in the original construction.

To show the upper bound on the lines we note that the lines correspond to sets of d lines on the surface $f(x,y,z)=cw^d$. Finally, the surface contains at most $d(11d-24)$ by the positive characteristic analogue of Salmon's theorem, proved in [4].

The construction of the theorem also works for $d=2$ provided that the surface $f(x,y,z)=cw^2$ in the proof is an elliptic quadric so that it has no lines. The resulting arcs probably coincide with the Barlotti arcs (see [1]) but we have not checked this.

By taking a pencil of lines through a point of a (k,n)-arc, we get that $k\leqslant(n-1)q+n$ and there are only small improvements known to this bound in general (see [2]). A way to produce (k,n)-arcs is to consider algebraic curves of degree n with no linear components. For small n those are usually complete as arcs but k is much smaller than the above bound. We note that, in [3], we constructed smooth algebraic curves of degree n with $k=n(q+1+n)/2$ points for $n=q-1-2d$ for the divisors $d<(q-1)/2$ of $q-1$. So these are somewhat large as (k,n)-arcs and it would be interesting to know whether they are complete.

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