d-Wise generation of prosolvable groups

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Let $G$ be a (topological) group. For $2 \leq d \in \mathbb{N}$, denote by $\mu_d(G)$ the largest $m$ for which there exists an $m$-tuple of elements of $G$ such that any of its $d$ entries generate $G$ (topologically). We obtain a lower bound for $\mu_d(G)$ in the case when $G$ is a prosolvable group. Our result implies in particular that if $G$ is $d$-generated then the difference $\mu_d(G) - d$ tends to infinity when the smallest prime divisor of the order of $G$ tends to infinity. One of the aim of the paper is to draw the attention to an intriguing question in linear algebra whose solution would allow to improve our bounds and determine the precise value for $\mu_d(G)$ in several relevant cases, for example when $d = 2$ and $G$ is a prosolvable group.

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1. Introduction

Let $d$ be a positive integer greater than or equal to 2, and let $G$ be a discrete or profinite group that can be topologically generated by $d$ elements. If there is a largest integer $m$ with the property that there exists an $m$-tuple of elements of $G$ such that any $d$ entries together (topologically) generate $G$ then denote this number by $\mu_d(G)$, and otherwise set $\mu_d(G)$ equal to $\infty$. If $G$ cannot be generated by $d$ elements then set $\mu_d(G) = 0$.

The case $d = 2$ has a particular significance, since it is related to the study of the generating graph $\Gamma(G)$ of $G$. This is the graph defined on the elements of $G$ in such a way that two distinct vertices are connected by an edge if and only if they generate $G$. The number $\mu_2(G)$ is the clique number of $\Gamma(G)$, i.e. the maximum size of a complete subgraph in $\Gamma(G)$.

It is very difficult to determine the precise value of $\mu_d(G)$, even in the case of abelian groups. For example if $G$ is an elementary abelian $p$-group of rank $d$, then it is not difficult to show that $\mu_d(G) = d + 1$ if $p < d$, while only recently S. Ball [3, Theorem 1.7] proved that $\mu_d(G) = p + 1$ if...
If $G = (g_1, \ldots, g_d)$, then any $d$ entries of the $(d+1)$-uple $(g_1, \ldots, g_d, g_1 \cdots g_d)$ generate $G$, hence $\mu_d(G) \geq d + 1$ whenever $G$ is $d$-generated. The results obtained in this paper show that $\mu_d(G)$ is much larger than $d + 1$ if $|G|$ has not “small” prime divisors. For example we will prove:

**Theorem 1.1.** If $G$ is a $d$-generated profinite group with $d \geq 2$ and $p$ is the smallest prime divisor of $|G|$, then $(\mu_d(G)) > p$.

The previous result is a corollary of a more precise statement. We say that $A = X/Y$ is a chief factor of $G$ if $X$ and $Y$ are open normal subgroups of $G$ and $X/Y$ is a minimal normal subgroup of $G/Y$. Let $A$ be the set of the chief factors $X/Y$ of $G$ with the property that $X/Y$ has more than one complement in $G/Y$.

**Theorem 1.2.** Let $G$ be a $d$-generated prosolvable group with $d \geq 2$. Assume that a positive integer $t$ satisfies the following property:

1. $t \leq |A|$ for each $A \in A$ with $C_G(A) = G$.
2. $(t-1)^{d-1} \leq |\text{End}_G(A)|$ for each $A \in A$ with $C_G(A) \neq G$.

Then $\mu_G(d) \geq t$.

The previous result allows to compute the precise value of $\mu_d(G)$ in some relevant cases: for example, in the case $d = 2$ we obtain the following corollary.

**Corollary 1.1.** Let $\pi$ be a set of prime numbers and let $p$ be the smallest prime in $\pi$. If $G$ is the free pro-$\pi$-group of rank 2, then $\mu_2(G) = p + 1$.

Assume that $G$ is a 2-generated finite group and let $\sigma(G)$ denote the least number of proper subgroups of $G$ whose union is $G$. Since a set that generates $G$ pairwise cannot contain two elements of any proper subgroup, we must have that $\mu_2(G) \leq \sigma(G)$. In general $\mu_2(G) \neq \sigma(G)$; for example $\mu(\text{Alt}(5)) = 8$ and $\sigma(\text{Alt}(5)) = 10$. However no example is known of a finite 2-generated solvable non-cyclic group $G$ with $\mu_2(G) \neq \sigma(G)$ and in [7] it was proved that $\mu_2(G) = \sigma(G)$ if $G$ has Fitting length 2 (we will see in Section 4 that this result can be easily obtained also as a corollary of Theorem 1.2). The exact value of $\sigma(G)$ when $G$ is a 2-generated non-cyclic solvable group was determined by Tomkinson [9]; he proved that $\sigma(G) = q + 1$, where $q$ is the minimal size of a chief factor of $G$ having more than one complement. Combined with Theorem 1.2, this implies:

**Corollary 1.2.** If $G$ is a finite, 2-generated, non-cyclic, solvable group and $A$ is the set of the chief factors $G$ having more than one complement, then

$$\min_{A \in A} (1 + |\text{End}_G(A)|) \leq \mu_2(G) \leq \sigma(G) = \min_{A \in A} (1 + |A|).$$

A question in linear algebra plays a crucial role in the study of the value of $\mu_d(G)$ when $G$ is solvable. Denote by $M_{r \times s}(F)$ the set of the $r \times s$ matrices with coefficients over the field $F$. Let $A_1, \ldots, A_d \in M_{n \times n}(F)$. If the $n \times nd$ matrix $A = (A_1 \cdots A_d)$ has rank $n$, then the $n$ rows of $A$ are linearly independent and can be completed to a basis of the vector space $F^{nd}$, hence there exist $B_1, \ldots, B_d \in M_{n(d-1) \times n}(F)$ with the property that

$$\det \begin{pmatrix} A_1 & \cdots & A_d \\ B_1 & \cdots & B_d \end{pmatrix} \neq 0.$$
More in general let \( t \geq d \) and assume that \((A_1, \ldots, A_t)\) is a family of \( n \times n\) matrices with the property that for all \( 1 \leq i_1 < i_2 < \cdots < i_d \leq t \), the \( n \times dn \) matrix \((A_{i_1} \cdots A_{i_d})\) has rank \( n \). Can we find \( B_1, \ldots, B_t \in M_{n(d-1)\times n}(F) \) with the property that
\[
\det \begin{pmatrix} A_{i_1} & \cdots & A_{i_d} \\ B_{i_1} & \cdots & B_{i_d} \end{pmatrix} \neq 0
\]
whenever \( 1 \leq i_1 < i_2 < \cdots < i_d \leq t \)? The answer is not in general affirmative: for example if the matrices \( A_1, \ldots, A_t \) are all equal and \( t \) is larger than the number of possible choices for \( B_i \), then there is no solution to the problem. In Section 3 we will prove that the answer is affirmative if \( \binom{t-1}{d-1} \leq |F| \), except in the case when \((n, t) = (1, |F| + 1)\) and \( A_i \neq 0 \) for each \( 1 \leq i \leq t \). However this result is not best possible. The discussion of some concrete examples indicates that the answer is still affirmative even when \( \binom{t-1}{d-1} \) is much larger than \( |F| \) and in any case a more satisfactory result should take into account also the value of \( n \). Unfortunately this seems a quite difficult problem but we hope that this paper can draw the attention to this intriguing question in linear algebra. Any positive result in this direction could immediately lead to an improvement of the bound given by Theorem 1.2.

2. The critical case

Let \( V \) be a finite dimensional vector space over a finite field of prime order. Let \( H \) be a linear solvable group acting irreducibly and faithfully on \( V \). Suppose that \( H \) can be generated by \( d \) elements. For a positive integer \( u \) consider the semidirect product \( G = V^u \rtimes H \) where \( H \) acts in the same way on each of the \( u \) direct factors. Put \( F = \text{End}_H(V) \).

**Proposition 2.1.** Assume \( H = \langle h_1, \ldots, h_d \rangle \) and let \( w_i = (v_{i,1}, \ldots, v_{i,u}) \in V^u \) with \( 1 \leq i \leq d \). The following are equivalent.

1. \( G \neq \langle h_1w_1, \ldots, h_dw_d \rangle \);
2. there exist \( \lambda_1, \ldots, \lambda_u \in F \) and \( w \in V \) with \( (\lambda_1, \ldots, \lambda_u, w) \neq (0, \ldots, 0, 0) \) such that \( \sum_{1 \leq j \leq u} \lambda_j v_{i,j} = w - wh_i \) for each \( i \in \{1, \ldots, d\} \).

**Proof.** Let \( K = \langle h_1w_1, \ldots, h_dw_d \rangle \). First we prove, by induction on \( u \), that if \( K \neq G \) then (2) holds. Let \( \tilde{h}_i = h_i(v_{i,1}, \ldots, v_{i,u-1}, 0) \) and let \( \tilde{K} = \langle \tilde{h}_1, \ldots, \tilde{h}_d \rangle \). If \( \tilde{K} \cong V^{u-1} H \), then, by induction, there exist \( \lambda_1, \ldots, \lambda_{u-1} \in F \) and \( w \in V \) with \( (\lambda_1, \ldots, \lambda_{u-1}, w) \neq (0, \ldots, 0, 0) \) such that \( \sum_{1 \leq j \leq u-1} \lambda_j v_{i,j} = w - wh_i \) for each \( i \in \{1, \ldots, d\} \). In this case \( \lambda_1, \ldots, \lambda_{u-1}, 0 \) and \( w \) are the requested elements. So we may assume \( K \cong V^{u-1} H \). Set \( V_u = \{0, \ldots, 0, v \} \mid v \in V \} \). We have \( \bar{K} V_u = K V_u = G \) and \( K \neq G \); this implies that \( K \) is a complement of \( V_u \) in \( G \) and therefore there exists \( \delta \in \text{Der}(\bar{K}, V_u) \) such that \( \delta(h_i) = v_{i,u} \) for each \( i \in \{1, \ldots, d\} \). However, by Propositions 2.7 and 2.10 of [1], we have \( H^1(\bar{K}, V_u) \cong F^{u-1} \). More precisely if \( \delta \in \text{Der}(\bar{K}, V_u) \), then there exist an inner derivation \( \delta_w \in \text{Der}(H, V) \) and \( \lambda_1, \ldots, \lambda_{u-1} \in F \) such that for each \( h(v_{1,1}, \ldots, v_{u-1,0}) \in \bar{K} \) we have \( \delta(h(v_{1,1}, \ldots, v_{u-1,0})) = \delta_w(h) + \lambda_1 v_{1,1} + \cdots + \lambda_{u-1} v_{u-1} = wh-w+\lambda_1 v_{1,1} + \cdots + \lambda_{u-1} v_{u-1} \). In particular \( \sum_{1 \leq j \leq u-1} \lambda_j v_{i,j} - v_{i,u} = w - wh_i \) for each \( i \in \{1, \ldots, d\} \).

Conversely, if (2) holds then \( (h(v_{1,1}, \ldots, v_{u,0}) \mid w - wh = \lambda_1 v_{1,1} + \cdots + \lambda_u v_{u,0}) \) is a proper subgroup of \( G \) containing \( K \). \( \square \)

Let \( n \) be the dimension of \( V \) over \( F \). We may identify \( H = \langle h_1, \ldots, h_d \rangle \) with a subgroup of \( GL(n, F) \). In this identification \( h_i \) becomes an \( n \times n \) matrix \( X_i \) with coefficients in \( F \). Let \( w_i = (v_{i,1}, \ldots, v_{i,u}) \in V^u \). Then every \( v_{i,j} \) can be viewed as a \( 1 \times n \) matrix. Denote the \( u \times n \) matrix with rows \( v_{i,1}, \ldots, v_{i,u} \) by \( A_i \). By Proposition 2.1, the elements \( h_1w_1, \ldots, h_dw_d \) generate a proper subgroup of \( G \) if and only if there exists a non-zero vector \( (\lambda_1, \ldots, \lambda_u; \mu_1, \ldots, \mu_n) \) in \( F^{u+n} \) such that
\[
(\lambda_1, \ldots, \lambda_u)A_i = (\mu_1, \ldots, \mu_n)(1 - X_i) \quad \text{for each } 1 \leq i \leq d.
\]
This is equivalent to saying that there exist elements $X_1, \ldots, X_d$ in $G$ such that $(X_1, \ldots, X_d) = G$ with the property that $X_i$ maps to $h_i$ under the projection from $G$ to $H$ if and only if there exist $u \times n$ matrices $A_1, \ldots, A_d$ with

$$\text{rank} \begin{pmatrix} 1 - X_1 & \cdots & 1 - X_d \\ A_1 & \cdots & A_d \end{pmatrix} = n + u. \quad (1)$$

From this it follows that $G$ cannot be generated by $d$ elements if $n + u > nd$. Notice also that if $X_1, \ldots, X_d$ are $n \times n$ matrices generating the matrix group $H$, then the linear map $\alpha : F^n \to (F^n)^d$, $w \mapsto (w(1-X_1), \ldots, w(1-X_d))$ is injective (if $w \in \ker \alpha$ then $wX_i = w$ for each $i \in [1, \ldots, d]$), against the fact that $X_1, \ldots, X_d$ generate a non-trivial irreducible group; the matrix $(1 - X_1 \cdots 1 - X_d)$ has rank $n$, and so it is possible to find $A_1, \ldots, A_d$ satisfying (1) whenever $n + u \leq nd$. Hence $d(V^u \times H) \leq d$ whenever $u \leq n(d - 1)$. The case $u = n(d - 1)$ is of special importance. In this case our observations yield:

**Proposition 2.2.** Let $u = n(d - 1)$. Assume that $(X_1, \ldots, X_t)$ is a $t$-tuple of elements of $H$ such that any $d$-entries together generate $H$. Then there exists a $t$-tuple $(X_1, \ldots, X_t)$ of elements of $G = V^u \times H$ such that any $d$-entries together generate $G$ (so that for all $i$ with $1 \leq i \leq t$ the element $X_i$ is the projection of $X_i$ under the projection from $G$ to $H$) if and only if there exist $n \times n$ matrices $A_1, \ldots, A_t$ such that for all $1 \leq i_1 < i_2 < \cdots < i_d \leq t$ we have

$$\det \begin{pmatrix} 1 - X_{i_1} & \cdots & 1 - X_{i_d} \\ A_{i_1} & \cdots & A_{i_d} \end{pmatrix} \neq 0.$$

**Lemma 2.1.** If $G$ is an elementary abelian $p$-group of rank $d$, then $\mu_d(G) \geq p + 1$.

**Proof.** It follows from a more general fact. Consider the vector space $F_q^d$ of dimension $d$ over the field $F_q$ of size $q$. If $q + 1 \geq d$, then

$$\{(1, t, t^2, \ldots, t^{d-1}) \mid t \in F_q\} \cup \{(0, \ldots, 0, 1)\}$$

is a set of size $q + 1$, with the property that every subset of size $d$ is a basis. $\square$

3. A problem in linear algebra

Let $M_{r \times d}(F)$ be the set of the $r \times s$ matrices over the field $F$. Assume that $d, t$ are positive integers with $2 \leq d \leq t$. We say that $(A_1, \ldots, A_t)$ is a $(t, d)$-family in $M_{n \times n}(F)$ if, for all $1 \leq i_1 < \cdots < i_d \leq m$, the $n \times dn$ matrix $(A_{i_1} \cdots A_{i_d})$ has rank $n$. Moreover we say that $(B_1, \ldots, B_t)$ is a completion for the $(t, d)$-family $(A_1, \ldots, A_t)$ if $B_1, \ldots, B_t$ are elements of $M_{n(d-1) \times n}(F)$ with the property that, whenever $1 \leq i_1 < i_2 < \cdots < i_d \leq t$, we have

$$\det \begin{pmatrix} A_{i_1} & \cdots & A_{i_d} \\ B_{i_1} & \cdots & B_{i_d} \end{pmatrix} \neq 0.$$

In this section we want to discuss the following question.

**Question 3.1.** Under which assumptions does a $(t, d)$-family $(A_1, \ldots, A_t)$ admit a completion $(B_1, \ldots, B_t)$?

A first easy remark shows that we can find a completion for a $(t, d)$-family in $M(n, F)$ if $|F|$ is large with respect to $t$. Precisely we have:
Lemma 3.1. Let \((A_1, \ldots, A_d)\) be a \((t, d)\)-family in \(M_{n \times n}(F)\). If \(\binom{d}{d-1} < |F|\), then \((A_1, \ldots, A_d)\) admits a completion. In particular any \((t, d)\)-family over \(M(n, F)\) admits a completion if \(F\) is an infinite field.

**Proof.** Consider \(n^2 \cdot t \cdot (d - 1)\) indeterminates \(x_\delta\) with

\[ \delta = (i, j, k) \in \Delta = \{1, \ldots, t\} \times \{1, \ldots, n(d - 1)\} \times \{1, \ldots, n\} \]

and let \(R\) be the polynomial ring \(F[x_\delta | \delta \in \Delta]\). For \(1 \leq i \leq t\), consider the \(n \times n\) matrix \(X_i\) over \(R\), whose \((j, k)\)-th entry is the indeterminate \(x_{i,j,k}\). Let \(\Omega\) be the set of subsets of \(\{1, \ldots, t\}\) of cardinality \(d\). If \(\omega = \{i_1, \ldots, i_d\} \in \Omega\), then

\[ f_\omega = \det \left( \begin{array}{ccc} A_{i_1} & \cdots & A_{i_d} \\ X_{i_1} & \cdots & X_{i_d} \end{array} \right) \in R. \]

Since \((A_{i_1} \cdots A_{i_d})\) has rank \(n\), there exists a family \(\{C_{i_1}, \ldots, C_{i_d}\}\) of \(n(d - 1) \times n\) matrices with

\[ \det \left( \begin{array}{ccc} A_{i_1} & \cdots & A_{i_d} \\ C_{i_1} & \cdots & C_{i_d} \end{array} \right) \neq 0, \]

hence \(f_\omega \neq 0\). In particular \(f = \prod_{\omega \in \Omega} f_\omega \neq 0\). Assume that either \(F\) is infinite or the degree of the polynomial \(f\) in \(x_\delta\) is smaller than \(|F|\) for each \(\delta \in \Delta\). In this case the polynomial function corresponding to \(f\) assume non-zero values, so we can find \(\{b_\omega | \omega \in \Delta\} \subseteq F^\Delta\) with \(f(b_\omega | \delta \in \Delta) \neq 0\). In particular \(f_\omega(b_\omega | \delta \in \Delta) \neq 0\) for each \(\omega \in \Omega\), hence the matrices \(B_1, \ldots, B_t\), obtained from \(X_1, \ldots, X_t\) giving to \(x_\delta\) the value \(b_\delta\), are a completion for the family \((A_1, \ldots, A_d)\). Let \(\delta = (i, j, k) \in \Delta\). The degree of the polynomial \(f_\omega\) in \(x_\delta\) is at most 1 and it is zero if \(i \notin \omega\). Since there are precisely \(\binom{t-1}{d-1}\) elements \(\omega \in \Omega\) with \(i \in \delta\), we conclude that the degree of the \(f\) in \(x_\delta\) is at most \(\binom{t-1}{d-1}\). \(\square\)

Lemma 3.2. Let \(F\) be a finite field of cardinality \(q\) and denote by \(I_n\) the \(n \times n\) identity matrix. If \(t \leq q^n\), then the \((t, d)\)-family \((I_n, \ldots, I_n)\) admits a completion.

**Proof.** The ring \(M_{n \times n}(F)\) contains a subring \(\{F_1, \ldots, F_{q^n}\}\) isomorphic to the finite field of order \(q^n\). For each \(1 \leq i \leq q^n\), consider the \(n(d - 1) \times n\) matrix

\[ B_i = \left( \begin{array}{c} F_i \\ F_i^2 \\ \vdots \\ F_i^{d-1} \end{array} \right). \]

If \(1 \leq i_1 < \cdots < i_d \leq t\), then

\[ \det \left( \begin{array}{ccc} I_n & \cdots & I_n \\ B_{i_1} & \cdots & B_{i_d} \end{array} \right) = \prod_{1 \leq i_1 < i_2 \leq i_d} \det(F_{i_2} - F_{i_1}) \neq 0. \]

Therefore \((B_1, \ldots, B_t)\) is a completion for the \((t, d)\)-family \((I_n, \ldots, I_n)\). \(\square\)

Lemma 3.3. Assume that \((A_1, \ldots, A_t)\) is a \((t, d)\)-family in \(M_{n \times n}(F)\) and that \((Y_1, \ldots, Y_t)\) are invertible matrices in \(M_{n \times n}(F)\). If \((A_1, \ldots, A_t)\) admits a completion, then \((A_1 Y_1, \ldots, A_t Y_t)\) admits a completion also.
Proof. Assume that \((B_1, \ldots, B_t)\) is a completion for the family \((A_1, \ldots, A_t)\). Whenever \(1 \leq i_1 < \cdots < i_d \leq t\), we have

\[
det\left( \begin{array}{ccc} A_{i_1}Y_{i_1} & \cdots & A_{i_d}Y_{i_d} \\ B_{i_1}Y_{i_1} & \cdots & B_{i_d}Y_{i_d} \end{array} \right) = \det\left( \begin{array}{ccc} A_{i_1} & \cdots & A_{i_d} \\ B_{i_1} & \cdots & B_{i_d} \end{array} \right) \begin{pmatrix} Y_{i_1} & 0 & \cdots & 0 \\ 0 & Y_{i_2} & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Y_{i_d} \end{pmatrix} \prod_{1 \leq j \leq d} \det Y_{i_j} \neq 0.
\]

This implies that \((B_1Y_1, \ldots, B_tY_t)\) is a completion for \((A_1Y_1, \ldots, A_tY_t)\).  

Lemma 3.4. Let \((A_1, \ldots, A_t)\) be a \((t, d)\)-family in \(M_{n \times n}(F)\), where \(F\) is the finite field with \(q\) elements. Assume that the following conditions are satisfied:

1. \(\binom{t-1}{d-1} \leq q\);
2. If \((n, t) = (1, q + 1)\) then \(A_i = 0\) for some \(1 \leq i \leq t\).

Then \((A_1, \ldots, A_t)\) admits a completion.

Proof. Certainly \((A_1, A_2, \ldots, A_t)\) admits a completion if \(t = d\), so we assume \(t > d\).

First suppose \(\det(A_i) \neq 0\) for each \(1 \leq i \leq t\). We have \(t - 1 \leq \binom{t-1}{d-1} \leq q\) and \((n, t) \neq (1, q + 1)\), hence \(t < n\). By Lemma 3.2, \((I_n, \ldots, I_n) = (A_1A_1^{-1}, \ldots, A_tA_t^{-1})\) admits a completion. By Lemma 3.3, \((A_1, \ldots, A_t)\) admits a completion also.

For the rest of this proof we assume \(\det(A_1) = 0\). By Lemma 3.1, we may assume \(\binom{t-1}{d-1} = q\). Moreover since \(\binom{t-2}{d-1} < \binom{t-1}{d-1} = q\), again by Lemma 3.1, there exists a completion \((B_2, \ldots, B_t)\) for the \((t - 1, d)\)-family \((A_2, \ldots, A_t)\). Let now \(X\) be an \(n(d - 1) \times n\) matrix. Let us introduce some notations:

1. \(\Delta = \{(i_1, i_2, \ldots, i_{d-1}) \in [d-1] | 2 \leq i_1 < i_2 < \cdots < i_{d-1} \leq t\};
2. to each \(\delta = (i_1, i_2, \ldots, i_{d-1}) \in \Delta\), we associate the \(n \times nd\) matrix

\[
A_{\delta} = (A_1 \ A_{i_1} \cdots \ A_{i_{d-1}})
\]

and the \(n(d - 1) \times nd\) matrix

\[
B_{\delta} = (X \ B_{i_1} \cdots \ B_{i_{d-1}}).
\]

For each positive integer \(m\), we denote by \(V_m\) the \(m\)-dimensional vector space over \(F\) whose elements are the \(m \times 1\) matrices with coefficients over \(F\) and for \(1 \leq i \leq m\), let \(E_i\) be the matrix in \(V_m\) whose only non-zero entry is 1 in position \((i, 1)\).

For \(\delta \in \Delta\), let

\[
K_{\delta} = (K_{\delta, 1} \cdots K_{\delta, n(d-1)})
\]

be an \(nd \times n(d-1)\) matrix whose columns are a basis of the kernel of the linear map \(\gamma_{\delta} : V_{nd} \to V_n\) defined by \(X \mapsto A_{\delta}X\). Since \(\operatorname{rank}(A_{\delta}) = n\), there exists an \(nd \times n\) matrix \(J_{\delta}\) such that \(A_{\delta}J_{\delta} = I_n\). Notice that

\[
\begin{pmatrix} A_{\delta} \\ B_{\delta} \end{pmatrix} (K_{\delta} \ J_{\delta}) = \begin{pmatrix} A_{\delta}K_{\delta} & A_{\delta}J_{\delta} \\ B_{\delta}K_{\delta} & B_{\delta}J_{\delta} \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ B_{\delta}K_{\delta} & B_{\delta}J_{\delta} \end{pmatrix}.
\]
Since \((K_\delta \quad J_\delta)\) is an invertible matrix, we have

\[
det \begin{pmatrix} A_\delta \\ B_\delta \end{pmatrix} \neq 0 \quad \text{if and only if} \quad det(B_\delta K_\delta) \neq 0.
\]

Since \((B_2, \ldots, B_t)\) is a completion for \((A_2, \ldots, A_t)\), we deduce that \((X, B_2, \ldots, B_t)\) is a completion for \((A_1, A_2, \ldots, A_t)\) if and only if

\[
det(B_i K_\delta) \neq 0 \quad \text{for each} \ \delta \in \Delta.
\]

Let \(u = n - \text{rank}(A)\). It follows from Lemma 3.3 that it is not restrictive to assume that the first \(u\) columns of \(A\) are equal to zero, i.e. \(E_j \in \ker \gamma_\delta\) for \(1 \leq j \leq u\). Let

\[
w_\delta = n(d - 1) - \text{rank} (A_{i_1} \cdots A_{i_{d-1}}) \quad \text{and} \quad v_\delta = n(d - 1) - (w_\delta + u).
\]

We may choose \(K_\delta\) with the following properties:

1. \(K_{\delta,i} = E_i\) if \(1 \leq i \leq u\);
2. there is \((m_{\delta,1}, \ldots, m_{\delta,v_\delta})\) \(\in \mathbb{N}^{v_\delta}\) with \(u < m_{\delta,1} < \cdots < m_{\delta,v_\delta} \leq n\) such that, for \(1 \leq i \leq v_\delta\), \(K_{\delta,u+i} = E_{m_{\delta,i}} + L_{\delta,i}\) with \(L_{\delta,i} \in \langle E_i \mid m_{\delta,i} < i \leq nd\rangle\);
3. \(K_{\delta,i} \in \langle E_i \mid n+1 \leq i \leq nd\rangle\) if \(1+u+v_\delta < i \leq n(d-1)\).

Define \(\Gamma_\delta = \{1, \ldots, u, m_{\delta,1}, \ldots, m_{\delta,v_\delta}\}\) and let \(R_\delta = B_\delta K_\delta\). Denote by \(X_1, \ldots, X_n\) the columns of the matrix \(X\). The previous remarks imply that

\[
R_\delta = (X_1 \cdots X_u \quad X_{m_{\delta,1}} + Y_{\delta,m_{\delta,1}} \cdots X_{m_{\delta,v_\delta}} + Y_{\delta,m_{\delta,v_\delta}} \quad Z_{\delta,1} \cdots Z_{\delta,w_\delta})
\]

where \(Z_{\delta,k}\) does not depend on the choice of \(X\) while \(Y_{\delta,j}\) depends on \(X_{j+1}, \ldots, X_{n}\) but not on \(X_1, \ldots, X_j\). For \(i \in \Gamma_\delta\) let \(R_{\delta,i}\) be the matrix obtained from \(R_\delta\) removing the first \(i-1\) columns if \(i \leq u\), the first \(u+j-1\) columns if \(i = m_{\delta,j} > u\).

Our task is to prove that we can choose \(X_1, \ldots, X_n\) in such a way that the columns of the matrix \(R_{\delta}\) are linearly independent for each \(\delta \in \Delta\). To do that, we prove by induction on \(n-r\) the following claim:

Let \(1 \leq r \leq n\). We can choose \(X_r, \ldots, X_n\) in such a way that the columns of the matrix \(R_{\delta,r}\) are linearly independent for each \(\delta \in \Lambda_r = \{\delta \in \Delta \mid r \in \Gamma_\delta\}\).

First notice that the fact that \(B_2, \ldots, B_t\) is a completion for the family \(A_2, \ldots, A_t\) ensures that the elements of \((Z_{\delta,1}, \ldots, Z_{\delta,w_\delta})\) are linearly independent for each \(\delta \in \Delta\). Now assume that \(X_{r+1}, \ldots, X_n\) are chosen so that the columns of \(R_{\delta,s}\) are linearly independent for each \(s > r\) and \(\delta \in \Lambda_s\). For \(\delta \in \Lambda_r\) let \(R_{\delta,r}\) be the matrix obtained from \(R_{\delta,r}\) by removing the first column and let \(W_{\delta,r}\) be the vector subspace of \(F^{n(d-1)}\) spanned by the columns of \(R_{\delta,r}^*\). The first column of \(R_{\delta,r}\) is \(X_r + Y_{\delta,r}^*\), with \(Y_{\delta,r}^* = 0\) if \(r \leq u\), \(Y_{\delta,r}^* = Y_{\delta,r}\) otherwise. The choice of \(X_{r+1}, \ldots, X_n\) ensures that the columns of \(R_{\delta,r}^*\) are linearly independent, so in order to conclude that the columns of \(R_{\delta,r}\) are linearly independent it suffices to choose \(X_r + Y_{\delta,r}^* \notin W_{\delta,r}\), i.e. \(X_r \notin -Y_{\delta,r}^* + W_{\delta,r}\). First assume \(r > 1\). For \(\delta \in \Lambda_r\), we have \(\dim W_{\delta,r} \leq n(d-1) - 2\) hence \(\left| -Y_{\delta,r}^* + W_{\delta,r}\right| \leq q^{n(d-1)-2}\). In particular

\[
\left| \bigcup_{\delta \in \Lambda_r} (-Y_{\delta,r}^* + W_{\delta,r}) \right| \leq |\Delta|q^{n(d-1)-2} \leq \binom{t-1}{d-1}q^{n(d-1)-2} \leq q^{n(d-1)-1}.
\]
Hence $\bigcup_{\delta \in \Delta} (-Y^*_{\delta,r} + W_{\delta,r}) \neq V_{n(d-1)}$ and we may choose $X_r \notin \bigcup_{\delta \in \Delta} (-Y^*_{\delta,r} + W_{\delta,r})$. Finally assume $r = 1$. We have $\dim W_{\delta,1} \leq n(d-1) - 1$ hence $|W_{\delta,1}| \leq q^{n(d-1) - 1}$. In particular, since $0 \in W_{\delta,r}$ for each $\delta \in \Delta$, we have

$$\left| \bigcup_{\delta \in \Delta} W_{\delta,1} \right| \leq 1 + |\Delta|(q^{n(d-1) - 1} - 1) < 1 + q(q^{n(d-1) - 1} - 1) < q^{n(d-1)}.$$ 

Hence $\bigcup_{\delta \in \Delta} W_{\delta,1} \neq V_{n(d-1)}$ and we may choose $X_1 + Y^*_1 = X_1 \notin \bigcup_{\delta \in \Delta} W_{\delta,1}$. □

4. The main theorem

Let $G$ be a finite solvable group, and let $\mathcal{A}_G$ be a set of representatives for the irreducible $G$-groups that are $G$-isomorphic to a complemented chief factor of $G$. For $A \in \mathcal{A}_G$ let $R_G(A)$ be the smallest normal subgroup contained in $C_G(A)$ with the property that $C_G(A)/R_G(A)$ is $G$-isomorphic to a direct product of copies of $A$ and it has a complement in $G/R_G(A)$. The factor group $C_G(A)/R_G(A)$ is called the $A$-crown of $G$. The non-negative integer $\delta_G(A)$ defined by $C_G(A)/R_G(A) \cong_G A^{\delta_G(A)}$ is called the $A$-rank of $G$ and it coincides with the number of complemented factors in any chief series of $G$ that are $G$-isomorphic to $A$. If $\delta_G(A) \neq 0$, then the $A$-crown is the socle of $G/R_G(A)$. The notion of crown was introduced by Gaschütz in [6]. The same notion can be given in the context of profinite groups (see for example [5]). Indeed a countably based profinite group $G$ has a chain \{ $G_n \}_n \leq \mathbb{N}$ of open normal subgroups with the properties that $\bigcap_{n \in \mathbb{N}} G_n = 1$ and $G_n/G_{n+1}$ is a chief factor of $G/G_{n+1}$ for each $n \in \mathbb{N}$. If $G$ is finitely generated profsolvable group and $A$ is an irreducible $G$-module, then the number $\delta_G(A)$ of $n \in \mathbb{N}$ such that $G_n/G_{n+1}$ is complemented in $G/G_{n+1}$ and $G$-isomorphic to $A$ is finite and independent on the choice of the chain \{ $G_n \}_n \leq \mathbb{N}$.

Lemma 4.1. (See [2, Lemma 1.3.6].) Let $G$ be a finite solvable group with trivial Frattini subgroup. There exists a crown $C/R$ and a non-trivial normal subgroup $U$ of $G$ such that $C = R \times U$.

Lemma 4.2. (See [4, Proposition 11].) Assume that $G$ is a finite solvable group with trivial Frattini subgroup and let $C$, $R$, $U$ be as in the statement of Lemma 4.1. If $HU = HR = G$, then $H = G$.

For $A \in \mathcal{A}_G$ let $q_A = |\text{End}_G(A)|$ and define

$$\alpha(G) = \min \{ q_A \mid A \in \mathcal{A}_G, C_G(A) = G \text{ and } \delta_G(A) > 1 \},$$

$$\beta(G) = \min \{ q_A \mid A \in \mathcal{A}_G, C_G(A) \neq G \},$$

setting $\alpha(G) = \infty$ if $\delta_G(A) \leq 1$ for each $A \in \mathcal{A}_G$ with $C_G(A) = G$ and $\beta(G) = \infty$ if $C_G(A) = G$ for all $A \in \mathcal{A}_G$.

Theorem 4.1. Let $d$ be a positive integer greater than or equal to 2, and let $G$ be a finitely generated prosolvable group that can be generated by $d$ elements. Assume that $t \in \mathbb{N}$ satisfies the conditions

$$t \leq \alpha(G) + 1 \text{ and } \left( \frac{t - 1}{d - 1} \right) \leq \beta(G). \quad (*)$$

Then $\mu_d(G) \geq t$.

Proof. First we prove the theorem in the case when $G$ is a finite solvable group. Let $\pi^*_G$ be the set of the primes $p$ with the property that the Sylow $p$-subgroup of $G/G'$ is cyclic and not trivial and let $G^*/G'$ be the $(\pi^*_G)'$-Hall subgroup of $G/G'$. We prove the following claim making induction on the order of $G$. If $t$ satisfies $(*)$, then there exists $(g_1, \ldots, g_t) \in G'$ such that
If \( F = \text{Frat}(G) \neq 1 \), then there exists \((g_1,\ldots,g_t) \in (G/F)^t \) satisfying the two properties (1) and (2). In particular \( g_t F \in (G/F)^* \leq G^F/F \), hence there exists \( f \in F \) with \( g_t f \in G^* \). Clearly \((g_1,\ldots,g_{t-1},g_t)\) is a \( t \)-uple of elements of \( G \) with the requested properties. So we may assume \( \text{Frat}(G) = 1 \). In this case, by Lemma 4.1, there exist a crown \( C/R \) of \( G \) and a normal subgroup \( U \) of \( G \) such that \( C = R \times U \). We have \( R = R_C(A) \) for \( A \in A_G \) and \( U \cong_C A^\delta \) for \( \delta = \delta_C(A) \). We distinguish two cases:

(a) \( C_G(A) = G \). In this case \( G = C = R \times U \) and \( U \cong (C_G)^\delta \) for \( q = q_A \). Since \( \alpha(R) = \alpha(G/U) \not\equiv \alpha(G) \) and \( \beta(R) = \beta(G/U) \not\equiv \beta(G) \), there exists a \( t \)-uple \((r_1,\ldots,r_t) \in \mathbb{R}^t \) such that \( r_i \in R \) and any \( d \)-entries together generate \( R \). If \( d = 1 \), then \( U = \{ u \} \) is cyclic; we consider the \( t \)-uple \((g_1,\ldots,g_t) \) is cyclic and \( g_t \) is cyclic. If \( \delta = 2 \), then \( \delta = d(U) \leq d(G) = 2 \) and, by Lemma 2.1, \( \mu_d(U) = \mu_d(C_G^\delta) \geq \mu_d(C_G) \), hence there exists a \( t \)-uple \((u_1,\ldots,u_t)\) of elements of \( U \) such that any \( d \)-entries together generate \( U \); we consider the \( t \)-uple \((g_1,\ldots,g_t) \) is cyclic and \( g_t \) is cyclic. In both the cases, if \( 1 \leq i_1 < \cdots < i_d \leq t \), then \((g_{i_1},\ldots,g_{i_d}) \) is a \( t \)-uple of elements of \( H \) with the property that every finite subfamily has non-empty intersection. As \( \Omega = -1^{d-1} < 0 \), we conclude from Proposition 2.2, that there exists a \( t \)-tuple \((x_1,\ldots,x_t) \) of elements in \(-1^{d-1} < 0 \). By Lemma 3.4, \((1 - X_1,\ldots,1 - X_t)\) admits a completion. But then we deduce from Proposition 2.2, that there exist \( b_i,\bar{b}_t \in \mathbb{U} \) with the property that \((b_i x_1,\ldots,b_ix_{i_d}) = \bar{G} \) whenever \( 1 \leq i_1 < \cdots < i_d \leq t \). Let \((g_1,\ldots,g_t) = (u_1 x_1,\ldots,u_t x_t) \). If \( 1 \leq i_1 < \cdots < i_d \leq t \), then \((g_{i_1},\ldots,g_{i_d})R = (u_{i_1} x_{i_1},\ldots,u_{i_d} x_{i_d})R = G \) and \((g_{i_1},\ldots,g_{i_d})U = (x_{i_1},\ldots,x_{i_d})U = U \); therefore by Lemma 4.2, \((g_{i_1},\ldots,g_{i_d}) = G \).

Now assume that \( G \) is a \( d \)-generated profinite group and let \( N \) be the family of the open normal subgroups of \( G \). For \( N \in \mathcal{N} \), let \( \Omega_N \) be the subset of the \( t \)-uple \((x_1,\ldots,x_t) \in \mathbb{C}^t \) with the property that \((x_1,\ldots,x_t)\) is a \( t \)-uple of elements of \( N \) whenever \( 1 \leq i_1 < \cdots < i_d \leq t \). Since \( \alpha(G/N) \not\equiv \alpha(G) \) and \( \beta(G/N) \not\equiv \beta(G) \), we have that \( \mu_d(G/N) \geq \mu_d(G) \), hence \( \Omega_N \neq \emptyset \). Notice that if \((x_1,\ldots,x_t) \in \Omega_N \), then \( x_1 N \times \cdots \times x_t \in \Omega_N \) and actually \( \Omega_N \) is the (finite) union of all subsets of that type, thus \( \Omega_N \) is closed in \( G^t \). Moreover, if we choose \( N_1,\ldots,N_t \in \mathcal{N} \), then \( \emptyset \neq \Omega_{N_1 \cap \cdots \cap N_t} \subseteq \Omega_{N_1} \cap \cdots \cap \Omega_{N_t} \), so the family \( \{ \Omega_N \}_{N \in \mathcal{N}} \) has the property that every finite subfamily has non-empty intersection. As \( G^t \) is compact, the whole family has non-empty intersection. Assume \((g_1,\ldots,g_t) \in \bigcap_{N \in \mathcal{N}} \Omega_N \). If \( 1 \leq i_1 < \cdots < i_d \leq t \), then \((g_{i_1},\ldots,g_{i_d})N = G \) for each \( N \in \mathcal{N} \), i.e. \((g_1,\ldots,g_t) \) is a dense subgroup of \( G \).

**Corollary 4.1.** Let \( G \) be a \( d \)-generated profinite group, with \( d \geq 2 \) and let \( p \) be the smallest prime divisor of the order of \( G \) (i.e. the smallest prime dividing the order of some finite epimorphic image of \( G \)). Then \((\mu_d(G))_{d-1} > p \).

**Proof.** Certainly \((\mu_d(G))_{d-1} \geq \mu_d(G) \geq d + 1 \geq 3 \), hence the statement is true if \( p = 2 \). So we may assume that \( p \neq 2 \). In particular, by the Odd Order Theorem, \( G \) is solvable and we may apply Theorem 4.1. Let \( \mu = \mu_d(G) \); we have that either \( \mu + 1 > \alpha(G) + 1 \geq p + 1 \) or \((\mu + 1)_{d-1} > \beta(G) \geq p \), otherwise we would have \( \mu_d(G) \geq \mu + 1 \). In both the cases, since \((\mu)_{d-1} > \mu \), we conclude \((\mu)_{d-1} > p \). \( \square \)

Let \( \sigma(G) \) denote the least number of proper subgroups of \( G \) whose union is \( G \). Theorem 4.1 allows us to give a shorter proof of a result that has already been obtained in [7], concerning the exact value of \( \mu_2(G) \) when the Fitting height of \( G \) is equal to 2.
Corollary 4.2. Let $G$ be a finite solvable group of Fitting height equal to 2. If $d(G) = 2$, then $\mu_2(G) = \sigma(G)$.

Proof. Let $\mathcal{A}$ be the set of the chief factors of $G$ having more than one complement. By Theorem 4.1 and [9]

$$\min(1 + q_A) \leq \mu_2(G) \leq \sigma(G) = \min(1 + |A|).$$

Fix $A \in \mathcal{A}$ with the property that $q_A \leq q_B$ for each $B \in \mathcal{A}$ and let $H = G/C_G(A)$. Consider the Fitting subgroup $\text{Fit}(G)$ of $G$: we have that $\text{Fit}(G) \leq C_G(A)$ and by hypothesis $G/\text{Fit}(G)$ is nilpotent, hence $H$ is nilpotent. If $H$ is not cyclic, then $H$, and consequently $G$, admits a central factor with more than one complement and $q_A \leq p$, where $p$ is the order of this central factor. On the other hand $Z(H)$ has an element of order $p$ and this implies that $p$ divides the order of the multiplicative group of the field $\text{End}_C(A)$, hence $p \leq q_A - 1$, a contradiction. So $H$ is cyclic, but then $\dim_{\text{End}_C(A)} A = 1$ and $q_A = |A|$. It follows that $\sigma(G) \leq \sigma(A \rtimes H) \leq |A| + 1 = q_A + 1 \leq \mu_2(G)$ and therefore $\sigma(G) = \mu_2(G)$. □

Corollary 4.3. Let $\pi$ be a set of prime numbers and let $p$ be the smallest prime in $\pi$. If $G$ is the free pro-$\pi$-group of rank 2, then $\mu_2(G) = p + 1$.

Proof. There exists an open subgroup $N$ of $G$ with $G/N \cong C_p \times C_p$, hence $\mu_2(G) \leq \mu_2(G/N) \leq \sigma(G/N) \leq p + 1$. Moreover $\mu_2(G) \geq d(G) + 1 = 3$. If $p = 2$ then $\mu_2(G) = 3$. If $p > 2$, then $G$ is pro-solvable by the Odd Order Theorem and $\beta(G) \geq \alpha(G) = p$, so $\mu_2(G) \geq p + 1$ by Theorem 4.1. □

As we noticed in the introduction, our lower bound for $\mu_4(G)$ is not the best possible and in the particular case of 2-generated finite solvable groups it remains open the question how large can be the difference $\sigma(G) - \mu_2(G)$. In this context, it can be interesting the discussion of the following example. Let $H = \Gamma L(1,8)$: $H$ is a solvable absolutely irreducible subgroup of $GL(V)$, with $V$ a 3-dimensional vector space over the field $F_2$ with 2 elements. We are in the situation described in Section 2. The semidirect product $G = V^3 \rtimes H$ (where $H$ acts in the same way on each of the 3 direct factors) is 2-generated. The set $\mathcal{A}$ of the chief factors with more than one complement consists only of 2-elements: $A_1 \cong C_3$ and $A_2 \cong C_7$. Using Corollary 1.2, we can only conclude

$$3 = |\text{End}_G(A_1)| + 1 \leq \mu_2(G) \leq \sigma(G) \leq |A_2| + 1 = 8.$$

However, the arguments introduced in Sections 2 and 3, allow us to prove that $\mu_2(G) = \sigma(G) = 8$. We have $H = (f, \sigma \mid f^3 = 1, \sigma^3 = 1, f^6 = f^2)$ and the 8 elements $h_1 = \sigma$, $h_2 = \sigma f$, $\ldots$, $h_7 = \sigma f^6$, $h_8 = f$ pairwise generate $H$. For each $1 \leq i \leq 8$, let $A_i$ be the $3 \times 3$ matrix over $F_2$ representing the endomorphism $1 - h_i$ with respect to a fixed basis of $V$. We have that $(A_1, \ldots, A_8)$ is an $(8,2)$-family in $M_{3 \times 3}(F_2)$ and, by Proposition 2.2, $\mu_2(G) = 8$ if $(A_1, \ldots, A_8)$ admits a completion. Notice that $C_V(h_1)$ has dimension 1 if $1 \leq i \leq 7$ while $C_V(h_8) = |\emptyset|$, hence $\text{rank}(A_i) = 2$ if $1 \leq i \leq 7$ and $\text{rank}(A_8) = 3$. For $1 \leq i \leq 8$, let $V_i$ be the subspace of $F_2^3$ spanned by the columns of the matrix $A_i$. If $1 \leq i < j \leq 7$ then, since $\text{rank}(A_i A_j) = 3$, it must be $V_i \neq V_j$. Therefore $(V_1, \ldots, V_7)$ coincides with the set of all the 2-dimensional subspaces of $F_2^3$ and by Lemma 3.3 $(A_1, \ldots, A_8)$ admits a completion if and only if $(A_1^*, \ldots, A_8^*)$ admits a completion, where

$$A_1^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_3^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_4^* = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A_5^* = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_6^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_7^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_8^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$
The following matrices represent a completion for \((A_1^*, \ldots, A_8^*)\):

\[
\begin{align*}
B_1^* &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & B_2^* &= \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, & B_3^* &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, & B_4^* &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \\
B_5^* &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, & B_6^* &= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & B_7^* &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, & B_8^* &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. 
\end{align*}
\]

References