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The Jacobi matrices approach to Nevanlinna–Pick problems

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Abstract

A modification of the well-known step-by-step process for solving Nevanlinna–Pick problems in the class of $\mathbf{R}_0$-functions gives rise to a linear pencil $H-\lambda J$, where H and J are Hermitian tridiagonal matrices. First, we show that J is a positive operator. Then it is proved that the corresponding Nevanlinna–Pick problem has a unique solution iff the densely defined symmetric operator $J^{-\frac{1}{2}}HJ^{-\frac{1}{2}}$ is self-adjoint and some criteria for this operator to be self-adjoint are presented. Finally, by means of the operator technique, we obtain that multipoint diagonal Padé approximants to a unique solution φ of the Nevanlinna–Pick problem converge to φ locally uniformly in $\mathbb{C}\setminus\mathbb{R}$. The proposed scheme extends the classical Jacobi matrix approach to moment problems and Padé approximation for $\mathbf{R}_0$-functions.

Keywords: Jacobi matrix; Linear pencil; Nevanlinna–Pick problem; Multipoint Padé approximants

1. Introduction

The connection with Jacobi matrices has led to numerous applications of spectral techniques for self-adjoint operators in the theory of moment problems, orthogonal polynomials on the real line, and Padé approximation. Let us recall some basic ideas of this interplay. First, note that one of the key tools in relating these theories is the class $\mathbf{R}_0$ of all functions having the representation

$$\varphi(\lambda)=\int_{\mathbb{R}}\frac{\mathrm{d}\sigma(t)}{t-\lambda}, \tag{1.1}$$

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where σ is a probability measure, that is, $\int_{\mathbb{R}} \mathrm{d}\sigma(t) = 1$. If the support $\operatorname{supp}\sigma$ of σ is contained in $[\alpha, \beta]$ we will say that $\varphi \in \mathbf{R}[\alpha, \beta]$.

Consider a probability measure σ such that all the moments

$$s_n := \int_{\mathbb{R}} t^n \mathrm{d}\sigma(t), \quad n \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\} \tag{1.2}$$

are finite. In this case, the corresponding function φ has the following asymptotic expansion:

$$\varphi(\lambda) = -\frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \cdots - \frac{s_{2n}}{\lambda^{2n+1}} + o\left(\frac{1}{\lambda^{2n+1}}\right), \quad \lambda \widehat{\rightarrow} \infty, \tag{1.3}$$

for every $n \in \mathbb{Z}_+$ (here and throughout in the sequel $\lambda \widehat{\rightarrow} \infty$ means that λ tends to ∞ non-tangentially, that is, inside the sector $\varepsilon < \arg\lambda < \pi - \varepsilon$ for some $\varepsilon > 0$). In view of the Hamburger–Nevanlinna theorem [1], the classical moment problem reads as follows.

Hamburger moment problem. Is the function $\varphi \in \mathbf{R}_0$ satisfying (1.3) uniquely determined by the sequence $\{s_j\}_{j=0}^{\infty}$ of moments?

The moment problem is called determinate if φ is uniquely determined. Otherwise the moment problem is said to be indeterminate. In fact, one can give an answer to the question in terms of the underlying Jacobi operators generated by Jacobi matrices. To see Jacobi matrices in this context, note that one can expand φ into the following continued fraction:

$$\varphi(\lambda) = -\cfrac{1}{\lambda - a_0 - \cfrac{b_0^2}{\lambda - a_1 - \cfrac{b_1^2}{\ddots}}} = -\frac{1\,|}{|\,\lambda - a_0} - \frac{b_0^2\,|}{|\,\lambda - a_1} - \frac{b_1^2\,|}{|\,\lambda - a_2} - \cdots, \tag{1.4}$$

where a_j are real numbers, b_j are positive numbers (see [1,40,49]). Moreover, numbers a_j and b_j can be explicitly expressed in terms of the moments $s_0, \ldots, s_{2j+1}$ [1]. Continued fractions of the form (1.4) are called J-fractions [35,49]. With the continued fraction (1.4) one can associate a Jacobi matrix H and its truncation $H_{[0,n-1]}$:

$$H = \begin{pmatrix} a_0 & b_0 & & \\ b_0 & a_1 & b_1 & \\ & b_1 & a_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \qquad H_{[0,n-1]} = \begin{pmatrix} a_0 & b_0 & & \\ b_0 & a_1 & \ddots & \\ & \ddots & \ddots & b_{n-2} \\ & & b_{n-2} & a_{n-1} \end{pmatrix}.$$

Let $\ell^2_{[0,\infty)}$ denote a Hilbert space of complex square summable sequences $(x_0, x_1, \ldots)$ equipped with the inner product

$$(x, y) = \sum_{i=0}^{\infty} x_i \overline{y}_i, \quad x, y \in \ell^2_{[0,\infty)}.$$

Now, in the standard way, we can define a minimal closed operator H acting in ℓ^2 generated by the matrix H [1,12]. We will denote the domain of H and the range of H by $\operatorname{dom} H$ and $\operatorname{ran} H$, respectively. It is easy to see that H is symmetric, i.e.

$$(Hx, y) = (x, Hy), \quad x, y \in \operatorname{dom} H.$$

Moreover, it is well known that H is self-adjoint if and only if the corresponding moment problem is determinate and the solution of the problem admits the representation

$$\varphi(\lambda) = \Big((H-\lambda)^{-1}e_0, e_0\Big)$$

where $e = (1, 0, \dots)^\top$ is a column vector (see [1,42]). In the indeterminate case, a description of all $\varphi \in \mathbf{R}_0$ satisfying (1.3) can be found in [1,15,42] (see also [24] where the operator approach to truncated moment problems was proposed). In both cases, we have

$$-\frac{Q_n(\lambda)}{P_n(\lambda)} = \Big((H_{[0,n-1]}-\lambda)^{-1}e_0, e_0\Big) = -\frac{1}{\lambda - a_0} - \dots - \frac{b_{n-2}^2}{\lambda - a_{n-1}},$$

where P_n are orthogonal polynomials with respect to σ, and Q_n are polynomials of the second kind (see [1,40,42]). It is an elementary fact of the continued fraction theory (see, for instance, [1,5,35]) that

$$\varphi(\lambda) + \frac{Q_n(\lambda)}{P_n(\lambda)} = O\left(\frac{1}{\lambda^{2n+1}}\right), \quad \lambda \widehat{\to} \infty. \tag{1.5}$$

In other words, relation (1.5) means that the rational function $-Q_n/P_n$ is the nth diagonal Padé approximant to φ at ∞ (for more details on Padé approximants see [5]). Now, we see that in the self-adjoint case, convergence of diagonal Padé approximants appears as the strong resolvent convergence of the finite matrix approximations $H_{[0,n]}$ to H. So, if the moment problem is determinate then the corresponding diagonal Padé approximants converge to the solution φ locally uniformly in $\mathbb{C} \setminus \mathbb{R}$. This statement for the class $\mathbf{R}[\alpha, \beta]$ is known as the Markov theorem [40]. The above-described scheme has been recently extended to the case of rational perturbations of Nevanlinna functions [20–22]. Also, the scheme was adapted to the case of complex Jacobi matrices [10] and generalized to the case of band matrices [9].

The main goal of this paper is to generalize the scheme to the case of Nevanlinna–Pick problems and to prove convergence of related multipoint diagonal Padé approximants. To show our purpose more precisely, let us recall that the classical Hamburger moment problem is the limiting case of the following problem (see [1,27,36]).

Nevanlinna–Pick problem. Let $\{z_k\}_{k=0}^\infty$ be a sequence of distinct numbers from the upper half-plane $\mathbb{C}_+$ and let $\varphi \in \mathbf{R}_0$. Define numbers $w_j := \varphi(z_j)$. Is the function $\varphi \in \mathbf{R}_0$ satisfying the interpolation relation $\varphi(z_j) = w_j$, $j \in \mathbb{Z}_+$, uniquely determined by the given data $\{z_k\}_{k=0}^\infty$, $\{w_k\}_{k=0}^\infty$?

In view of the classical uniqueness theorem for analytic functions, the answer to this question is trivial if the sequence $\{z_k\}_{k=0}^\infty$ has at least one accumulation point in $\mathbb{C}_+$. So, *in what follows we will suppose that the sequence* $\{z_k\}_{k=0}^\infty$ *does not have any accumulation point in* $\mathbb{C}_+$. In other words, all the accumulation points of the sequence $\{z_k\}_{k=0}^\infty$ lie in $\mathbb{R}$.

Like for the moment problem case, the Nevanlinna–Pick problem is called determinate if φ is uniquely determined. Otherwise the Nevanlinna–Pick problem is said to be indeterminate. We should also note that diagonal Padé approximants at ∞ are the limiting case of the following multipoint diagonal Padé approximants.

Definition 1.1 (*[5]*)**.** The nth multipoint diagonal Padé approximant for the function φ at the points $\{z_0, \overline{z}_0, \dots, z_j, \overline{z}_j, \dots\}$ is defined as a ratio $-Q_n/P_n$ of two polynomials Q_n, P_n of degrees at most $n-1$ and n, respectively, such that the function $P_n\varphi + Q_n$ vanishes at the points $z_0, \overline{z}_0, \dots, z_{n-1}, \overline{z}_{n-1}$.

It appears that the problem of finding multipoint diagonal Padé approximants for the $\mathbf{R}_0$-function φ at the points $\{z_0, \overline{z}_0, \dots, z_j, \overline{z}_j, \dots\}$ is closely related to a continued fraction expansion of the following type

$$-\frac{1}{\left| a_0^{(2)}\lambda - a_0^{(1)}\right.} - \frac{b_0^2(\lambda - z_0)(\lambda - \overline{z}_0)\Big|}{\left| a_1^{(2)}\lambda - a_1^{(1)}\right.} - \frac{b_1^2(\lambda - z_1)(\lambda - \overline{z}_1)\Big|}{\left| a_2^{(2)}\lambda - a_2^{(1)}\right.} - \cdots, \tag{1.6}$$

where $a_j^{(1)}$ are real numbers and $a_j^{(2)}$, b_j are positive numbers. This continued fraction gives rise to a tridiagonal linear pencil $H - \lambda J$, where H and J are semi-infinite tridiagonal matrices [23] (see also [50] where tridiagonal linear pencils associated with general continued fractions of type (1.6) were introduced). In this paper, we firstly obtain that J generates a positive operator. Then we introduce a densely defined symmetric operator $J^{-\frac{1}{2}} H J^{-\frac{1}{2}}$ and present criteria for this operator to be self-adjoint. Next, we prove that the Nevanlinna–Pick problem in question has a unique solution if and only if $J^{-\frac{1}{2}} H J^{-\frac{1}{2}}$ is self-adjoint. Finally, we show that if $J^{-\frac{1}{2}} H J^{-\frac{1}{2}}$ is self-adjoint then the locally uniform convergence of the multipoint diagonal Padé approximants

$$-\frac{Q_{n+1}(\lambda)}{P_{n+1}(\lambda)} = \left(\left(J_{[0,n]}^{-\frac{1}{2}} H_{[0,n]} J_{[0,n]}^{-\frac{1}{2}} - \lambda \right)^{-1} J_{[0,n]}^{-\frac{1}{2}} e_0, J_{[0,n]}^{-\frac{1}{2}} e_0 \right)$$

to the unique solution

$$\varphi(\lambda) = \left(\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}} - \lambda \right)^{-1} J^{-\frac{1}{2}} e_0, J^{-\frac{1}{2}} e_0 \right)$$

of the Nevanlinna–Pick problem arises as the resolvent convergence.

The paper is organized as follows. In Section 2 we present the step-by-step process for solving the Nevanlinna–Pick problems and associated sequences of polynomials. In Section 3, a tridiagonal linear pencil is introduced and basic properties of the operator J are given. The one-to-one correspondence between tridiagonal linear pencils and the Nevanlinna–Pick problems in question is shown in Section 4. The next section is concerned with the Weyl circles. Section 6 reveals the underlying symmetric operators. In Section 7, we characterize the determinacy of the underlying Nevanlinna–Pick problems in terms of the self-adjointness of $J^{-\frac{1}{2}} H J^{-\frac{1}{2}}$. After that, in Section 8, for the determinate case, we prove the locally uniform convergence of multipoint diagonal Padé approximants for $\mathbf{R}_0$-functions.

2. The modified multipoint Schur algorithm

As is known, the Schur transformation is a powerful tool in solving moment and interpolation problems (see [1,3]). The starting point of our analysis is the following modification of the Schur transformation.

Proposition 2.1 (*Cf. [23]*)**.** *Let $\varphi \in \mathbf{R}_0$ and let $z \in \mathbb{C}_+$ be a fixed number. Then there exist unique numbers $a^{(1)}, a^{(2)} \in \mathbb{R}$ and $b > 0$ such that the function φ_1 defined by the equality*

$$\varphi(\lambda) = -\frac{1}{a^{(2)}\lambda - a^{(1)} + b^2(\lambda - z)(\lambda - \overline{z})\varphi_1(\lambda)} \tag{2.1}$$

belongs to $\mathbf{R}_0 \cup \{0\}$, *that is,* φ_1 *has the representation* (1.1) *with a probability measure in the case* $\varphi_1 \not\equiv 0$. *Moreover, we have that*

$$b^2 = a^{(2)} - 1. \tag{2.2}$$

Proof. To see that the numbers $a^{(1)}, a^{(2)}$ are uniquely determined, let us substitute λ for z and $\overline{z}$ in (2.1). We thus get

$$a^{(2)}z - a^{(1)} = -\frac{1}{\varphi(z)}, \qquad a^{(2)}\overline{z} - a^{(1)} = -\frac{1}{\varphi(\overline{z})}. \tag{2.3}$$

Eliminating from the above relations $a^{(1)}$ and $a^{(2)}$, one can obtain the following formulas:

$$a^{(1)} = \left(\int_{\mathbb{R}} \frac{t\,\mathrm{d}\sigma(t)}{|t-z|^2}\right)\left|\int_{\mathbb{R}} \frac{\mathrm{d}\sigma(t)}{t-z}\right|^{-2}, \qquad a^{(2)} = \left(\int_{\mathbb{R}} \frac{\mathrm{d}\sigma(t)}{|t-z|^2}\right)\left|\int_{\mathbb{R}} \frac{\mathrm{d}\sigma(t)}{t-z}\right|^{-2}. \tag{2.4}$$

Further, it follows from the Schwarz lemma that

$$\widetilde{\varphi}_1(\lambda) = -\frac{\frac{1}{\varphi(\lambda)} + a^{(2)}\lambda - a^{(1)}}{(\lambda - z)(\lambda - \overline{z})} = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{t-\lambda} \tag{2.5}$$

(the proof of this fact is in line with that of [23, Lemma 3.1]). Choosing $b > 0$ in the following way:

$$b^2 = \int_{\mathbb{R}} \mathrm{d}\mu(t)$$

and defining $\varphi_1 := \widetilde{\varphi}_1/b^2$ we get that the function φ_1 possesses the integral representation (1.1) with a probability measure. Finally, by taking $\lambda = \mathrm{i}y$ and $y \to \infty$ in (2.5) we get (2.2). □

Remark 2.2. It should be noted that for $\varphi \in \mathbf{R}[\alpha, \beta]$ this modification of the Schur algorithm was presented in [23, Lemma 3.1]. However, its proof is valid for $\varphi \in \mathbf{R}_0$. A similar transformation for Caratheodory functions was proposed in [19].

Let φ be a non-rational function of the class $\mathbf{R}_0$, i.e. φ admits the representation (1.1) with a probability measure which has an infinite support. Let also an infinite sequence $\{z_k\}_{k=0}^{\infty} \subset \mathbb{C}_+$ of distinct numbers be given. Since φ is not rational the given data give rise to infinitely many steps of the step-by-step process. So, we have infinitely many linear fractional transformations of the form (2.1) which lead to the following continued fraction:

$$-\frac{1}{\left|a_0^{(2)}\lambda - a_0^{(1)}\right.} - \frac{b_0^2(\lambda - z_0)(\lambda - \overline{z}_0)}{\left|a_1^{(2)}\lambda - a_1^{(1)}\right.} - \frac{b_1^2(\lambda - z_1)(\lambda - \overline{z}_1)}{\left|a_2^{(2)}\lambda - a_2^{(1)}\right.} - \cdots \tag{2.6}$$

(for more details, see [23]). It should be noted that general continued fractions associated with finding multipoint Padé approximants were introduced in [32] and studied in [33,34].

It is immediate from the construction that the $(n+1)$th convergent of (2.6)

$$-\frac{Q_{n+1}(\lambda)}{P_{n+1}(\lambda)} = -\frac{1}{\left|a_0^{(2)}\lambda - a_0^{(1)}\right.} - \cdots - \frac{b_{n-1}^2(\lambda - z_{n-1})(\lambda - \overline{z}_{n-1})}{\left|a_n^{(2)}\lambda - a_n^{(1)}\right.}$$

satisfies the following interpolation relation:

$$\varphi(z_j) = -\frac{Q_{n+1}(z_j)}{P_{n+1}(z_j)}, \quad j = 0, \ldots, n. \tag{2.7}$$

Since $\varphi \in \mathbf{R}_0$ and the coefficients $a_j^{(1)}, a_j^{(2)}, b_j$ are real, one also has

$$\varphi(\overline{z}_j) = -\frac{Q_{n+1}(\overline{z}_j)}{P_{n+1}(\overline{z}_j)}, \quad j = 0, \ldots, n.$$

So, we have just concluded the following.

Proposition 2.3. *The rational function $-Q_{n+1}/P_{n+1}$ is the $(n+1)$th multipoint diagonal Padé approximant to φ at the points $\{z_0, \overline{z}_0, \ldots, z_j, \overline{z}_j, \ldots\}$.*

It is well known that denominators and numerators of the convergents of a continued fraction satisfy a three-term recurrence relation (see, for instance, [35]). In particular, for the continued fraction (2.6) the recurrence relation takes the following form:

$$u_{j+1} - (a_j^{(2)}\lambda - a_j^{(1)})u_j + b_{j-1}^2(\lambda - z_{j-1})(\lambda - \overline{z}_{j-1})u_{j-1} = 0, \quad j \in \mathbb{N}. \tag{2.8}$$

Further, the polynomials P_j of the first kind are solutions $u_j = P_j(\lambda)$ of the system (2.8) with the initial conditions

$$u_0 = 1, \qquad u_1 = a_0^{(2)}\lambda - a_0^{(1)}. \tag{2.9}$$

Similarly, the polynomials of the second kind $Q_j(\lambda)$ are solutions $u_j = Q_j(\lambda)$ of the system (2.8) subject to the following initial conditions:

$$u_0 = 0, \qquad u_1 = -1. \tag{2.10}$$

Remark 2.4. Note that the polynomials P_j are orthogonal with respect to the varying measures $\frac{d\sigma(t)}{\prod_{k=0}^{j-1}|t-z_k|^2}$ (see [29,37], [47, Section 6.1]). Moreover, for $\varphi \in \mathbf{R}[\alpha, \beta]$ an operator treatment of the relation of the polynomials P_j to orthogonal rational functions was presented in [23] (see [17, Section 9.5], where this relation is also discussed). It should be also remarked that some orthogonality relations for polynomials and rational functions related to general continued fractions of type (2.6) were obtained in [34,50] (see also [51], where biorthogonality properties of rational functions related to multipoint Padé approximation were studied and concrete examples connected with generalized hypergeometric functions were constructed).

3. Tridiagonal linear pencils associated with $\mathbf{R}_0$-functions

In order to see linear pencils in our context, let us note that the recurrence relation (2.8) can be renormalized to the following one:

$$(\overline{\mathfrak{b}}_{j-1} - \lambda\mathfrak{d}_{j-1})\widehat{u}_{j-1} + (\mathfrak{a}_j - \lambda\mathfrak{c}_j)\widehat{u}_j + (\mathfrak{b}_j - \lambda\mathfrak{d}_j)\widehat{u}_{j+1} = 0, \quad j \in \mathbb{N}, \tag{3.1}$$

where the numbers $\mathfrak{a}_j, \mathfrak{b}_j, \mathfrak{c}_j, \mathfrak{d}_j$ are defined as follows:

$$\mathfrak{a}_j = a_j^{(1)}, \qquad \mathfrak{b}_j = z_j b_j, \qquad \mathfrak{c}_j = a_j^{(2)}, \qquad \mathfrak{d}_j = b_j, \quad j \in \mathbb{Z}_+,$$

and the transformation $u \to \widehat{u}$ has the following form:

$$\widehat{u}_0 = u_0, \qquad \widehat{u}_j = \frac{u_j}{b_0 \cdots b_{j-1}(z_0 - \lambda) \cdots (z_{j-1} - \lambda)}, \quad j \in \mathbb{N}. \tag{3.2}$$

Thus, we have two associated sequences $\widehat{P}_j$ and $\widehat{Q}_j$ of rational functions obtained from the polynomial sequences P_j and Q_j, respectively, by means of the transformation (3.2). In contrast to the polynomial case, the rational functions $\widehat{P}_j$ are not orthogonal with respect to the original measure σ since

$$\int_{\mathbb{R}} \widehat{P}_0(t)\widehat{P}_1(t)\mathrm{d}\sigma(t) = \int_{\mathbb{R}} \widehat{P}_1(t)\mathrm{d}\sigma(t) = 1 - a_0^{(2)} \tag{3.3}$$

and, due to (2.4), $1 - a_0^{(2)} \neq 0$ for any $z_0 \in \mathbb{C}_+$. Despite this, some orthogonality properties remain valid (see [11, Theorem 2.10]). It should also be noted that some orthogonal proper rational functions satisfy a relation similar to (3.1) [4, p. 541] (see also [17] for the recurrence relations for orthogonal rational functions).

The relation (3.1) naturally leads to a linear pencil $H - \lambda J$, where

$$H = \begin{pmatrix} \mathfrak{a}_0 & \mathfrak{b}_0 & & \\ \overline{\mathfrak{b}}_0 & \mathfrak{a}_1 & \mathfrak{b}_1 & \\ & \overline{\mathfrak{b}}_1 & \mathfrak{a}_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \qquad J = \begin{pmatrix} \mathfrak{c}_0 & \mathfrak{d}_0 & & \\ \mathfrak{d}_0 & \mathfrak{c}_1 & \mathfrak{d}_1 & \\ & \mathfrak{d}_1 & \mathfrak{c}_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

are Jacobi matrices. For an infinite matrix A, we denote by $A_{[j,k]}$ the square sub-matrix obtained by taking rows and columns $l = j, j+1, \ldots, k \leq \infty$. For example, for finite j and k we have that

$$H_{[j,k]} = \begin{pmatrix} \mathfrak{a}_j & \mathfrak{b}_j & \mathbf{0} \\ \overline{\mathfrak{b}}_j & \ddots & \\ \mathbf{0} & & \mathfrak{a}_k \end{pmatrix}, \qquad J_{[j,k]} = \begin{pmatrix} \mathfrak{c}_j & \mathfrak{d}_j & \mathbf{0} \\ \mathfrak{d}_j & \ddots & \\ \mathbf{0} & & \mathfrak{c}_k \end{pmatrix}.$$

By J we also denote the minimal closed operator on $\ell^2_{[0,\infty)}$ generated by the matrix J [1]. Obviously, J is a symmetric operator. Besides, due to (2.2), we have the relation $\mathfrak{c}_j = 1 + \mathfrak{d}_j^2$, which gives us the following factorization of J:

$$J = L^*L = \begin{pmatrix} 1 & \mathfrak{d}_0 & & \\ 0 & 1 & \mathfrak{d}_1 & \\ & 0 & 1 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & & \\ \mathfrak{d}_0 & 1 & 0 & \\ & \mathfrak{d}_1 & 1 & \ddots \\ & & \ddots & \ddots \end{pmatrix}. \tag{3.4}$$

The factorization of J allows us to say a bit more about J.

Proposition 3.1. *The operator J is self-adjoint and positive, that is,*

$$(Jx, x) > 0, \quad x \in \operatorname{dom} J \setminus \{0\}.$$

In particular, $\ker J = \{0\}$.

Proof. Let us consider the Hermitian form $(J\xi, \xi)$ on finitely supported sequences ξ, that is, $\xi = (\xi_0, \xi_1, \ldots, \xi_n, 0, 0, \ldots)^\top$. By virtue of (3.4), we have that

$$(J\xi, \xi) = (L\xi, L\xi) \geq 0.$$

Further, let us prove that $\ker J^* = \{0\}$. Suppose the converse, that is, there exists $\eta \in \ell^2$ such that $J^*\eta = 0$ and $\eta \neq 0$. Taking into account the structure of J we get the equality

$$0 = (J^*\eta, \eta) = |\eta_0|^2 + |\mathfrak{d}_0\eta_0 + \eta_1|^2 + \cdots + |\mathfrak{d}_{n-1}\eta_{n-1} + \eta_n|^2 + \cdots,$$

which implies $\eta = 0$. So, $\ker J = \ker J^* = \{0\}$. This contradiction also shows that

$$\sum_{k=0}^{\infty} |p_k(0)|^2 = \infty, \tag{3.5}$$

where p_j are polynomials of the first kind associated with J. Since the relation (3.5) doesn't hold true for Jacobi operators with deficiency indices (1, 1) (see [12,42]), we obtain that J is self-adjoint. The statement of the proposition also immediately follows from [12, Theorem VII.1.4]. □

Remark 3.2. It has been recently proved [11] that if $\varphi \in \mathbf{R}[\alpha, \beta]$ and $z_k \to \infty$ then

$$(Jx, x) \geq \delta(x, x), \quad x \in \ell^2,$$

for some $\delta > 0$. Furthermore, in this case the operator J is a compact perturbation of I and, in fact, the linear pencil $H - \lambda J$ is a compact perturbation of the classical pencil $H_0 - \lambda I$ (which corresponds to the limiting case $z_k = \infty$ for $k = 0, 1, 2, \ldots$). It should be noted that for the case of orthogonal Laurent polynomials a similar tridiagonal pencil was considered in [18]. Roughly speaking, the case of orthogonal Laurent polynomials corresponds to the multiple interpolation at 0 and ∞, which is known as the strong moment problem on the real line [35]. An operator approach to the strong moment problem was given in [31]. It is also worth noting that, in the matrix case, Jacobi type symmetric operators related to the matrix strong moment problems were presented and studied in [44,45].

Since $\ker J = \{0\}$ and J is self-adjoint, we can consider the self-adjoint operator $J^{-\frac{1}{2}}$, which is not necessarily bounded. However, the following statement holds true.

Proposition 3.3. *We have that*

$$e_j \in \operatorname{dom} J^{-\frac{1}{2}}, \quad j \in \mathbb{Z}_+, \tag{3.6}$$

where the vectors $e_0 = (1, 0, 0, \ldots)^\top, e_1 = (0, 1, 0, \ldots)^\top, \ldots$ *form the standard basis in* ℓ^2.

Proof. It is the basic spectral theory that for the positive operator J there exists a resolution of the identity E_t such that

$$Jf = \int_0^\infty t \, \mathrm{d}E_t f, \quad f \in \operatorname{dom} J,$$

and $f \in \operatorname{dom} J$ if and only if $\int_0^\infty t^2 \mathrm{d}(E_t f, f) < \infty$ [2, Section 66]. Moreover, we also have that

$$J^{-\frac{1}{2}} f = \int_0^\infty \frac{1}{\sqrt{t}} \mathrm{d}E_t f, \quad f \in \operatorname{dom} J^{-\frac{1}{2}},$$

and $f \in \operatorname{dom} J^{-\frac{1}{2}}$ if and only if $\int_0^\infty \frac{1}{t} \mathrm{d}(E_t f, f) < \infty$. Now, (3.6) is equivalent to

$$\int_0^\infty \frac{1}{t} \mathrm{d}(E_t e_j, e_j) < \infty, \quad j \in \mathbb{Z}_+.$$

First we will prove that

$$\int_0^\infty \frac{1}{t} \mathrm{d}(E_t e_0, e_0) < \infty. \tag{3.7}$$

For simplicity, let us define $\nu = (E_. e_0, e_0)$ and introduce the similar measures $\nu_n = (E_.^{(n)} e_0, e_0)$ for the truncations $J_{[0,n]}$, where $E_.^{(n)}$ is such that

$$J_{[0,n]} = \int_0^\infty t \mathrm{d}E_t^{(n)}, \quad n \in \mathbb{Z}_+.$$

Next, it is a standard fact of theory of moment problems [1] that

$$\int_0^\infty \psi(t) \mathrm{d}\nu_n(t) \to \int_0^\infty \psi(t) \mathrm{d}\nu(t), \quad n \to \infty,$$

for any simple function ψ (that is, ψ is measurable and assumes only a finite number of values). Now, recall that in [23, Lemma 6.1] it was proved that

$$\int_0^\infty \frac{1}{t} \mathrm{d}\nu_n(t) = \left(J_{[0,n]}^{-1} e_0, e_0\right) \le 1, \quad n \in \mathbb{Z}_+. \tag{3.8}$$

Thus, Fatou's lemma for varying measures [41, Proposition 17, p. 231] and (3.8) yield

$$\int_0^\infty \frac{1}{t} \mathrm{d}\nu(t) \le \liminf_{n\to\infty} \int_0^\infty \frac{1}{t} \mathrm{d}\nu_n(t) \le 1. \tag{3.9}$$

The rest is a consequence of (3.7). Indeed, it is well known that for any λ from the resolvent set $\rho(J)$ of the operator J we have the following formula for the diagonal Green function:

$$\left((J-\lambda)^{-1} e_j, e_j\right) = p_j(\lambda)\left(p_j(\lambda)\left((J-\lambda)^{-1} e_0, e_0\right) + q_j(\lambda)\right), \quad j \in \mathbb{Z}_+, \tag{3.10}$$

where p_j and q_j are polynomials of the first and second kinds, respectively, associated with the Jacobi matrix J (see for example [10, Theorem 2.10], [28, Proposition 2.2]). Putting $\lambda = -x$, $x > 0$, into formula (3.10), it can be rewritten as follows:

$$\int_0^\infty \frac{1}{t+x} \mathrm{d}(E_t e_j, e_j) = p_j(-x)\left(p_j(-x) \int_0^\infty \frac{1}{t+x} \mathrm{d}\nu(t) + q_j(-x)\right), \quad j \in \mathbb{N},$$

where $p_j(-x) = \frac{\det(J_{[0,j-1]}+x)}{\mathfrak{d}_0 \cdots \mathfrak{d}_{j-1}} > 0$ for $x \ge 0$. Now, it remains to apply the Fatou lemma to $\int_0^\infty \frac{1}{t+x} \mathrm{d}(E_t e_j, e_j)$ as $x \to 0$ and to use (3.7). □

Remark 3.4. The main ingredient in the proof was obtaining (3.7). Another way to prove it is through the Darboux transformations. Namely, let us consider a Jacobi matrix $J_1 = LL^*$ and let ν^* be a corresponding probability measure associated with J_1. Then it follows from [16, Theorem 3.4] that

$$\mathrm{d}\nu(t) = ct \mathrm{d}\nu^*(t), \quad c > 0.$$

The latter relation immediately implies (3.7).

To end this section, note that we can now say more about the sequence $\left(J_{[0,n]}^{-1}e_0, e_0\right)$. Namely, the following relation holds true:

$$\left(J_{[0,n]}^{-1}e_0, e_0\right) \to 1, \quad \text{as } n \to \infty. \tag{3.11}$$

Indeed, by applying [28, Formula (2.15)] we see that $\left(J_{[0,n]}^{-1}e_0, e_0\right)$, $n \in \mathbb{Z}_+$, are convergents of the continued fraction

$$\frac{1}{\lceil \mathfrak{c}_0} - \frac{\mathfrak{d}_0}{\lceil \mathfrak{c}_1} - \frac{\mathfrak{d}_1}{\lceil \mathfrak{c}_2} - \cdots.$$

For as long as $\mathfrak{c}_j = 1 + \mathfrak{d}_j^2$, applying the remark to the Śleszyński–Pringsheim theorem given on [35, p. 93] implies (3.11).

4. Relations between Nevanlinna–Pick problems and linear pencils

In this section we show that there exists a one-to-one correspondence between the linear pencils under consideration and the Nevanlinna–Pick problems in question. We also re-examine some facts for the polynomials P_j and Q_j which are well known for orthogonal polynomials.

We begin with the following connection between the polynomials of the first and second kinds P_j, Q_j and the truncated linear pencils $\lambda J_{[0,j]} - H_{[0,j]}$, which in the classical case can be found in [12, Section 7.1.2] and [4, Section 6.1].

Proposition 4.1. *The polynomials P_j and Q_j, $j \in \mathbb{N}$, can be found by using the formulas*

$$P_j(\lambda) = \det(\lambda J_{[0,j-1]} - H_{[0,j-1]}), \qquad Q_j(\lambda) = \det(\lambda J_{[1,j-1]} - H_{[1,j-1]}). \tag{4.1}$$

The zeros of the polynomials P_j and Q_j are real. Moreover, the polynomials P_j and Q_j do not have common zeros.

Proof. Formula (4.1) immediately follows from the definition of P_j and Q_j by using the Laplace expansions of the determinants in terms of the last row. Since $J_{[0,j-1]}$ is strictly positive, one can rewrite the first relation in (4.1) as follows:

$$P_j(\lambda) = \det J_{[0,j-1]}^{1/2} \det\left(\lambda - J_{[0,j-1]}^{-1/2} H_{[0,j-1]} J_{[0,j-1]}^{-1/2}\right) \det J_{[0,j-1]}^{1/2}.$$

Clearly, $J_{[0,j-1]}^{-1/2} H_{[0,j-1]} J_{[0,j-1]}^{-1/2}$ is a self-adjoint matrix. Thus, the latter relation yields the fact that the zeros of P_j are real. Similarly, one can show that the zeros of Q_j are real. The last statement follows by induction via applying the Laplace expansion of the determinant $\det(\lambda J_{[0,j-1]} - H_{[0,j-1]})$ in terms of the first row. $\square$

By induction, one easily gets from (3.1) the Liouville–Ostrogradsky formula

$$Q_{n+1}(\lambda)P_n(\lambda) - Q_n(\lambda)P_{n+1}(\lambda) = \prod_{k=0}^{n-1} b_k^2(\lambda - z_k)(\lambda - \overline{z}_k), \tag{4.2}$$

for every $n \in \mathbb{Z}_+$ (see [11]). Going further in this direction, we should note that, sometimes, it is very useful to have (3.1) in the following matrix form:

$$(H - \lambda J)\pi_{[0,j]}(\lambda) = -(\mathfrak{b}_j - \lambda\mathfrak{d}_j)\widehat{P}_{j+1}(\lambda)e_j + (\overline{\mathfrak{b}}_j - \lambda\mathfrak{d}_j)\widehat{P}_j(\lambda)e_{j+1}, \tag{4.3}$$

$$(H - \lambda J)\xi_{[0,j]}(\lambda) = -(\mathfrak{b}_j - \lambda\mathfrak{d}_j)\widehat{Q}_{j+1}(\lambda)e_j + (\overline{\mathfrak{b}}_j - \lambda\mathfrak{d}_j)\widehat{Q}_j(\lambda)e_{j+1} + e_0, \tag{4.4}$$

where the vectors $\pi_{[0,j]}(\lambda)$ and $\xi_{[0,j]}(\lambda)$ are defined as follows:

$$\pi_{[0,j]}(\lambda) = \left(\widehat{P}_0(\lambda), \widehat{P}_1(\lambda), \ldots, \widehat{P}_j(\lambda), 0, 0, \ldots\right)^\top,$$
$$\xi_{[0,j]}(\lambda) = \left(\widehat{Q}_0(\lambda), \widehat{Q}_1(\lambda), \ldots, \widehat{Q}_j(\lambda), 0, 0, \ldots\right)^\top.$$

For example, by virtue of (4.3) we get the following generalization of the Christoffel–Darboux formula.

Proposition 4.2. *We have that for $j \in \mathbb{Z}_+$*

$$(\lambda - \overline{\zeta}) \sum_{k=0}^{j} (\widehat{P}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{P}_{k-1}(\lambda))\overline{(\widehat{P}_k(\zeta) + \mathfrak{d}_{k-1}\widehat{P}_{k-1}(\zeta))}$$
$$= \frac{P_{j+1}(\lambda)\overline{P_j(\zeta)} - \overline{P_{j+1}(\zeta)}P_j(\lambda)}{\prod_{k=0}^{j-1} b_k^2(\lambda - z_k)(\overline{\zeta} - \overline{z}_k)}, \tag{4.5}$$

where $\mathfrak{d}_{-1} = 0$ for convenience and $\lambda, \zeta \in \mathbb{C}_+ \setminus \{z_k\}_{k=0}^{j}$.

Proof. It clearly follows from (4.3) that

$$\left((H - \lambda J)\pi_{[0,j]}(\lambda), \pi_{[0,j]}(\zeta)\right) = -(\mathfrak{b}_j - \lambda\mathfrak{d}_j)\widehat{P}_{j+1}(\lambda)\overline{\widehat{P}_j(\zeta)}, \tag{4.6}$$

$$\left((H - \overline{\zeta} J)\pi_{[0,j]}(\lambda), \pi_{[0,j]}(\zeta)\right) = -(\overline{\mathfrak{b}}_j - \overline{\zeta}\mathfrak{d}_j)\overline{\widehat{P}_{j+1}(\zeta)}\widehat{P}_j(\lambda). \tag{4.7}$$

Subtracting (4.6) from (4.7) and using (3.2) we get the following relation:

$$(\lambda - \overline{\zeta})\left(J\pi_{[0,j]}(\lambda), \pi_{[0,j]}(\zeta)\right) = \frac{P_{j+1}(\lambda)P_j(\overline{\zeta}) - P_{j+1}(\overline{\zeta})P_j(\lambda)}{\prod_{k=0}^{j-1} b_k^2(\lambda - z_k)(\overline{\zeta} - \overline{z}_k)}. \tag{4.8}$$

Now, observe that due to (3.4) we have

$$\left(J\pi_{[0,j]}(\lambda), \pi_{[0,j]}(\zeta)\right) = \left(L\pi_{[0,j]}(\lambda), L\pi_{[0,j]}(\zeta)\right)$$

and, so, from (4.8) we obtain (4.5). □

Remark 4.3. To see how it is related to the classical Christoffel–Darboux relation [1] let us note that, according to (2.4) and (2.2), we have that $\mathfrak{d}_k \to 0$ and $b_k^2/|z_k|^2 \to \widetilde{b}_k^2 \neq 0$ as $z_k \to \infty, k = 0, \ldots, j$, provided that the numbers $\int_{\mathbb{R}} t^k d\sigma(t)$ are finite for $k = 0, \ldots, j$. Consequently, the classical Christoffel–Darboux formula is the limiting case of (4.5). Moreover, it is shown in [23, Theorem 2.2] (see also [11, Section 4]) that the sequence $\{\widehat{P}_k + \mathfrak{d}_{k-1}\widehat{P}_{k-1}\}_{k=0}^{\infty}$ is a sequence of rational functions orthogonal with respect to the original measure σ (see [17] for further information on orthogonal rational functions).

In what follows we will also need the following relation:

$$\sum_{k=0}^{j} |\omega(\widehat{P}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{P}_{k-1}(\lambda)) + \widehat{Q}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{Q}_{k-1}(\lambda)|^2 - \frac{\omega - \overline{\omega}}{\lambda - \overline{\lambda}}$$
$$= \left(J(\omega\pi_{[0,j]}(\lambda) + \xi_{[0,j]}(\lambda)), (\omega\pi_{[0,j]}(\lambda) + \xi_{[0,j]}(\lambda))\right) - \frac{\omega - \overline{\omega}}{\lambda - \overline{\lambda}}$$
$$= \frac{1}{\operatorname{Im}\lambda} \frac{|\omega P_j(\lambda) + Q_j(\lambda)|^2}{\prod_{k=0}^{j-1} b_k^2|\lambda - z_k|^2} \operatorname{Im} \frac{\omega P_{j+1}(\lambda) + Q_{j+1}(\lambda)}{\omega P_j(\lambda) + Q_j(\lambda)}, \tag{4.9}$$

where $\omega \in \mathbb{C}_+$ and $\lambda \in \mathbb{C}_+ \setminus \{z_k\}_{k=0}^{j}$. Formula (4.9) can be easily obtained by straightforward manipulations with (4.3) and (4.4) (for the classical case see [1, Section I.2.1]).

Next, by following [28], let us introduce m-functions of the truncated linear pencils.

Definition 4.4. Let j and n be nonnegative integers such that $j \le n$. The function

$$m_{[j,n]}(\lambda) = \left((H_{[j,n]} - \lambda J_{[j,n]})^{-1} e_j, e_j\right) \tag{4.10}$$

will be called the m-function of the linear pencil $H_{[j,n]} - \lambda J_{[j,n]}$.

To see the correctness of the above given definition it is sufficient to recall that $J_{[j,n]}$ is positive definite in view of Proposition 3.1 and to rewrite (4.10) in the following form:

$$m_{[j,n]}(\lambda) = \left(\left(J_{[j,n]}^{-\frac{1}{2}} H_{[j,n]} J_{[j,n]}^{-\frac{1}{2}} - \lambda\right)^{-1} J_{[j,n]}^{-\frac{1}{2}} e_j, J_{[j,n]}^{-\frac{1}{2}} e_j\right). \tag{4.11}$$

Literally as in the classical case (see for instance [28]), one obtains that m-functions satisfy the Riccati equation.

Proposition 4.5 *([23]). The m-functions $m_{[j,n]}$ and $m_{[j+1,n]}$ are related by the equality*

$$m_{[j,n]} = -\frac{1}{a_j^{(2)}\lambda - a_j^{(1)} + b_j^2(\lambda - z_j)(\lambda - \overline{z}_j) m_{[j+1,n]}(\lambda)}. \tag{4.12}$$

The latter statement allows us to see the relation of m-functions to multipoint diagonal Padé approximants.

Proposition 4.6. *Let $\theta_n = \det J_{[0,n]} / \det J_{[1,n]}$ and $\eta_n = \det J_{[0,n]} / \det J_{[0,n-1]}$. Then the function $\theta_n m_{[0,n]}$ is an $\mathbf{R}_0$-function and*

$$m_{[0,n]}(\lambda) = -\frac{Q_{n+1}(\lambda)}{P_{n+1}(\lambda)}, \tag{4.13}$$

that is, $m_{[0,n]}$ is the $(n+1)$th multipoint diagonal Padé approximant for φ. Moreover, we have that $-\eta_n P_n / P_{n+1} \in \mathbf{R}_0$.

Proof. Formula (4.13) is implied by the relation (4.12). Now, from Proposition 2.3 we see that $m_{[0,n]}$ is the $(n+1)$th multipoint diagonal Padé approximant for φ. To see that $\theta_n m_{[0,n]} \in \mathbf{R}_0$, it is enough to recall that $\Phi \in \mathbf{R}_0$ if and only if

$$\frac{\operatorname{Im} \Phi(\lambda)}{\operatorname{Im} \lambda} > 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

and $\sup_{y>0} |y \Phi(iy)| = 1$ [1, Section III.1.1]. The first condition is easily verified by means of (4.11) and the second one follows from (4.13). In the same way, by noticing that

$$-\frac{P_n(\lambda)}{P_{n+1}(\lambda)} = -\frac{\det(\lambda J_{[0,n-1]} - H_{[0,n-1]})}{\det(\lambda J_{[0,n]} - H_{[0,n]})} = \left((H_{[0,n]} - \lambda J_{[0,n]})^{-1} e_n, e_n\right)$$

one can check that $-\eta_n P_n / P_{n+1} \in \mathbf{R}_0$ since $\eta_n > 0$. $\square$

Due to $-\theta_n Q_{n+1}/P_{n+1} \in \mathbf{R}_0$ and $-\eta_n P_n / P_{n+1} \in \mathbf{R}_0$, we get the following.

Corollary 4.7. *We have that:*

(i) *The zeros of Q_{n+1} and P_{n+1} interlace.*
(ii) *The zeros of P_n and P_{n+1} interlace.*

Summing up Propositions 2.1 and 4.6, we conclude the following.

Theorem 4.8. *There is a one-to-one correspondence between the linear pencils in question and the data $\{z_k\}_{k=0}^{\infty}$, $\{w_k\}_{k=0}^{\infty}$ of the Nevanlinna–Pick problems.*

Proof. It follows from formulas (2.2) and (2.4) that the data $\{z_k\}_{k=0}^{\infty}$, $\{w_k\}_{k=0}^{\infty}$ uniquely determine the linear pencil, that is, the following numbers:

$$\mathfrak{a}_j = a_j^{(1)}, \qquad \mathfrak{b}_j = z_j b_j, \qquad \mathfrak{c}_j = a_j^{(2)}, \qquad \mathfrak{d}_j = b_j, \quad j \in \mathbb{Z}_+, \tag{4.14}$$

where $a_j^{(1)} \in \mathbb{R}$, $a_j^{(2)} > 0$, $b_j > 0$, $z_j \in \mathbb{C}_+$, and $\mathfrak{c}_j = 1 + \mathfrak{d}_j^2$. Let us suppose that we are given a set of numbers that can be represented as above. Then we see from (4.14) that $z_j = \mathfrak{b}_j/\mathfrak{d}_j$. Finally, by virtue of Proposition 4.6 we get that the numbers w_j are uniquely determined by the formula

$$w_j = -\frac{Q_n(z_j)}{P_n(z_j)}$$

for large enough n. It remains to note that in view of the precompactness of the family $-Q_n/P_n$ (see Proposition 8.1) and (3.11) there exists a function $\varphi \in \mathbf{R}_0$ which satisfies the underlying interpolation relation $\varphi(z_j) = w_j$, $j \in \mathbb{Z}_+$. $\square$

5. The Weyl circles

The classical Weyl circles approach to Nevanlinna–Pick problems can be found in [27, Section IV.6]. In this section, following [1, Section I.2.3], we adapt the notion of the Weyl circles to the linear pencil case.

Let us begin by considering the function

$$\omega_j(\lambda, \tau) = -\frac{Q_j(\lambda) - \tau Q_{j-1}(\lambda)}{P_j(\lambda) - \tau P_{j-1}(\lambda)}, \tag{5.1}$$

where $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $\tau \in \mathbb{R} \cup \{\infty\}$, and $j \in \mathbb{N}$. Obviously, from the definition we have that

$$\omega_j(\lambda, \infty) = \omega_{j-1}(\lambda, 0).$$

Moreover, in view of (2.7) we have that $\omega_j(z_k, \tau) = w_k$ and $\omega_j(\overline{z}_k, \tau) = \overline{w}_k$ for $j = k+2, k+3, \dots$. So, formula (5.1) gives a parametrization of $[j-1/j]$ rational solutions to the truncated Nevanlinna–Pick problems. Another such a parametrization is given in [17, Theorem 6.1.3] in terms of orthogonal rational functions of the first and second kinds.

Due to Proposition 4.6, the number $-\frac{P_{j-1}(\lambda)}{P_j(\lambda)}$ is not real for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and, therefore, we see that the set

$$K_j(\lambda) = \{\omega_j(\lambda, \tau) : \tau \in \mathbb{R} \cup \{\infty\}\}$$

is a circle. In addition, we have that $K_j(\overline{\lambda}) = \overline{K_j(\lambda)}$. So, we can consider only the case when $\lambda \in \mathbb{C}_+$. The following statement contains a characterization of the circle $K_j(\lambda)$.

Theorem 5.1. *Let $\lambda \in \mathbb{C}_+ \setminus \{z_k\}_{k=0}^{j-1}$ be a fixed number. Then the center of $K_j(\lambda)$ is*

$$-\frac{Q_j(\lambda)\overline{P_{j-1}(\lambda)} - Q_{j-1}(\lambda)\overline{P_j(\lambda)}}{P_j(\lambda)\overline{P_{j-1}(\lambda)} - P_{j-1}(\lambda)\overline{P_j(\lambda)}}, \tag{5.2}$$

and the radius of $K_j(\lambda)$ is

$$\frac{1}{|\lambda - \overline{\lambda}|} \frac{1}{\sum_{k=0}^{j-1} |\widehat{P}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{P}_{k-1}(\lambda)|^2}. \tag{5.3}$$

Besides, the equation of $K_j(\lambda)$ can be represented as follows (setting $\mathfrak{d}_{-1} = 0$):

$$\sum_{k=0}^{j-1} |\omega(\widehat{P}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{P}_{k-1}(\lambda)) + \widehat{Q}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{Q}_{k-1}(\lambda)|^2 - \frac{\omega - \overline{\omega}}{\lambda - \overline{\lambda}} = 0. \tag{5.4}$$

Proof. By the same reasoning as in the proof of [1, Theorem 1.2.3] we conclude that

$$\omega_j(\lambda, \tau) = -\frac{Q_j(\lambda)\overline{P_{j-1}(\lambda)} - Q_{j-1}(\lambda)\overline{P_j(\lambda)}}{P_j(\lambda)\overline{P_{j-1}(\lambda)} - P_{j-1}(\lambda)\overline{P_j(\lambda)}} + \left|\frac{Q_j(\lambda)P_{j-1}(\lambda) - Q_{j-1}(\lambda)P_j(\lambda)}{P_j(\lambda)\overline{P_{j-1}(\lambda)} - P_{j-1}(\lambda)\overline{P_j(\lambda)}}\right| e^{i\theta},$$

where $\theta = \theta(\tau)$ is real. The latter relation immediately gives us (5.2) and the formula for the radius of $K_j(\lambda)$

$$\left|\frac{Q_j(\lambda)P_{j-1}(\lambda) - Q_{j-1}(\lambda)P_j(\lambda)}{P_j(\lambda)\overline{P_{j-1}(\lambda)} - P_{j-1}(\lambda)\overline{P_j(\lambda)}}\right|,$$

which by means of (4.2) and (4.5) can be reduced to (5.3).

The rest of the proof is identical to the proof of [1, Theorem 1.2.3]. □

Denote by $\mathbf{K}_j(\lambda)$ the closure of the interior of $K_j(\lambda)$. Then the following statement holds true.

Corollary 5.2. *Let $\lambda \in \mathbb{C}_+ \setminus \{z_k\}_{k=0}^{j-1}$ be a fixed number. Then the set $\mathbf{K}_j(\lambda)$ is a set of numbers $\omega \in \mathbb{C}$ satisfying the inequality*

$$\sum_{k=0}^{j-1} |\omega(\widehat{P}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{P}_{k-1}(\lambda)) + \widehat{Q}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{Q}_{k-1}(\lambda)|^2 \leq \frac{\omega - \overline{\omega}}{\lambda - \overline{\lambda}}. \tag{5.5}$$

Furthermore, we can get a relation between the discs $\mathbf{K}_{j+1}(\lambda)$ and $\mathbf{K}_j(\lambda)$.

Corollary 5.3. *We have that*

$$\mathbf{K}_{j+1}(\lambda) \subseteq \mathbf{K}_j(\lambda), \quad j \in \mathbb{N}.$$

Besides this, the circles $K_{j+1}(\lambda)$ and $K_j(\lambda)$ have at least one common point.

Proof. The proof of the both corollaries is in line with the proof of the analogous statements given in [1, Section 2.3]. □

Now, we see that there are two options for the sequence $\mathbf{K}_j(\lambda)$. Namely, we can have a limit point or a limit circle.

Theorem 5.4. *Let $\lambda \in \mathbb{C}_+ \setminus \{z_k\}_{k=0}^{\infty}$ be a fixed number. Then the sequence $\mathbf{K}_j(\lambda)$ converges to a point iff*

$$\sum_{k=0}^{\infty} |\widehat{P}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{P}_{k-1}(\lambda)|^2 = \infty.$$

Proof. The proof is immediate from Corollary 5.3 and (5.3). □

Next, we obtain the existence of the Weyl solution.

Theorem 5.5. *For every $\lambda \in \mathbb{C}_+ \setminus \{z_k\}_{k=0}^{\infty}$ there exists a number $\omega = \omega(\lambda) \in \mathbb{C}_+$ such that*

$$\sum_{k=0}^{\infty} |\omega(\widehat{P}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{P}_{k-1}(\lambda)) + \widehat{Q}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{Q}_{k-1}(\lambda)|^2 \leq \frac{\omega - \overline{\omega}}{\lambda - \overline{\lambda}}. \tag{5.6}$$

Proof. The statement is a straightforward consequence of Corollary 5.3 and the inequality (5.5). □

Finally, it should be noticed that the above-mentioned parametrization from [17] leads to a slightly different but very similar theory of nested disks [17, Section 10]. That theory is equivalent to the presented one in the sense that the underlying Nevanlinna–Pick problems are the same.

6. The underlying symmetric operators

In this section we reduce the linear pencil in question to an operator generated by the formal matrix expression $J^{-\frac{1}{2}}HJ^{-\frac{1}{2}}$. Namely, we show that this operator is a densely defined symmetric operator.

Since $e_j \in \operatorname{dom} J \subset \operatorname{dom} J^{\frac{1}{2}}$ the vectors $f_j := J^{\frac{1}{2}}e_j$, $j \in \mathbb{Z}_+$, belong to ℓ^2. The relation $\ker J^{\frac{1}{2}} = \{0\}$ implies that the linear span

$$\mathcal{F} = \operatorname{span}\{f_j\}_{j=0}^{\infty} = \left\{ \sum_{k=0}^{n} c_k f_k : c_k \in \mathbb{C}, n \in \mathbb{Z}_+ \right\}$$

is dense in ℓ^2. In view of (3.6), we can also introduce the vectors $g_j := J^{-\frac{1}{2}}e_j$, $j \in \mathbb{Z}_+$, which lie in ℓ^2. Moreover, the linear span $\mathcal{G} = \operatorname{span}\{g_j\}_{j=0}^{\infty}$ is dense in ℓ^2. Besides, we have that the systems $\{f_j\}_{j=0}^{\infty}$ and $\{g_j\}_{j=0}^{\infty}$ are biorthogonal, i.e.

$$(f_j, g_k) = \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases}$$

As a consequence, we get that there is a one-to-one correspondence between $h \in \ell^2$ and the formal series

$$\sum_{k=0}^{\infty} (h, g_k) f_k, \qquad \sum_{k=0}^{\infty} (h, f_k) g_k.$$

In this case, we will write $h \sim \sum_{k=0}^{\infty}(h, g_k) f_k$ or $h \sim \sum_{k=0}^{\infty}(h, f_k) g_k$. Next, we see that (setting $\mathfrak{b}_{-1} = 0$ for convenience)

$$J^{-\frac{1}{2}} H J^{-\frac{1}{2}} f_j = \mathfrak{b}_{j-1} g_{j-1} + \mathfrak{a}_j g_j + \overline{\mathfrak{b}}_j g_{j+1}, \quad j \in \mathbb{Z}_+.$$

So, we have that $J^{-\frac{1}{2}} H J^{-\frac{1}{2}} : \mathcal{F} \mapsto \mathcal{G}$. Thus the domain of the matrix expression $J^{-\frac{1}{2}} H J^{-\frac{1}{2}}$ is dense in ℓ^2.

Proposition 6.1. *The formal matrix expression* $J^{-\frac{1}{2}} H J^{-\frac{1}{2}}$ *generates a densely defined symmetric operator with the deficiency indices either* $(1, 1)$ *or* $(0, 0)$.

Proof. It is easy to see that

$$\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}} f_j, f_k\right) = \left(f_j, J^{-\frac{1}{2}} H J^{-\frac{1}{2}} f_k\right), \quad j, k \in \mathbb{Z}_+,$$

that is, $J^{-\frac{1}{2}} H J^{-\frac{1}{2}}$ is symmetric in ℓ^2. Thus, the operator is closable and, in what follows, by $J^{-\frac{1}{2}} H J^{-\frac{1}{2}}$ we denote the minimal closed operator defined by the matrix expression $J^{-\frac{1}{2}} H J^{-\frac{1}{2}}$. Let $\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}}\right)^*$ be adjoint to $J^{-\frac{1}{2}} H J^{-\frac{1}{2}}$ in ℓ^2. By the definition, a vector $h \in \operatorname{dom}\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}}\right)^*$ if and only if there exists a vector $h^* \in \ell^2$ such that

$$\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}} f_k, h\right) = (f_k, h^*), \quad f \in k \in \mathbb{Z}_+.$$

Further, it can be rewritten as follows:

$$(\mathfrak{b}_{k-1} g_{k-1} + \mathfrak{a}_k g_k + \overline{\mathfrak{b}}_k g_{k+1}, h) = (f_k, h^*), \quad k \in \mathbb{Z}_+,$$

which actually implies that

$$y_k = \overline{\mathfrak{b}}_{k-1} x_{k-1} + \mathfrak{a}_k x_k + \mathfrak{b}_k x_{k+1}, \quad k \in \mathbb{Z}_+,$$

where $h \sim \sum_{k=0}^{\infty} x_k f_k$ and $h^* \sim \sum_{k=0}^{\infty} y_k g_k$. Thus, $h \in \operatorname{dom}\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}}\right)^*$ if and only if there exists $h^* \in \ell^2$ such that

$$h^* \sim \sum_{k=0}^{\infty} (\overline{\mathfrak{b}}_{k-1} x_{k-1} + \mathfrak{a}_k x_k + \mathfrak{b}_k x_{k+1}) g_k.$$

The next step is to determine the deficiency indices. In order to do that we should find nontrivial solutions of the equation

$$\left(\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}}\right)^* - \overline{\lambda}\right) h = 0, \qquad \operatorname{Im} \lambda \neq 0. \tag{6.1}$$

Let $h \sim \sum_{k=0}^{\infty} x_k f_k$ be a solution to (6.1). Then we obviously have that

$$\left(f_k, \left(\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}}\right)^* - \overline{\lambda}\right) h\right) = 0, \quad k \in \mathbb{Z}_+,$$

which reduces to the following:

$$\overline{\mathfrak{b}}_{k-1} \overline{x}_{k-1} + \mathfrak{a}_k \overline{x}_k + \mathfrak{b}_k \overline{x}_{k+1} = \lambda (f_k, h), \quad k \in \mathbb{Z}_+.$$

Observing that $(f_k, h) = \mathfrak{d}_{k-1} \overline{x}_{k-1} + \mathfrak{c}_k \overline{x}_k + \mathfrak{d}_k \overline{x}_{k+1}$, we arrive at

$$(\overline{\mathfrak{b}}_{k-1} - \lambda \mathfrak{d}_{k-1}) \overline{x}_{k-1} + (\mathfrak{a}_k - \lambda \mathfrak{c}_k) \overline{x}_k + (\mathfrak{b}_k - \lambda \mathfrak{d}_{k+1}) \overline{x}_{k+1} = 0, \quad k \in \mathbb{Z}_+.$$

In view of (3.1), (3.2) and (2.9), we conclude that $\overline{x}_k = c\widehat{P}_k(\lambda)$. So, the linear space $\mathcal{N}_\lambda$ of the solutions to (6.1) has dimension 1 if there exists an element $h \in \ell^2$ such that

$$h \sim \sum_{k=0}^{\infty} \overline{\widehat{P}_k(\lambda)} f_k. \tag{6.2}$$

Otherwise, the linear space $\mathcal{N}_\lambda$ has dimension 0.

Let us find the condition for h from (6.2) to belong to ℓ^2. First, we should check the weak convergence of the sequence $h_n = \sum_{k=0}^{n} \overline{\widehat{P}_k(\lambda)} f_k$. Obviously, we have that $(h_n, g_k) \to (h, g_k) = \overline{\widehat{P}_k(\lambda)}$ as $n \to \infty$. Furthermore, $\overline{\mathcal{G}} = \overline{\operatorname{span}\{g_j\}_{j=0}^{\infty}} = \ell^2$. Consequently, according to the criterion of the weak convergence we get that the convergence of (6.2) is implied by the uniform boundedness of the following sequence:

$$\begin{aligned}\left\| \sum_{k=0}^{n} \overline{\widehat{P}_k(\lambda)} f_k \right\| &= (J\pi_{[0,n]}(\lambda), \pi_{[0,n]}(\lambda)) \\ &= (L\pi_{[0,n]}(\lambda), L\pi_{[0,n]}(\lambda)) = \sum_{k=0}^{n} |\widehat{P}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{P}_{k-1}(\lambda)|^2. \end{aligned}\tag{6.3}$$

From (6.3) we see that the condition

$$\sum_{k=0}^{\infty} |\widehat{P}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{P}_{k-1}(\lambda)|^2 < \infty \tag{6.4}$$

guarantees the existence of h satisfying (6.2). It turns out that this condition is also necessary. Indeed, let us suppose the converse, that $\sum_{k=0}^{\infty} |\widehat{P}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{P}_{k-1}(\lambda)|^2 = \infty$ and there exists $h \in \ell^2$ having the representation (6.2). Then it follows from (6.1) that $h \in \operatorname{ran} J^{-\frac{1}{2}}$ and, therefore, $h = J^{\frac{1}{2}}h_0$ for some $h_0 \in \ell^2$. The latter means that

$$\|h\| = \|J^{\frac{1}{2}}h_0\| = \|Lh_0\| = \sum_{k=0}^{\infty} |\widehat{P}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{P}_{k-1}(\lambda)|^2 = \infty,$$

which yields the contradiction. So, $\dim \mathcal{N}_\lambda = 1$ if and only if (6.4) holds true.

It is well known that for symmetric operators the deficiency index $d_\lambda = \dim \mathcal{N}_\lambda$ is the same for each $\lambda \in \mathbb{C}_+$ as well as for each $\lambda \in \mathbb{C}_-$. Further, it follows from (5.3) that

$$\sum_{k=0}^{n-1} |\widehat{P}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{P}_{k-1}(\lambda)|^2 = \sum_{k=0}^{n-1} |\widehat{P}_k(\overline{\lambda}) + \mathfrak{d}_{k-1}\widehat{P}_{k-1}(\overline{\lambda})|^2$$

since the radii of $K_n(\lambda)$ and $K_n(\overline{\lambda})$ are equal. The latter relation implies that $d_\lambda = d_{\overline{\lambda}}$. □

Now we are in a position to formulate criteria for $J^{-\frac{1}{2}}HJ^{-\frac{1}{2}}$ to be self-adjoint (for the classical case see [1,12,42]).

Theorem 6.2. *The following statements are equivalent:*

(i) *The operator $J^{-\frac{1}{2}}HJ^{-\frac{1}{2}}$ is self-adjoint.*
(ii) *The sequence $\mathbf{K}_j(\lambda)$ converges to a point for some $\lambda \in \mathbb{C}_+ \setminus \{z_k\}_{k=0}^{\infty}$.*

(iii) *We have that*

$$\sum_{k=0}^{\infty} |\widehat{P}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{P}_{k-1}(\lambda)|^2 = \infty \tag{6.5}$$

for some $\lambda \in \mathbb{C}_+ \setminus \{z_k\}_{k=0}^{\infty}$.

Proof. The equivalence of (ii) and (iii) is established in Theorem 5.4. The equivalence of (i) and (iii) is actually proved in the proof of Proposition 6.1 by showing that the defect vector (6.2) belongs to ℓ^2 if and only if (6.5) holds true. □

Remark 6.3. It is well known that for symmetric operators the dimension of the defect space $\mathcal{N}_\lambda$ remains the same for all $\lambda \in \mathbb{C}_+$. Thus, if (6.5) holds for some $\lambda_0 \in \mathbb{C}_+ \setminus \{z_k\}_{k=0}^{\infty}$ then it holds for all $\lambda \in \mathbb{C}_+ \setminus \{z_k\}_{k=0}^{\infty}$. The same is true for the limit point case.

We should emphasize that in our approach the operator $J^{-\frac{1}{2}}HJ^{-\frac{1}{2}}$ plays exactly the same role as the Jacobi matrix for a moment problem. We should also stress here that if the original measure has finite moments of all nonnegative orders and we have a collection of interpolation sequences $\{z_k^{(n)}\}_{k=0}^{\infty}$ such that for every $k \in \mathbb{Z}_+$

$$z_k^{(n)} \to \infty, \quad \text{as } n \to \infty,$$

then the corresponding matrices $J^{(n)}$ converge to the identity I, as $n \to \infty$, elementwise (see (2.2) and (2.4)). So, roughly speaking, in this case, the operator $(J^{(n)})^{-\frac{1}{2}}H^{(n)}(J^{(n)})^{-\frac{1}{2}}$ approaches the classical Jacobi matrix (see also [11]).

To complete this section, it should be remarked that, in recent years, a lot of attention has been paid to the study of orthogonal polynomials on the unit circle via the spectral theory of CMV matrices (see [43] and references therein). Roughly speaking, orthogonal polynomials on the unit circle correspond to the multiple interpolation problem at 0 and ∞ for the Schur class (actually, there is only one interpolation point since ∞ is symmetric to 0 with respect to the unit circle). The multiple interpolation at two points is, in some sense, the limiting case of the case under consideration. Also note that an operator approach to orthogonal rational functions on the unit circle via CMV matrices can be found in [48]. It is also worth mentioning that Jacobi type normal matrices associated with complex moment problems were introduced and studied in [13,14].

7. The uniqueness of Nevanlinna–Pick problems

In this section, by mimicking the proofs of [42, Theorem 2.10] and [42, Theorem 2.11], we characterize the determinacy of the Nevanlinna–Pick problems in question in terms of the self-adjointness of $J^{-\frac{1}{2}}HJ^{-\frac{1}{2}}$.

Let $\varphi \in \mathbf{R}_0$ and let a sequence of distinct numbers $\{z_k\}_{k=0}^{\infty} \subset \mathbb{C}_+$ be given. According to (2.3) and (2.2), the pencil $H - \lambda J$ in question is uniquely determined by the sequences $\{z_k\}_{k=0}^{\infty}$ and $w_k := \varphi(z_k)$, $k \in \mathbb{Z}_+$. So, as we already mentioned, the following question naturally arises.

Nevanlinna–Pick problem. Is the function $\varphi \in \mathbf{R}_0$ satisfying the interpolation relation

$$\varphi(z_k) = w_k, \quad k \in \mathbb{Z}_+, \tag{7.1}$$

uniquely determined by the data $\{z_k\}_{k=0}^{\infty}$, $\{w_k\}_{k=0}^{\infty}$?

More details about Nevanlinna–Pick problems can be found in [1,27,36].

Remark 7.1. Recall that an $\mathbf{R}$-function is a function which is holomorphic in the open upper half-plane $\mathbb{C}_+$ and maps $\mathbb{C}_+$ onto $\mathbb{C}_+$. For convenience, it is supposed that every $\varphi \in \mathbf{R}$ is extended to the lower half-plane $\mathbb{C}_-$ by the symmetry relation $\varphi(\lambda) = \overline{\varphi(\overline{\lambda})}$, $\lambda \in \mathbb{C}_-$. Clearly, $\mathbf{R}_0$ is a subclass of $\mathbf{R}$. In fact, the condition $\varphi \in \mathbf{R}_0$ means that φ is an $\mathbf{R}$-function and satisfies the following tangential interpolation condition:

$$\varphi(\lambda) = -\frac{1}{\lambda} + o\left(\frac{1}{\lambda}\right), \quad \lambda \widehat{\to} \infty. \tag{7.2}$$

Roughly speaking, (7.2) can be interpreted as the interpolation conditions $\varphi(\infty) = 0$, $\varphi'(\infty) = -1$. So, the Nevanlinna–Pick problem in question is a subclass of Nevanlinna–Pick problems in $\mathbf{R}$.

Before answering the question of the Nevanlinna–Pick problem we will prove the following auxiliary statement.

Lemma 7.2. *We have that for $j \in \mathbb{Z}_+$*

$$\begin{aligned} e_0 &= (H - \overline{z}_j J)(\xi_{[0,j]}(\overline{z}_j) + m_{[0,j]}(\overline{z}_j)\pi_{[0,j]}(\overline{z}_j)) \\ &= (H_{[0,j]} - \overline{z}_j J_{[0,j]})(\xi_{[0,j]}(\overline{z}_j) + m_{[0,j]}(\overline{z}_j)\pi_{[0,j]}(\overline{z}_j)). \end{aligned} \tag{7.3}$$

Moreover, if $J^{-\frac{1}{2}}HJ^{-\frac{1}{2}}$ is self-adjoint in ℓ^2 then the systems $\{(J^{-\frac{1}{2}}HJ^{-\frac{1}{2}} - \overline{z}_j)^{-1}J^{-\frac{1}{2}} e_0\}_{j=0}^{\infty}$ and $\{J^{\frac{1}{2}}e_j\}_{j=0}^{\infty}$ are equivalent, that is,

$$\operatorname{span}\left\{\left(J^{-\frac{1}{2}}HJ^{-\frac{1}{2}} - \overline{z}_0\right)^{-1}, \dots, \left(J^{-\frac{1}{2}}HJ^{-\frac{1}{2}} - \overline{z}_k\right)^{-1} e_0\right\} = \operatorname{span}\left\{J^{\frac{1}{2}}e_0, \dots, J^{\frac{1}{2}}e_k\right\}$$

for every $k \in \mathbb{Z}_+$.

Proof. Notice that $\overline{\mathfrak{b}}_j - \overline{z}_j\mathfrak{d}_j = 0$. Then it follows from (4.3) and (4.4) that

$$\begin{aligned} (H - \overline{z}_j J)\pi_{[0,j]}(\overline{z}_j) &= (H_{[0,j]} - \overline{z}_j J_{[0,j]})\pi_{[0,j]}(\overline{z}_j) = -(\mathfrak{b}_j - \overline{z}_j\mathfrak{d}_j)\widehat{P}_{j+1}(\overline{z}_j)e_j, \\ (H - \overline{z}_j J)\xi_{[0,j]}(\overline{z}_j) &= (H_{[0,j]} - \overline{z}_j J_{[0,j]})\xi_{[0,j]}(\overline{z}_j) = -(\mathfrak{b}_j - \overline{z}_j\mathfrak{d}_j)\widehat{Q}_{j+1}(\overline{z}_j)e_j + e_0. \end{aligned} \tag{7.4}$$

Now, (7.3) is immediate from (7.4) on taking into account

$$m_{[0,j]}(\overline{z}_j) = -\frac{Q_{j+1}(\overline{z}_j)}{P_{j+1}(\overline{z}_j)} = -\frac{\widehat{Q}_{j+1}(\overline{z}_j)}{\widehat{P}_{j+1}(\overline{z}_j)}.$$

If $J^{-\frac{1}{2}}HJ^{-\frac{1}{2}}$ is a self-adjoint operator in ℓ^2 then (7.3) implies that

$$\left(J^{-\frac{1}{2}}HJ^{-\frac{1}{2}} - \overline{z}_j\right)^{-1} J^{-\frac{1}{2}}e_0 = J^{\frac{1}{2}}(\xi_{[0,j]}(\overline{z}_j) + m_{[0,j]}(\overline{z}_j)\pi_{[0,j]}(\overline{z}_j)). \tag{7.5}$$

Now, the equivalence follows from (7.5) for $j = 0, \dots, k$ and the fact that $\widehat{Q}_j(\overline{z}_j) + m_{[0,j]}(\overline{z}_j)\widehat{P}_j(\overline{z}_j) \neq 0$ for $j = 0, \dots, k$. The latter fact immediately follows from (3.2) and (4.13), and the Liouville–Ostrogradsky formula (4.2). □

Proposition 7.3. *If the operator $J^{-\frac{1}{2}}HJ^{-\frac{1}{2}}$ is self-adjoint in ℓ^2 then the corresponding Nevanlinna–Pick problem (7.1) has the unique solution*

$$\varphi(\lambda) = m(\lambda) := \left(\left(J^{-\frac{1}{2}}HJ^{-\frac{1}{2}} - \lambda I\right)^{-1} J^{-\frac{1}{2}}e_0, J^{-\frac{1}{2}}e_0\right).$$

Proof. Clearly, for every $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ there exists a sequence $r_n(\lambda) \in \operatorname{span}\left\{J^{\frac{1}{2}}e_0, \dots, J^{\frac{1}{2}}e_n\right\}$ $\subset \operatorname{dom}\left(J^{\frac{1}{2}}HJ^{\frac{1}{2}}\right)$ such that

$$\left\|\left(J^{-\frac{1}{2}}HJ^{-\frac{1}{2}} - \lambda\right) r_n(\lambda) - J^{-\frac{1}{2}}e_0\right\| \to 0, \quad n \to \infty. \tag{7.6}$$

It follows from Lemma 7.2 that

$$r_n(\lambda) = \sum_{k=0}^{n} c_k(\lambda)\left(J^{-\frac{1}{2}}HJ^{-\frac{1}{2}} - \overline{z}_k\right)^{-1} J^{-\frac{1}{2}}e_0. \tag{7.7}$$

Further, let $HJ^{-1} = \int_{\mathbb{R}} t\mathrm{d}E_t$ be a spectral decomposition of $J^{-\frac{1}{2}}HJ^{-\frac{1}{2}}$. Then the function

$$m(\lambda) = \int_{\mathbb{R}} \frac{\mathrm{d}(E_t e_0, e_0)}{t-\lambda} = \left(\left(J^{-\frac{1}{2}}HJ^{-\frac{1}{2}} - \lambda\right)^{-1} J^{-\frac{1}{2}}e_0, J^{-\frac{1}{2}}e_0\right)$$

is a solution of the Nevanlinna–Pick problem (7.1). In fact, according to (7.3) we have

$$m(\overline{z}_j) = \left(\left(J^{-\frac{1}{2}}HJ^{-\frac{1}{2}} - \overline{z}_j\right)^{-1} J^{-\frac{1}{2}}e_0, J^{-\frac{1}{2}}e_0\right)_{\ell^2} = m_{[0,j]}(\overline{z}_j).$$

Further, due to (2.7) and (4.13) one easily gets that $m(z_j) = w_j$ for $j \in \mathbb{Z}_+$. Suppose that there is another solution $\varphi_\rho(\lambda) = \int_{\mathbb{R}} \frac{\mathrm{d}\rho(t)}{t-\lambda}$. Then we have

$$\begin{aligned}
&\int_{\mathbb{R}} \left|(t-\lambda)\sum_{k=0}^{n} \frac{c_k(\lambda)}{t-\overline{z}_j} - 1\right|^2 \mathrm{d}\rho(t) \\
&= \int_{\mathbb{R}} \left|(t-\lambda)\sum_{k=0}^{n} \frac{c_k(\lambda)}{t-\overline{z}_j} - 1\right|^2 \mathrm{d}(E_t e_0, e_0) \\
&= \left\|\left(J^{-\frac{1}{2}}HJ^{-\frac{1}{2}} - \lambda\right)\sum_{k=0}^{n} c_k(\lambda)\left(J^{-\frac{1}{2}}HJ^{-\frac{1}{2}} - \overline{z}_j\right)^{-1} J^{-\frac{1}{2}}e_0 - J^{-\frac{1}{2}}e_0\right\| \to 0,
\end{aligned}$$

as $n \to \infty$. Now, $1/(t-\lambda)$ is bounded for $t \in \mathbb{R}$ since $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$. Thus

$$\int_{\mathbb{R}} \left|\sum_{k=0}^{n} \frac{c_k(\lambda)}{t-\overline{z}_k} - \frac{1}{t-\lambda}\right|^2 \mathrm{d}\rho(t) \to 0, \quad n \to \infty.$$

Finally, it follows that

$$\varphi_\rho(\lambda) = \lim_{n\to\infty} \int_{\mathbb{R}} \sum_{k=0}^{n} \frac{c_k(\lambda)}{t-\overline{z}_k} \mathrm{d}\rho(t)$$

is independent of ρ. Since φ_ρ determines ρ (see for instance [1, Chapter III]), all ρ's must be the same. □

Proposition 7.4. *If the operator $J^{-\frac{1}{2}}HJ^{-\frac{1}{2}}$ is not self-adjoint in ℓ^2 then the corresponding Nevanlinna–Pick problem (7.1) has an infinite number of solutions.*

Proof. Since the deficiency indices of $J^{-\frac{1}{2}}HJ^{-\frac{1}{2}}$ are equal it has self-adjoint extensions in ℓ^2. Let H_1 and H_2 be two different self-adjoint extensions of $J^{-\frac{1}{2}}HJ^{-\frac{1}{2}}$ in ℓ^2. Then the following two functions:

$$\varphi_1(\lambda) = \left((H_1 - \lambda)^{-1} J^{-\frac{1}{2}} e_0, J^{-\frac{1}{2}} e_0\right), \qquad \varphi_2(\lambda) = \left((H_2 - \lambda)^{-1} J^{-\frac{1}{2}} e_0, J^{-\frac{1}{2}} e_0\right)$$

are solutions of (7.1). In fact, according to Lemma 7.2 we have

$$\varphi_k(\overline{z}_j) = \left((H_k - \overline{z}_j)^{-1} J^{-\frac{1}{2}} e_0, J^{-\frac{1}{2}} e_0\right) = ((H_{[0,j]} - \overline{z}_j J_{[0,j]})^{-1} e_0, e_0) = \overline{w}_j$$

for every $j \in \mathbb{Z}_+$ and $k = 1, 2$. Since $\varphi_k \in \mathbf{R}_0$, one also has $\varphi_k(z_j) = w_j$.

Further, let $\lambda \in \mathbb{C}_+ \setminus \{z_j\}_{j=0}^{\infty}$. Note that $g_0 = J^{-\frac{1}{2}} e_0 \notin \operatorname{ran}\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}} - \lambda\right)$. To see this, suppose the contrary: that there exists $x \in \operatorname{dom}\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}} - \lambda\right)$ such that $g_0 = \left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}} - \lambda\right) x$ and that $\left(\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}}\right)^* - \overline{\lambda}\right) y = 0$. Then

$$(g_0, y) = \left(\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}} - \lambda\right) x, y\right) = \left(x, \left(\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}}\right)^* - \overline{\lambda}\right) y\right) = 0.$$

We thus see that $(g_0, y) = 0$ and $\left(\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}}\right)^* - \overline{\lambda}\right) y = 0$. As a consequence, the coefficients $\widehat{u}_k = (g_k, y)$ of the vector $y \sim \sum_{k=0}^{\infty} \widehat{u}_k f_k$ solve (3.1) with the initial conditions $\widehat{u}_{-1} = \widehat{u}_0 = 0$. Therefore, $y = 0$, that is, $J^{-\frac{1}{2}} H J^{-\frac{1}{2}}$ is self-adjoint in ℓ^2. By hypothesis, this is false, so $J^{-\frac{1}{2}} e_0 \notin \operatorname{ran}\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}} - \lambda\right)$. Thus $(H_1 - \lambda)^{-1} J^{-\frac{1}{2}} e_0$ and $(H_2 - \lambda)^{-1} J^{-\frac{1}{2}} e_0$ are in $\operatorname{dom}\left(\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}}\right)^*\right) \setminus \operatorname{dom}\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}}\right)$. So, we have $(H_1 - \lambda)^{-1} J^{-\frac{1}{2}} e_0 \neq (H_2 - \lambda)^{-1} J^{-\frac{1}{2}} e_0$ because otherwise, according to the fact that $J^{-\frac{1}{2}} H J^{-\frac{1}{2}}$ has deficiency indices $(1, 1)$ and the von Neumann formulas, we would have $H_1 = H_2$.

Let $\eta = (H_1 - \lambda)^{-1} J^{-\frac{1}{2}} e_0 - (H_2 - \lambda)^{-1} J^{-\frac{1}{2}} e_0$. Then one has $\left(\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}}\right)^* - \lambda\right) \eta = 0$ and, so, the coefficients $\widehat{\eta}_k = (g_k, \eta)$ of the vector $\eta \sim \sum_{k=0}^{\infty} \widehat{\eta}_k f_k$ give a solution of (3.1) with the initial conditions

$$\widehat{\eta}_{-1} = 0, \qquad \widehat{\eta}_0 = (g_0, \eta).$$

Since $\eta \neq 0$ we get $(g_0, \eta) \neq 0$. As a consequence, we have $\varphi_1 \not\equiv \varphi_2$. To complete the proof it remains to observe that the function

$$\varphi_\alpha(\lambda) = \alpha \varphi_1(\lambda) + (1 - \alpha) \varphi_2(\lambda)$$

is also a solution of (7.1) for every $\alpha \in (0, 1)$. □

Remark 7.5. It follows from the proof that every self-adjoint extension of the symmetric operator $J^{-\frac{1}{2}} H J^{-\frac{1}{2}}$ generates a solution of the corresponding Nevanlinna–Pick problem. Moreover, by using the standard technique of theory of extensions of symmetric operators (see [1, 24,39]), one can get the description of all solutions of the Nevanlinna–Pick problem and this will be done elsewhere. The description of all solutions can be found, for instance, in [27].

The following theorem immediately follows from Propositions 7.3 and 7.4.

Theorem 7.6. *The Nevanlinna–Pick problem (7.1) has a unique solution iff the corresponding operator $J^{-\frac{1}{2}} H J^{-\frac{1}{2}}$ is self-adjoint in ℓ^2.*

Remark 7.7. Other criteria for the Nevanlinna–Pick problems to be determinate can be found in [27,36]. It is worth noting that, in the matrix case, the Stieltjes type criteria for Nevanlinna–Pick problems to be completely indeterminate were obtained by Yu.M. Dyukarev in his second doctorate thesis (see [25,26]).

8. Convergence of multipoint Padé approximants

In this section we prove a Markov type result on convergence of multipoint diagonal Padé approximants for $\mathbf{R}_0$-functions.

First, let us recall that for the symmetric matrix $J_{[0,j]}^{-\frac{1}{2}} H_{[0,j]} J_{[0,j]}^{-\frac{1}{2}}$ the following estimate holds true:

$$\left\| \left(J_{[0,j]}^{-\frac{1}{2}} H_{[0,j]} J_{[0,j]}^{-\frac{1}{2}} - \lambda \right)^{-1} \right\| \leq \frac{1}{|\operatorname{Im} \lambda|}, \quad j \in \mathbb{Z}_+. \tag{8.1}$$

Before showing the convergence result, it is natural to obtain the precompactness.

Proposition 8.1. *The family $\{m_{[0,j]}\}_{j=0}^{\infty}$ is precompact in the topology of locally uniform convergence in $\mathbb{C} \setminus \mathbb{R}$.*

Proof. Let us rewrite the function $m_{[0,j]}$ as follows:

$$m_{[0,j]}(\lambda) = \left(\left(J_{[0,j]}^{-\frac{1}{2}} H_{[0,j]} J_{[0,j]}^{-\frac{1}{2}} - \lambda \right)^{-1} J_{[0,j]}^{-\frac{1}{2}} e_0, J_{[0,j]}^{-\frac{1}{2}} e_0 \right).$$

It follows from the Cauchy–Schwarz inequality and (3.8) that

$$|m_{[0,j]}(\lambda)| = \frac{(J_{[0,j]}^{-1} e_0, e_0)}{|\operatorname{Im} \lambda|} \leq \frac{1}{|\operatorname{Im} \lambda|}, \tag{8.2}$$

which, in view of the Montel theorem, implies the precompactness of $\{m_{[0,j]}\}_{j=0}^{\infty}$. □

Now we are ready to prove the main result of this section.

Theorem 8.2. *Let a sequence of distinct numbers $\{z_j\}_{j=0}^{\infty} \subset \mathbb{C}_+$ be given and let φ be a unique solution of the Nevanlinna–Pick problem (7.1). Then all the multipoint diagonal Padé approximants for φ at $\{z_0, \overline{z}_0, \dots, z_j, \overline{z}_j, \dots\}$ exist and converge to φ locally uniformly in $\mathbb{C} \setminus \mathbb{R}$.*

Proof. Eq. (4.13) says that the rational function $m_{[0,j]}$ is the $(j+1)$th multipoint diagonal Padé approximant. Further, according to Theorem 7.6, one obviously has that $J^{-\frac{1}{2}} H J^{-\frac{1}{2}}$ is self-adjoint in ℓ^2 and, therefore, $\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}} - \lambda \right)^{-1}$ is bounded for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Let ψ be a finite sequence, that is, $\psi = (\psi_1, \dots, \psi_k, 0, 0, \dots)^\top$. Then

$$(H - \lambda J)\psi = (H_{[0,j]} - \lambda J_{[0,j]})\psi = \phi$$

for sufficiently large $j \in \mathbb{Z}_+$ and ϕ is also a finite sequence. Further, one obviously has

$$\begin{aligned} &\left(\left(J^{-\frac{1}{2}} H J^{-\frac{1}{2}} - \lambda \right)^{-1} J^{-\frac{1}{2}} \phi, J^{-\frac{1}{2}} e_0 \right) \\ &\quad = \lim_{j \to \infty} \left(\left(J_{[0,j]}^{-\frac{1}{2}} H_{[0,j]} J_{[0,j]}^{-\frac{1}{2}} - \lambda \right)^{-1} J_{[0,j]}^{-\frac{1}{2}} \phi, J_{[0,j]}^{-\frac{1}{2}} e_0 \right). \end{aligned} \tag{8.3}$$

In particular, formula (8.3) is valid for

$$\phi_n = (HJ^{-\frac{1}{2}} - \lambda J^{\frac{1}{2}})r_n(\lambda),$$

where r_n is defined by (7.7). So, due to (7.6) we have that

$$J^{-\frac{1}{2}}\phi_n \to J^{-\frac{1}{2}}e_0 \quad \text{as } n \to \infty. \tag{8.4}$$

Moreover, the vectors ϕ_n satisfy the following relation:

$$J_{[0,j]}^{-\frac{1}{2}}\phi_n \to J_{[0,j]}^{-\frac{1}{2}}e_0 \quad \text{as } n \to \infty \tag{8.5}$$

for $j \in \mathbb{Z}_+$. To see the latter relation, note that (8.4) implies

$$\left(J^{-\frac{1}{2}}\phi_n, \eta\right) \to \left(J^{-\frac{1}{2}}e_0, \eta\right) \quad \text{as } n \to \infty$$

for every $\eta \in \ell^2$. Putting $\eta = J^{\frac{1}{2}}J_{[0,j]}^{-\frac{1}{2}}e_k, k = 0, \dots, j$, we get (8.5) from the fact that, in finite-dimensional spaces, the weak convergence is equivalent to the strong one. Now, taking into account (8.1) and (8.3)–(8.5), we obtain that (8.3) holds true for $\phi = e_0$, that is,

$$m_{[0,j]}(\lambda) \to m(\lambda) = \varphi(\lambda) = \left(\left(J^{-\frac{1}{2}}HJ^{-\frac{1}{2}} - \lambda\right)^{-1} J^{-\frac{1}{2}}e_0, J^{-\frac{1}{2}}e_0\right)$$

for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Finally, the statement of the theorem follows from the precompactness and the Vitali theorem. □

Remark 8.3. In the case when $\varphi \in \mathbf{R}[\alpha, \beta]$ and the interpolation points stay away from $[\alpha, \beta]$, an analog of the Markov theorem for multipoint diagonal Padé approximants is well known [30,47] (see also [23] where the operator approach was presented). For the case where the interpolation points belong to $[-\infty, 0)$, the locally uniform convergence of multipoint Padé approximants for $\varphi \in \mathbf{R}[0, +\infty)$ was proved under the Carleman type condition [37] (see also [38] where results in this direction are reviewed). It should also be remarked that there are some results on convergence of multipoint Padé approximants for rational perturbations of the Cauchy transforms of some complex measures [7,8].

It is a standard fact that the following condition:

$$\sum_{k=0}^{\infty} \frac{\operatorname{Im} z_k}{|z_k + \mathrm{i}|^2} = +\infty \tag{8.6}$$

implies the determinacy of the corresponding Nevanlinna–Pick problem in $\mathbf{R}_0$ [27,36]. Thus, the underlying operator $J^{-\frac{1}{2}}HJ^{-\frac{1}{2}}$ is self-adjoint in ℓ^2.

Corollary 8.4. *If the given sequence $\{z_j\}_{j=0}^{\infty}$ satisfies (8.6) then for every $\varphi \in \mathbf{R}_0$ all the multipoint diagonal Padé approximants for φ at $\{z_0, \overline{z}_0, \dots, z_j, \overline{z}_j, \dots\}$ exist and converge to φ locally uniformly in $\mathbb{C} \setminus \mathbb{R}$.*

Remark 8.5. First, note that (8.6) is sufficient for the Nevanlinna–Pick problem in $\mathbf{R}_0$ to be determinate but not necessary (see [27, Chapter IV, Example 4.2]). It should also be noted that, under the Szegö condition and the negation of the Blashcke type condition, the locally uniform convergence of multipoint diagonal Padé approximants for $\varphi \in \mathbf{R}[\alpha, \beta]$ was proved in [46] (see also [6]).

Now, we are also able to adapt Theorem 5.5 for the self-adjoint case.

Theorem 8.6. *If $J^{-\frac{1}{2}}HJ^{-\frac{1}{2}}$ is self-adjoint in ℓ^2 then for every $\lambda \in \mathbb{C}_+ \setminus \{z_k\}_{k=0}^{\infty}$ there holds*

$$\sum_{k=0}^{\infty} |m(\lambda)(\widehat{P}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{P}_{k-1}(\lambda)) + \widehat{Q}_k(\lambda) + \mathfrak{d}_{k-1}\widehat{Q}_{k-1}(\lambda)|^2 = \frac{m(\lambda) - \overline{m(\lambda)}}{\lambda - \overline{\lambda}}.$$

Proof. According to Theorem 8.2 and (5.1), we have that $\mathbf{K}_j(\lambda) \to m(\lambda)$ as $j \to \infty$. Now, the statement directly follows from Corollary 5.3 and the inequality (5.5). □

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