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Finiteness conditions on groups and quasi-isometries

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Abstract

We use *quasi-retractions* to show that the finiteness conditions F_n and FP_n are invariant under quasi-isometries, and give an application to group extensions

1. Introduction

Recall that a group G is said to be of type F_n ($n \geq 1$) if there exists an Eilenberg–Mac Lane complex $K(G,1)$ with *finite* n-skeleton. Conditions F_1, F_2 are equivalent, respectively, to G being finitely generated, finitely presented.

There is a corresponding homological version of this condition, defined as follows. A group G is said to be of type FP_n ($n \geq 1$) if the G-module $\mathbb{Z}$ admits a projective resolution which is finitely generated in all dimensions $\leq n$. It is easy to see that G is of type FP_1 if and only if it is finitely generated. Every finitely presented group is of type FP_2, but the converse is not known. Clearly F_n implies FP_n and, for $n \geq 3$, G is of type F_n if and only if it is finitely presented and of type FP_n.

In this paper we show that F_n and FP_n are preserved by *quasi-retractions*. This implies they are invariant under quasi-isometries. We use this together with a result from [3] (whose proof is included in an appendix) to give an application to group extensions.

2. Quasi-retractions

Given metric spaces (X, d) and (X', d'), recall that a (not necessarily continuous) function $f: X \to X'$ is called *(C, D)-Lipschitz* if there are constants $C \geq 1$ and $D \geq 0$

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such that for all $x,y \in X$, $d'(f(x), f(y)) \leq Cd(x, y) + D$. Note that the composition of Lipschitz functions is again Lipschitz.

Two metric spaces (X, d) and (X', d') are said to be *quasi-isometric* (abbreviated $X \sim_{\mathrm{qi}} X'$) if there exist (not necessarily continuous) functions $f: X \to X'$ and $f': X' \to X$ and constants $C \geq 1$ and $D \geq 0$ such that f and f' are (C, D)-Lipschitz, and the following two conditions are satisfied:

$$d(f'f(x), x) \leq D \text{ for all } x \in X, \quad \text{and} \quad d'(ff'(x'), x') \leq D \text{ for all } x' \in X'.$$

Definition. We say that X is a *quasi-retract* of X' (and write $X \preceq X'$) if there exist (C, D)-Lipschitz functions $X \xrightarrow{i} X' \xrightarrow{r} X$ such that for all $x \in X$, $d(ri(x), x) \leq D$. The pair (i, r) will be called a *quasi-retraction of* X' to X.

Remarks. (1) This notion is implicit in [1], where it is shown that if X and Y are Dehn complexes and X is a quasi-retract of Y, then $\delta_X \prec \delta_Y$, where δ_X is the Dehn or isoperimetric function of X (see [1] for details). Roughly speaking, $\delta_X \prec \delta_Y$ means that δ_X cannot grow faster than δ_Y. For instance, if $\delta_Y(n)$ $(n \in \mathbb{N})$ is bounded by a linear function of n, then so is δ_X.

(2) Our interest in quasi-retractions stems from the fact that if X and X' are quasi-isometric then $X \preceq X'$ and $X' \preceq X$.

Lemma 1. *Given metric spaces X, X' and X'', if $X \preceq X'$ and $X' \preceq X''$ then $X \preceq X''$.*

Proof. Let (i, r) be a quasi-retraction of X' to X and (i', r') a quasi-retraction of X'' to X'. Define $i'' = i' \circ i$ and $r'' = r \circ r'$. Then i'' and r'' are Lipschitz, and for all $x \in X$,

$$\begin{aligned} d_X(r''i''(x), x) &= d_X(rr'i'i(x), x) \\ &\leq d_X(rr'i'i(x), ri(x)) + d_X(ri(x), x) \\ &\leq Cd_{X'}(r'i'(i(x)), i(x)) + 2D \leq (C + 2)D, \end{aligned}$$

as desired. □

Corollary 2. *Quasi-retraction is a geometric property, i.e., it is preserved by quasi-isometries.*

Proof. Suppose that $X \preceq X'$, $X \sim_{\mathrm{qi}} Y$ and $X' \sim_{\mathrm{qi}} Y'$. Then $Y \preceq X \preceq X' \preceq Y'$ and Lemma 1 implies that $Y \preceq Y'$, as was to be proved. □

For a metric space (X, d) and an integer $p \geq 1$, the *Rips' complex* $P_p(X)$ of X is the simplicial complex whose simplices are all the finite subsets of X of diameter $\leq p$. For each pair (p, p') with $p \leq p'$, there are obvious (simplicial) inclusions $l_{(p,p')}: P_p(X) \hookrightarrow P_{p'}(X)$.

Suppose that $f: X \to X'$ is (C, D)-Lipschitz. Then f induces *simplicial* maps

$$f_{(p,q)}: P_p(X) \to P_q(X')$$

for all (p, q) with $q \geq Cp + D$. Indeed, if $\sigma = \{x_0, \dots, x_n\}$ is an n-simplex of $P_p(X)$, we set

$$f_{(p,q)}(\sigma) = \{f(x_0), \dots, f(x_n)\},$$

which is an m-simplex (for some $m \leq n$) of $P_q(X')$, since $d'(f(x_i), f(x_j)) \leq Cd(x_i, x_j) + D \leq Cp + D \leq q$. Moreover, if $X \xrightarrow{f} X' \xrightarrow{g} X''$ are (C, D)-Lipschitz functions, and (p, q) and (q, t) satisfy $q \geq Cp + D$ and $t \geq Cq + D$, then the maps $gf_{(p,t)}$, $g_{(q,t)}$ and $f_{(p,q)}$ are simplicial, and $gf_{(p,t)} = g_{(q,t)} \circ f_{(p,q)}$. We have thus proven the following lemma.

Lemma 3. *Let $f: X \to X'$ be a (C, D)-Lipschitz function. Then for every pair (p, q) with $q \geq Cp + D$, there is an induced simplicial map $f_{(p,q)}: P_p(X) \to P_q(X')$. Moreover, the diagram of simplicial maps*

$$\begin{tikzcd}
P_p(X) \arrow[r, "\iota_{(p,p')}"] \arrow[d, "f_{(p,q)}"'] & P_{p'}(X) \arrow[d, "f_{(p',q')}"] \\
P_q(X') \arrow[r, "\iota_{q,q'}"'] & P_{q'}(X')
\end{tikzcd}$$

is commutative, whenever $p' \geq p$, $q' \geq q$, and $q \geq Cp + D$, $q' \geq Cp' + D$. $\square$

Lemma 4. *Suppose that $g, f: X \to X'$ are (C, D)-Lipschitz, and that for all $x \in X$, $d'(f(x), g(x)) \leq M$. Then for all (p, q) with $q \geq Cp + D + M$, the maps $f_{(p,q)}$, $g_{(p,q)}: P_p(X) \to P_q(X')$ are simplicially homotopic.*

Proof. We need to prove that for all simplices $\sigma \in P_p(X)$, $f_{(p,q)}(\sigma) \cup g_{(p,q)}(\sigma)$ is contained in a simplex of $P_q(X')$. But if $\sigma = \{x_0, \dots, x_n\}$, then

$$f_{(p,q)}(\sigma) \cup g_{(p,q)}(\sigma) = \{f(x_0), \dots, f(x_n), g(x_0), \dots, g(x_n)\}$$

is itself a simplex of $P_q(X)$, since the distance between any two of its vertices is bounded by at most $Cp + D + M$, and this is $\leq q$. $\square$

Combining the previous two lemmas we obtain:

Corollary 5. *Suppose that (i, r) is a quasi-retraction of X' to X. Then for all pairs (p, q) and (q, t) with $q \geq Cp + D$ and $t \geq Cq + 2D$, the simplicial maps*

$$\iota_{(p,t)}, r_{(q,t)} \circ i_{(p,q)}: P_p(X) \to P_t(X)$$

are simplicially homotopic. $\square$

3. The case of groups

Let G be a group and A a finite set of semigroup generators not containing the identity. We use A to turn G into a metric space by setting $d(g,h) = |g^{-1}h|_A$, where $|-|_A$, is the word length in G relative to A. Then (G, d_A) is a metric space on which G acts by isometries (by left translation). Different choices of A produce very different metric spaces. However, they are all quasi-isometric (see Example (2) below). Metric properties that are invariant under quasi-isometries are important in group theory because whether or not they hold on (G, d_A) is independent of the particular A chosen. Thus they define genuinely group theoretical properties of G.

Examples. (1) Suppose that $f: G \to G'$ is a group homomorphism, and let $A \subset G$ and $A' \subset G'$ be finite generating sets as above. Then $f: (G, d_A) \to (G', d_{A'})$ is $(C, 0)$-Lipschitz for $C = \max\{|f(a)|_{A'} \mid a \in A\}$.

(2) In particular, given two different generating sets $A, B \subset G$, the identity on G gives $(G, d_A) \sim_{\mathrm{qi}} (G, d_B)$.

(3) If a subgroup H of G is a retract, then $H \preceq G$. This follows from (1) since by hypothesis there are group homomorphisms $H \xrightarrow{i} G \xrightarrow{r} H$ which satisfy $ri = \mathrm{id}_H$.

Definition. Given finitely generated groups G and G', we say that G is a *quasi-retract* of G', and write $G \preceq G'$, if $(G, d_A) \preceq (G, d_{A'})$ for some choice of finite generating sets $A \subset G$ and $A' \subset G'$.

In view of Corollary 2 and the above examples, we have the following lemma.

Lemma 6. *Quasi-retraction of groups is a geometric property. In particular, the above definition is independent of the chosen generating sets.* □

4. The Rips' filtration

We start by formulating two very useful criteria of Brown's [4]. Suppose that X is a G-CW-complex. X is called *n-good* for G if (a) X is acyclic in dimensions $< n$ (i.e. the reduced homology $\tilde{H}_k(X) = 0$ for $k < n$), and (b) for $0 \le p \le n$, the stabilizer G_σ of any p-cell σ of X is of type FP_{n-p}. A *filtration of* X is a family $\{X_\alpha\}_{\alpha \in D}$ of G-invariant subcomplexes such that D is a directed set, $X_\alpha \subseteq X_\beta$ when $\alpha \le \beta$, and $X = \bigcup_\alpha X_\alpha$. When the X_α have a finite n-skeleton mod G, the filtration will be said to be of finite *n-type*. Following Brown [4] we say that $\{X_\alpha\}$ is *$\tilde{H}_k$-essentially trivial* (resp. *π_k-essentially trivial*) if for each α there is $\beta \ge \alpha$ such that $\tilde{H}_k(\iota_{\alpha,\beta})$ (resp. $\pi_k(\iota_{\alpha,\beta})$) is trivial, where $\iota_{\alpha,\beta}: X_\alpha \hookrightarrow X_\beta$ is the inclusion. We can now formulate the following theorems.

Theorem (Brown). *Let X be an n-good G-complex with a filtration $\{X_\alpha\}$ of finite n-type. Then G is of type* FP_n *if and only if the directed system $\{X_\alpha\}$ is $\tilde{H}_k$-essentially trivial for each $k < n$.* $\square$

Theorem (Brown). *Let X be a 1-connected G-complex such that the vertex stabilizers are finitely presented and the edge stabilizers are finitely generated. Let $\{X_\alpha\}$ be a filtration of X such that each X_α has a finite 2-skeleton* mod G, *and let $v \in \bigcap X_\alpha$ be a basepoint. If G is finitely generated, then G is finitely presented if and only if $\{(X_\alpha, v)\}$ is π_1-essentially trivial.* $\square$

Let G be a finitely generated group, with some generating set A as before. Consider G as a metric space with $d = d_A$. Since we want to apply Brown's theorems, rather than being interested in some specific Rips' complex, we focus attention on the whole sequence $\{P_p(G)\}_{p \in \mathbb{N}}$ which filters the "infinite simplex" $P_\infty(G)$. Note that $P_\infty(G)$ is contractible.

In the case of groups, the Rips' complexes have some important special features. Indeed, G acts simplicially and with finite stabilizers on each $P_p(G)$. For $p \leq p'$, $P_p(G)$ is G-invariant in $P_{p'}(G)$, and we let $l_{(p,p')}$ denote the inclusion $P_p(G) \subset P_{p'}(G)$. Also, $P_\infty(G) = \bigcup_p P_p(G)$. The fact that the balls in G (of any radius and centered, say, at the identity of G) are finite, implies that $P_p(G)$ is finite-dimensional and locally finite (so that, in particular, $P_p(G)$ is finite mod G). In conclusion, we have the following lemma (cf. [2]).

Lemma 7. *Let G be a group. Then the complex $P_\infty(G)$ is an n-good G-complex for all $n \geq 1$. If G is finitely generated, then $\{P_p(G)\}_{p \in \mathbb{N}}$ is a filtration of $P_\infty(G)$ which is of finite n-type for all $n \geq 1$.*

We are now ready to prove the following.

Theorem 8. *Suppose that G is a quasi-retract of G', and let $n \geq 2$. Then*
(i) *if G' is of type* FP_n, *then so is G,*
(ii) *if G' is of type* F_n, *then so is G.*

Proof. We prove (i) first. For $0 \leq j \leq \infty$, set $Y_j = P_j(G)$, $Y'_j = P_j(G')$. By Brown's theorem and Lemma 7, it suffices to show that $\{Y_j\}_{j \geq 0}$ is $\tilde{H}_k$-essentially trivial for $k < n$.

Suppose that (i, r) is a quasi-retraction (with Lipschitz constants C and D) of G' to G, and that $q \geq Cj + D$. Consider the diagram:

$$\begin{tikzcd}
Y_j \arrow[r, "i_{(j,q)}"] \arrow[d, equal] & Y'_q \arrow[d, "l'_{(q,s)}"] \\
Y_j \arrow[r, "i_{(j,s)}"'] \arrow[d, "l_{(j,t)}"'] & Y'_s \arrow[d, "r_{(s,t)}"] \\
Y_t \arrow[r, equal] & Y_t
\end{tikzcd}$$

By hypothesis, G' is of type FP_n. This is equivalent, by Brown's theorem and Lemma 7, to the existence of $s \geq q$ such that $l'_{(q,s)}$ induces the trivial homomorphism on $\tilde{H}_k(Y'_q)$ for $k < n$. Choose $t \geq Cs + D$. Then the square on top commutes by Lemma 3, and $r_{(s,t)} \circ i_{(j,s)} \simeq l_{(j,t)}$ by Corollary 5. Thus $\tilde{H}_k(l_{(j,t)}) = \tilde{H}_k(r_{(s,t)} \circ i_{(j,s)}) = \tilde{H}_k(r_{(s,t)} \circ l'_{(q,s)} \circ i_{(j,q)}) = \tilde{H}_k(r_{(s,t)}) \circ \tilde{H}_k(l'_{(q,s)}) \circ \tilde{H}_k(i_{(j,q)}) = 0$ for $k < n$. This concludes the proof of (i).

As remarked in the Introduction, F_n is equivalent to F_2 together with FP_n. To prove (ii) for all $n \geq 2$ it is sufficient, in view of (i), to consider only the case $n = 2$. For this purpose, one repeats the above argument but using instead the second of Brown's theorems quoted above. We leave the details to the reader. This completes the proof. □

Corollary 9. *For $n \geq 2$, the following finiteness conditions:*

(1) *type* FP_n,

(2) *type* F_n,

are geometric *properties, i.e. they are invariant under quasi-isometries.* □

Remark. Condition F_2 is well-known to be geometric and can be proved directly, avoiding Brown's theorem (see [5]). That FP_n and F_{n+1} are geometric for $n \geq 2$ seems to be new.

5. Group extensions

Suppose given an extension of groups

$$H \rightarrowtail G \xrightarrow{\pi} Q$$

and let $\mu: Q \to G$ be a set-theoretic cross-section of π. Recall that this determines a cocycle $f: Q \times Q \to H$ (which measures the failure of μ to be a homomorphism), and a function $\varphi: Q \to \mathrm{Aut}(H)$ given by $\varphi(x)[a] = \mu(x)a\mu(x)^{-1}$, which is usually *not* a homomorphism. However, the composition of φ with $\mathrm{Aut}(H) \twoheadrightarrow \mathrm{Out}(H)$ is always a homomorphism $Q \to \mathrm{Out}(H)$. The section μ also allows us to view G as the set $H \times Q$. In fact, given φ and f as above, we can recover the original extension by describing G as the set $H \times Q$ with the product:

$$(a, x) \bullet (b, y) = (a\varphi(x)[b]f(x, y), xy).$$

In this situation, we say that the extension $H \rightarrowtail G \xrightarrow{\pi} Q$ is "defined" by f and φ. With this terminology we have the following

Theorem 10. *Let $H \rightarrowtail G \twoheadrightarrow Q$ be an extension defined by a homomorphism $\varphi: Q \to \mathrm{Out}(H)$ and a cocycle $f: Q \times Q \to H$, where H and Q are finitely generated groups. Suppose that* (im φ) *and* (im f) *are finite. Then for all $n \geq 2$ we have:*

(i) *G is of type* FP_n *if and only if H and Q are of type* FP_n,

(ii) *G is of type* F_n *if and only if H and Q are of type* F_n.

Proof. The hypothesis on f and φ imply that G is quasi-isometric to $H \times Q$ (see the Appendix). By Corollary 9, G is of type FP_n (resp. of type F_n) if and only if the same is true for $H \times Q$. Using Theorem 8 we conclude that both H and Q must then be of type FP_n (resp. of type F_n). The converse is clear. This concludes the proof. □

Appendix

For the convenience of the reader and to complete the proof of Theorem 10, we reproduce here a proof of a result of [3]. But first some preliminaries.

Lemma A.1. *Let A be a finite set of generators of a group G, and let $|-|$ denote the corresponding word length. Suppose that Γ is a finite subset of automorphisms of G. Then there exists a positive constant M such that*

$$\frac{1}{M}|x| \leq |\gamma(x)| \leq M|x|,$$

for all $\gamma \in \Gamma \cup \Gamma^{-1}$ and all $x \in G$.

Proof. Let $\Gamma' = \Gamma \cup \Gamma^{-1}$, and suppose that $A = \{a_1, \ldots, a_n\}$. Set $M = \max\{|\gamma(a)|: \gamma \in \Gamma', a \in A\}$. We suppose that $x \in G$ has length k, so $x = a_{i_1} \cdots a_{i_k}$ for some $a_{i_j} \in A$. Then, for any $\gamma \in \Gamma'$, $|\gamma(x)| = |\gamma(a_{i_1}) \cdots \gamma(a_{i_k})| \leq |\gamma(a_{i_1})| + \cdots + |\gamma(a_{i_k})| \leq Mk = M|x|$. Similarly, $|x| = |\gamma^{-1}(\gamma(x))| \leq M|\gamma(x)|$. This completes the proof. □

The conventions we use in the formulation and proof of the following proposition are explained in Section 5.

Proposition A.2. *Let $H \rightarrowtail G \twoheadrightarrow Q$ be an extension defined by a homomorphism $\varphi: Q \to \mathrm{Out}(H)$ and a cocycle $f: Q \times Q \to H$, where H and Q are finitely generated groups. Suppose that* $(\mathrm{im}\,\varphi)$ *and* $(\mathrm{im}\, f)$ *are finite. Then G is quasi-isometric to $H \times Q$.*

Proof. We shall denote the direct product of the groups H and Q by $H \times Q$, and we reserve G to denote the set $H \times Q$ with the operation $\bullet$ defined before the statement of Theorem 10. Let S be a set of generators for both G and $H \times Q$, chosen as follows. Let $S_H \subset H$ (resp. $S_Q \subset Q$) be a symmetric set of generators of H (resp. Q), and set $S = S_H \times \{1\} \cup \{1\} \times S_Q$. The metrics induced by S on G (resp. on $H \times Q$) will be denoted d_G (resp. $d_{H \times Q}$). Similarly, $|-|_G$ and $|-|_{H \times Q}$ will denote the corresponding length functions on G and $H \times Q$. Note that $|(a,x)|_{H \times Q} = |a|_H + |x|_Q$, and consequently $d_{H \times Q}((a,x),(b,y)) = d_H(a,b) + d_Q(x,y)$. Note also that it follows from our choice of S that

$$|(a,1)|_G \leq |a|_H \quad \text{and} \quad |x|_Q \leq |(a,x)|_G,$$

for all $a \in H$ and $x \in Q$. We prove first that there exists a constant C such that for all (a, x),

$$\frac{1}{C}|(a,x)|_{H\times Q} \le |(a,x)|_G \le C|(a,x)|_{H\times Q}. \tag{A.1}$$

Let M_1 be the constant of Lemma A.1 for $\Gamma = \operatorname{im}\varphi$, and $M_2 = \max\{|b|_H \mid b^{\pm 1} \in \operatorname{im} f\}$. We set $M = \max\{M_1, M_2\}$.

Suppose now that $|(a,x)|_G = n$, and let $(a,x) = (a_1, x_1)\bullet \cdots \bullet(a_n, x_n)$ with $(a_i, x_i) \in S$. Then

$$(a,x) = (a_1\,\varphi(x_1)[a_2]f(x_1,x_2)\cdots\varphi(x_1\cdots x_{n-1})[a_n]f(x_1\cdots x_{n-1},x_n), x_1\cdots x_n),$$

and hence

$$\begin{aligned}|(a,x)|_{H\times Q} &\le |a_1|_H + |\varphi(x_1)[a_2]|_H \\ &\qquad + |f(x_1,x_2)|_H + \cdots + |f(x_1\cdots x_{n-1},x_n)|_H + |x|_Q \\ &\le nM + (n-1)M + n \le (2M+1)|(a,x)|_G.\end{aligned}$$

To prove the other inequality, suppose that $|(a,x)|_{H\times Q} = n$, and let $(a,x) = (a_1,x_1)\cdots(a_n,x_n) = (a_1\cdots a_n, x_1\cdots x_n)$, where $(a_i, x_i) \in S$. From

$$(a,1) = (a_1,1)\bullet\cdots\bullet(a_n,1) \quad \text{and}$$

$$(1,x_1)\bullet\cdots\bullet(1,x_n) = (f(x_1,x_2)\cdots f(x_1\cdots x_{n-1},x_n), x) = (e,x),$$

where $e = f(x_1,x_2)\cdots f(x_1\cdots x_{n-1},x_n)$, it follows that $|(a,1)|_G \le n$ and $|(e,x)|_G \le n$. Since $(a,x) = (a,1)\bullet(e^{-1},1)\bullet(e,x)$, we have

$$\begin{aligned}|(a,x)|_G &\le 2n + |(e^{-1},1)|_G \le 2n + |e|_H \\ &\le 2n + (n-1)M \le (M+2)|(a,x)|_{H\times Q}.\end{aligned}$$

This concludes the proof of (A.1). It is now easy to establish the proposition. Indeed,

$$\begin{aligned}d_G((a,x),(b,y)) &= |([f(x^{-1},x)]^{-1}\varphi(x^{-1})[a^{-1}b]f(x^{-1},x), x^{-1}y)|_G \\ &\le |([f(x^{-1},x)]^{-1},1)|_G + |(\varphi(x^{-1})[a^{-1}b],1)|_G \\ &\qquad + |(f(x^{-1},x),1)|_G + |(1,x^{-1}y)|_G \\ &\le 2|f(x^{-1},x)|_H + |\varphi(x^{-1})[a^{-1}b]|_H + |(1,x^{-1}y)|_{H\times Q} \\ &\le 2M + M|a^{-1}b|_H + C|x^{-1}y|_Q \quad \text{(by Lemma A.1)} \\ &\le 2M + C'd_{H\times Q}((a,x),(b,y)),\end{aligned} \tag{A.2}$$

for $C' = \max\{M, C\}$. On the other hand,

$$\begin{aligned}
&d_{H\times Q}((a,x),(b,y))\\
&\quad = |a^{-1}b|_H + |x^{-1}y|_Q\\
&\quad \le M\,|\varphi(x^{-1})[a^{-1}b]|_H + |x^{-1}y|_Q \quad \text{(by Lemma 7)}\\
&\quad \le M\{|f(x^{-1},x)]^{-1}\varphi(x^{-1})[a^{-1}b]f(x^{-1},x)|_H + 2M\} + |x^{-1}y|_Q\\
&\quad \le M\{|([f(x^{-1},x)]^{-1}\varphi(x^{-1})[a^{-1}b]f(x^{-1},x),x^{-1}y)|_{H\times Q} + 2M\}\\
&\qquad + |([f(x^{-1},x)]^{-1}\varphi(x^{-1})[a^{-1}b]f(x^{-1},x),x^{-1}y)|_G\\
&\quad \le (MC+1)|([f(x^{-1},x)]^{-1}\varphi(x^{-1})[a^{-1}b]f(x^{-1},x),x^{-1}y)|_G + 2M^2\\
&\quad = (MC+1)d_G((a,x),(b,y)) + 2M^2. \qquad\qquad \text{(A.3)}
\end{aligned}$$

This completes the proof, since (A.2) and (A.3) show that the identity function on $H \times Q$ induces a quasi-isometry.

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