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Existence of homoclinic solutions to periodic orbits in a center manifold

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Abstract

Consider a Lagrangian of the form

$$L(x,\dot{x},q,\dot{q}) = \tfrac{1}{2}(\dot{x}^2 - x^2) + \tfrac{1}{2}\dot{q}^2 + (1+\delta(x))V(q),$$

where x, $q \in \mathbb{R}$. Assuming that δ is bounded and V, periodic in q, is such that $V'(0) = 0$, we prove existence of infinitely many solutions homoclinic to periodic orbits in the center manifold $q = 0$, $\dot{q} = 0$ of the corresponding system.

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Keywords: Homoclinic solutions; Center manifold; Saddle center fixed points

1. Introduction

Consider the Lagrangian

$$L(x,\dot{x},q,\dot{q}) = \frac{1}{2}(\dot{x}^2 - x^2) + \frac{1}{2}\dot{q}^2 + (1+\delta(x))V(q), \quad x, q \in \mathbb{R}.$$

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Under the assumption that V has a strict global minimum at $q=0$ and it is periodic in $q\in\mathbb{R}$ one knows that the point $P_0=(x=0,\dot{x}=0,q=0,\dot{q}=0)$ is a saddle-center stationary point for the associated Hamiltonian system.

Such a stationary point has a one-dimensional stable and unstable manifold which in general do not cross in the three-dimensional energy surface. So one does not expect to find solutions homoclinic to P_0.

On the other hand, associated to such a P_0, there is also a center manifold, which in this particular situation is simply the manifold $q=0$, $\dot{q}=0$, which is foliated by the periodic orbits $x_R(t)=R\cos(t+\varphi)$, $\dot{x}=-R\sin(t+\varphi)$. Such orbits are hyperbolic with respect to their energy surface.

A model of this kind (actually a rather more general one, when $(q,\dot{q})$ belongs to the tangent bundle of a compact manifold M of dimension $k\geqslant 1$) has been recently studied by Patrick Bernard [2], who has shown that there is at least one solution of the associated Lagrangian system homoclinic to one of the periodic solution in the center manifold. In the paper [2], upper estimates are also given on the energy of the solution found (that is, on the values of R for which there is a homoclinic solution to x_R).

On the other hand, one expects to find a lot of homoclinic solutions in this setting. Indeed, in a perturbative setting, assuming that the system has a homoclinic solution to P_0, existence of many homoclinic solutions and even chaotic behavior has been shown in the papers [4,5,10,12–14]. Another interesting result is contained in paper [3], where it is shown that there are systems which have solutions homoclinic to x_R for all $R>0$ and small (even if there might be no homoclinic to P_0).

A global, non-perturbative results has been obtained, always by Patrick Bernard, who has shown [1], for a class of first-order Hamiltonian systems in $\mathbb{C}^n$ having a saddle-center stationary point, that homoclinic solution to x_R can be found for a dense subset of $R\geqslant R_0>0$.

In this paper, we obtain existence of infinitely many homoclinic solutions for the above Lagrangian system. To be more precise, we look for solutions $(x(t),q(t))$ of

$$\begin{cases} \ddot{q}=(1+\delta(x))V'(q), \\ \ddot{x}+x=\delta'(x)V(q) \end{cases} \tag{1}$$

such that, for some $R>0$, φ_1, $\varphi_2\in[0,2\pi)$ we have

$$\begin{aligned} &\lim_{t\to-\infty}|x(t)-R\cos(t+\varphi_1)|=0, \\ &\lim_{t\to+\infty}|x(t)-R\cos(t+\varphi_2)|=0, \\ &\lim_{t\to-\infty}q(t)=0, \\ &\lim_{t\to+\infty}q(t)=2\pi. \end{aligned} \tag{2}$$

We assume that V and $\delta\in C^2(\mathbb{R})$ are such that

(V1) $V(q+2\pi)=V(q)$ for all $q\in\mathbb{R}$;

(V2) $0 = V(0) < V(q)$ for all $q \in \mathbb{R} \backslash 2\pi\mathbb{Z}$;
(V3) $V''(0) = \mu > 0$;
(V4) $V'(q) > 0$ for all $q \in (0, \bar{\eta}]$ and $V'(q) < 0$ for all $q \in [2\pi - \bar{\eta}, 2\pi)$;
(δ1) $-1 < \underline{\delta} \leqslant \delta(x) \leqslant \bar{\delta}$ for all $x \in \mathbb{R}$;
(δ2) $|\delta'(x)x| \leqslant 2\delta^*$ for all $x \in \mathbb{R}$ where $1 + \underline{\delta} - \delta^* > 0$;
(δ3)

$$\bar{\delta} - \underline{\delta} + \delta^* \leqslant \frac{\bar{\eta}(1 + \underline{\delta} - \delta^*)^{3/2}}{1 + 2\pi^2 + (1 + \bar{\delta})||V||_\infty} \sqrt{\frac{V_{\bar{\eta}/2}}{2}},$$

where $V_{\bar{\eta}/2} = \min\{V(s), s \in [\frac{\bar{\eta}}{2}, 2\pi - \frac{\bar{\eta}}{2}]\}$.

Remark 1. Let us point out, for future reference, that (V3) implies that there is a $\eta_0 \in (0, \bar{\eta}/2)$ such that

$$\frac{\mu}{2} \leqslant V''(q) \leqslant 2\mu \quad \text{for all } |q| \leqslant \eta_0. \tag{3}$$

Remark 2. The above assumptions are satisfied, for example, by $V(q) = 1 - \cos q$ and $\delta(x) = \delta_\infty \arctan x$ provided $\delta_\infty < 0.02$.

Remark 3. If $\delta(x) \equiv \delta_0$ is a constant, then under assumptions (V1)–(V2) there is a solution $q_0(t)$ of $\ddot{q} = (1 + \delta_0)V'(q)$ homoclinic to 0. (See, for example, [6,15]) and hence $(R\cos(t + \varphi_1), q_0(t))$ is a solution of our problem for all $R \geqslant 0$ and $\varphi_1 = \varphi_2 \in [0, 2\pi)$.

Remark 4. If $\delta'(0) = 0$, then, denoting by $q_0(t)$ a solution of $\ddot{q} = (1 + \delta(0))V'(q)$ homoclinic to 0, we have that $(0, q_0(t))$ is a solution of our problem with $R = 0$.

Our main result (see Theorem 17) holds provided (1) has no solution homoclinic to $q = 0$, $x = 0$. We prove that, given any $[\alpha, \beta] \subset (0, 2\pi)$, one can find a solution of (1) satisfying (2) for some $\varphi_2 - \varphi_1 \in [\alpha, \beta]$ and $R > 0$. An immediate consequence (see Corollary 18) is that we have infinitely many solutions of (1) satisfying (2).

Our result, although in a simpler setting, improves that of Bernard since, besides finding infinitely many solutions of (1)–(2), we also show that the solutions we find are such that $q(t) \in [0, 2\pi]$ for all t and that $\lim_{t \to -\infty} q(t) = 0$, $\lim_{t \to +\infty} q(t) = 2\pi$.

Solutions of our problem will be found using variational methods as limit of solutions of the following boundary value problem as $T \to +\infty$:

$$\begin{cases} \ddot{q} = (1 + \delta(x))V'(q), \\ \ddot{x} + x = \delta'(x)V(q), \\ q(0) = 0, \quad q(T) = 2\pi, \\ x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T). \end{cases} \tag{PT}$$

The method we use is close to the one used by Bernard in [2], and uses in an essential way Struwe's monotonicity trick (see [18] and also [11]). For other global (i.e. non-perturbative) results on existence of solutions homoclinic to periodic, let us quote [7–9,16,17].

2. Variational setting

Let

$$E = \{x \in H^1_{loc}(\mathbb{R}) \mid x \text{ is 1-periodic}\}$$

with scalar product $(x, y) = \int_0^1 (\dot{x}\dot{y} + xy)$ and

$$\Gamma = \{q \in H^1(0,1) \mid q(0) = 0, q(1) = 2\pi\}$$

and

$$\Gamma^* = \{q \in \Gamma \mid q(s) \in [0, 2\pi] \quad \forall s \in [0,1]\},$$

Γ^* is a closed subset of Γ. Take $T > 0$ and let, for all $(x, q) \in E \times \Gamma$

$$f_T(x, q) = \frac{1}{T^2}\int_0^1 \frac{\dot{x}^2 + \dot{q}^2}{2} - \int_0^1 \left[\frac{x^2}{2} - (1 + \delta(x))V(q)\right].$$

Lemma 5. *Under assumptions* (V1)–(V3) *and* (δ1) $f_T \in C^1(E \times \Gamma; \mathbb{R})$. *Moreover, if* $(x, q) \in E \times \Gamma$ *is a critical point for* f_T, *then, letting, for all* $t \in [0, T]$

$$\tilde{x}(t) = x(t/T), \quad \tilde{q}(t) = q(t/T)$$

we have that $(\tilde{x}, \tilde{q})$ *is a solution of* (PT).

Proof. It is well known that $f_T \in C^1$ and that critical points of f_T are solutions of

$$\begin{cases} \frac{1}{T^2}\ddot{q}(s) = (1 + \delta(x(s)))V'(q(s)), \\ \frac{1}{T^2}\ddot{x}(s) + x(s) = \delta'(x(s))V(q(s)), \\ q(0) = 0, \quad q(1) = 2\pi, \\ x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1). \end{cases}$$

Then

$$\ddot{\tilde{q}}(t) = \frac{1}{T^2}\ddot{q}(t/T) = (1 + \delta(x(t/T)))V'(q(t/T)) = (1 + \delta(\tilde{x}(t)))V'(\tilde{q}(t)),$$

etc. □

We now assume that $T = 2\pi N + \varphi$, $N \in \mathbb{N}$, $\varphi \in (0, 2\pi)$. Associated with the quadratic form

$$\int_0^1 \left(\frac{1}{T^2} \dot{x}^2 - x^2 \right) ds$$

on E there is a splitting of $E = E_N^- \oplus E_N^+$. More precisely, let

$$E_N^- = \left\{ x(s) = a_0 + \sum_{k=1}^{N} (a_k \cos 2\pi ks + b_k \sin 2\pi ks) \right\},$$
$$E_N^+ = \left\{ x(s) = \sum_{k=N+1}^{+\infty} (a_k \cos 2\pi ks + b_k \sin 2\pi ks) \right\}.$$

Then, for all $x \in E$, $x = x^+ + x^-$, $x^+ \in E_N^+$, $x^- \in E_N^-$ and $\int_0^1 x^+ x^- = 0$, $\int_0^1 \dot{x}^+ \dot{x}^- = 0$.

For all $x(s) = a_0 + \sum_{k=1}^{N} (a_k \cos 2\pi ks + b_k \sin 2\pi ks) \in E_N^-$ we have that

$$\begin{aligned} \int_0^1 \left(\frac{1}{T^2} \dot{x}^2 - x^2 \right) &= -a_0^2 + \frac{1}{2} \sum_{k=1}^{N} \left(\frac{4\pi^2 k^2}{T^2} - 1 \right) (a_k^2 + b_k^2) \\ &= -a_0^2 + \frac{1}{2} \sum_{k=1}^{N} \left(\frac{4\pi^2 k^2}{(2\pi N + \varphi)^2} - 1 \right) (a_k^2 + b_k^2) < 0. \end{aligned}$$

Remark also that, for such a $x \in E_N^-$

$$||x||^2 = \int_0^1 (\dot{x}^2 + x^2) = a_0^2 + \frac{1}{2} \sum_{k=1}^{N} (4\pi^2 k^2 + 1)(a_k^2 + b_k^2),$$

so that, for all $x \in E_N^-$

$$\begin{aligned} -\int_0^1 \left(\frac{1}{T^2} \dot{x}^2 - x^2 \right) &\geqslant \frac{1}{T^2} \frac{(2\pi N + \varphi)^2 - (2\pi N)^2}{1 + 4\pi^2 N^2} \int_0^1 (\dot{x}^2 + x^2) \\ &= \lambda^-(T) ||x||^2. \end{aligned} \tag{4}$$

Similarly, for all $x \in E_N^+$

$$\int_0^1 \left(\frac{1}{T^2} \dot{x}^2 - x^2 \right) = \frac{1}{2} \sum_{k=N+1}^{+\infty} \left(\frac{4\pi^2 k^2}{(2\pi N + \varphi)^2} - 1 \right) (a_k^2 + b_k^2) > 0$$

and

$$\int_0^1 \left(\frac{1}{T^2} \dot{x}^2 - x^2 \right) \geqslant \frac{1}{T^2} \frac{(2\pi(N+1))^2 - (2\pi N + \varphi)^2}{1 + 4\pi^2 (N+1)^2} \int_0^1 (\dot{x}^2 + x^2) = \lambda^+(T) ||x||^2.$$

Lemma 6. *Assume V and δ satisfy assumptions* (V1)–(V3), (δ1) *and* (δ2). *Let $T = 2\pi N + \varphi$, $N \in \mathbb{N}$, $\varphi \in (0, 2\pi)$. Then f_T satisfies the Palais Smale condition* (PS), *that is for all $(x_n, q_n) \in E \times \Gamma$ such that*

$$f_T(x_n, q_n) \to c \quad \nabla f_T(x_n, q_n) \to 0$$

there is a subsequence $(x_{n_k}, q_{n_k}) \to (x, q)$. Moreover

$$\frac{1}{2}\int_0^1 \frac{1}{T^2}\dot{q}_n^2 + (1 + \underline{\delta} - \delta^*)\int_0^1 V(q_n) \leqslant c + \frac{1}{T}$$

for all n large enough.

Proof. Using (δ2) we have, for n large

$$\begin{aligned} c + \frac{1}{2T} + \varepsilon_n \|x_n\| &\geqslant f_T(x_n, q_n) - \frac{1}{2}\langle \nabla f_T(x_n, q_n), (x_n, 0)\rangle \\ &= \int_0^1 \left[\frac{1}{2T^2}\dot{q}_n^2 + \left(1 + \delta(x_n) - \frac{1}{2}\delta'(x_n)x_n\right)V(q_n)\right] \\ &\geqslant \int_0^1 \left[\frac{1}{2T^2}\dot{q}_n^2 + (1 + \underline{\delta} - \delta^*)V(q_n)\right]. \end{aligned} \tag{5}$$

Let $x_n = x_n^+ + x_n^-$, $x_n^+ \in E_N^+$, $x_n^- \in E_N^-$. Then

$$\begin{aligned} \varepsilon_n \|x_n^-\| &\geqslant |\langle \nabla f_T(x_n, q_n), (x_n^-, 0)\rangle| \\ &= \left|\int_0^1 \left[\frac{1}{T^2}\dot{x}_n^{-2} - x_n^{-2} + \delta'(x_n)V(q_n)x_n^-\right]\right| \\ &\geqslant \left|\int_0^1 \left[\frac{1}{T^2}\dot{x}_n^{-2} - x_n^{-2}\right]\right| - \left|\int_0^1 \delta'(x_n)V(q_n)x_n^-\right| \end{aligned}$$

and, using (4),

$$\begin{aligned} \lambda^-(T)\|x_n^-\|^2 &\leqslant \varepsilon_n\|x_n^-\| + \left|\int_0^1 \delta'(x_n)V(q_n)x_n^-\right| \\ &\leqslant \left[\varepsilon_n + \|\delta'\|_\infty \left(\int_0^1 V(q_n)^2\right)^{\frac{1}{2}}\right]\|x_n^-\| \\ &\leqslant [\varepsilon_n + \|\delta'\|_\infty \|V\|_\infty]\|x_n^-\| \end{aligned}$$

which implies

$$\|x_n^-\| \leqslant \frac{1}{\lambda^-(T)}[\varepsilon_n + \|\delta'\|_\infty \|V\|_\infty]. \tag{6}$$

Similarly,

$$\varepsilon_n \|x_n^+\| \geqslant |\langle \nabla f_T(x_n, q_n), (x_n^+, 0) \rangle|$$

implies that

$$\|x_n^+\| \leqslant \frac{1}{\lambda^+(T)}[\varepsilon_n + \|\delta'\|_\infty \|V\|_\infty]. \tag{7}$$

Eq. (6) and (7) imply that

$$\|x_n\| \leqslant \|x_n^+\| + \|x_n^-\| \leqslant 2 \max\left\{\frac{1}{\lambda^+(T)}, \frac{1}{\lambda^-(T)}\right\}[\varepsilon_n + \|\delta'\|_\infty \|V\|_\infty]$$

and $\|x_n\|$ is bounded. Inserting this in (5), we find that

$$\int_0^1 \left[\frac{1}{2T^2}\dot{q}_n^2 + (1 + \underset{\sim}{\delta} - \delta^*)V(q_n)\right] \leqslant c + \frac{1}{2T} + \varepsilon_n \rho(T)[\varepsilon_n + \|\delta'\|_\infty \|V\|_\infty].$$

Since V is bounded, we immediately deduce that, for all n large enough

$$\int_0^1 \left[\frac{1}{2T^2}\dot{q}_n^2 + (1 + \underset{\sim}{\delta} - \delta^*)V(q_n)\right] \leqslant c + \frac{1}{T}.$$

This implies that $\|q_n\|_{H^1(0,1)}$ is bounded and it is then a standard fact to show that (PS) holds. □

3. The min–max procedure

As in the previous section, we fix $T = 2\pi N + \varphi$, $N \in \mathbb{N}$, $\varphi \in (0, 2\pi)$. We say that $h \in \mathcal{H}_N$ if:

(1) $h : E_N^- \to E \times \Gamma$ is continuous;
(2) there are $R > 0$ and $q_h \in \Gamma$ such that,

$$h(x) = (x, q_h) \quad \forall \|x\| \geqslant R.$$

We define

$$c(T) = \inf_{h \in \mathcal{H}_N} \sup_{x \in E_N^-} f_T(h(x)). \tag{8}$$

Lemma 7. *Assume V and δ satisfy assumptions* (V1)–(V3), (δ1) *and* (δ2).

Let $T = 2\pi N + \varphi$, $N \in \mathbb{N}$, $\varphi \in (0, 2\pi)$ *and* $c(T)$ *be defined as in* (8). *Then the following hold*:

(a) $\exists 0 < \underline{c} \leqslant \bar{c}$ *such that*

$$\frac{\underline{c}}{T} \leqslant c(T) \leqslant \frac{\bar{c}}{T} \quad \forall T \geqslant 1.$$

(b) $c(2\pi N + \varphi)$ *is non-increasing in* $\varphi \in (0, 2\pi)$.

(c) *Let* $[\alpha, \beta] \subset (0, 2\pi)$. *Then for all* $N \in \mathbb{N}$ *there is* $\varphi_N \in (\alpha, \beta)$ *such that* $c(T)$ *is differentiable at* $T = 2\pi N + \varphi_N$ *and*

$$|c'(T)| \leqslant \frac{1}{\beta - \alpha} \left(\frac{\bar{c}}{2\pi N + \alpha} - \frac{\underline{c}}{2\pi N + \beta} \right).$$

Proof. For all $T = 2\pi N + \varphi$, $N \in \mathbb{N}$, $\varphi \in (0, 2\pi)$, let

$$\underline{c}(T) = \min_{q \in \Gamma} \int_0^1 \left[\frac{1}{2T^2} \dot{q}^2 + (1 + \underline{\delta}) V(q) \right] dt,$$

$$\bar{c}(T) = \min_{q \in \Gamma} \int_0^1 \left[\frac{1}{2T^2} \dot{q}^2 + (1 + \bar{\delta}) V(q) \right] dt$$

and let $\bar{q}_T \in \Gamma$ such that

$$\int_0^1 \left[\frac{1}{2T^2} \dot{\bar{q}}_T^2 + (1 + \bar{\delta}) V(\bar{q}_T) \right] dt = \bar{c}(T).$$

Let us show that

$$\underline{c}(T) \leqslant c(T) \leqslant \bar{c}(T).$$

Let $\bar{h}(x) = (x, \bar{q}_T)$ for all $x \in E_N^-$. Then $\bar{h} \in \mathcal{H}_N$ and

$$\begin{aligned} \inf_{h \in \mathcal{H}_N} \sup_{x \in E_N^-} f_T(h(x)) &\leqslant \sup_{x \in E_N^-} f_T(\bar{h}(x)) = \sup_{x \in E_N^-} f_T(x, \bar{q}_T) \\ &\leqslant \sup_{x \in E_N^-} \int_0^1 \left[\frac{1}{2T^2} \dot{\bar{q}}_T^2 + (1 + \delta(x)) V(\bar{q}_T) \right] dt \\ &\leqslant \int_0^1 \left[\frac{1}{2T^2} \dot{\bar{q}}_T^2 + (1 + \bar{\delta}) V(\bar{q}_T) \right] dt = \bar{c}(T). \end{aligned}$$

Take now $h \in \mathcal{H}_N$ and consider

$$\bar{h} : E_N^- \to E_N^-, \quad \bar{h}(x) = \pi_{E_N^-}(\pi_E(h(x))),$$

where $\pi_E(x,q) = x$, $\pi_{E_N^-}(x) = x^-$. Since $\bar{h}_{|\partial B(0,R)} = Id$ for all R large enough, there is $\bar{x} \in E_N^-$ such that $\bar{h}(\bar{x}) = 0$, i.e.

$$\pi_E(h(\bar{x})) \in E_N^+.$$

Then, letting $q = \pi_\Gamma(h(\bar{x}))$,

$$\sup_{x \in E_N^-} f_T(h(x)) \geqslant f_T(h(\bar{x})) \geqslant \int_0^1 \left[\frac{1}{2T^2}\dot{q}^2 + (1+\underline{\delta})V(q)\right] dt$$

$$\geqslant \min_{q \in \Gamma} \int_0^1 \left[\frac{1}{2T^2}\dot{q}^2 + (1+\underline{\delta})V(q)\right] dt = \underline{c}(T).$$

Now to estimate $\bar{c}(T)$, let

$$\tilde{p}_T(t) = \begin{cases} \bar{q}_1(t), & t \in [0,1], \\ 2\pi, & t \in [1,T] \end{cases}$$

and $p_T(s) = \tilde{p}_T(sT) \in \Gamma$. Then

$$\int_0^1 \left[\frac{1}{2T^2}\dot{\bar{q}}_T^2 + (1+\bar{\delta})V(\bar{q}_T)\right] dt \leqslant \int_0^1 \left[\frac{1}{2T^2}\dot{p}_T^2 + (1+\bar{\delta})V(p_T)\right] dt$$

$$= \int_0^T \left[\frac{1}{2}\dot{\tilde{p}}_T^2 + (1+\bar{\delta})V(\tilde{p}_T)\right] \frac{dt}{T}$$

$$= \frac{1}{T}\int_0^1 \left[\frac{1}{2}\dot{\bar{q}}_1^2 + (1+\bar{\delta})V(\bar{q}_1)\right] dt$$

$$= \frac{\bar{c}(1)}{T} \leqslant \frac{1}{T}[2\pi^2 + (1+\bar{\delta})||V||_\infty].$$

Let us note, for future reference, that

$$\bar{c} = \bar{c}(1) \leqslant 2\pi^2 + (1+\bar{\delta})||V||_\infty. \tag{9}$$

Let us show that $\underline{c}(T) \geqslant \underline{c}/T$. Indeed, since for all $q \in \Gamma$ we have that $q(0) = 0$, $q(1) = 2\pi$, for all $\eta \in (0,\pi)$ there is an interval $[a,b] \subset [0,1]$ such that

$$0 < \eta \leqslant q(t) \leqslant 2\pi - \eta \quad \forall t \in [a,b], \quad q(a) = \eta, \quad q(b) = 2\pi - \eta.$$

Let $V_\eta = \min_{s \in [\eta, 2\pi-\eta]} V(s) > 0$. Then

$$2\pi - 2\eta = q(b) - q(a) \leqslant \int_a^b |\dot{q}(s)|\, ds \leqslant T\sqrt{2(b-a)}\left(\int_a^b \frac{1}{2T^2}\dot{q}^2\right)^{1/2}$$

and

$$\int_a^b V(q) \geqslant (b-a) V_\eta$$

so that

$$\int_0^1 \frac{1}{2T^2} \dot{q}^2 + (1+\underline{\delta}) V(q) \geqslant \left(\frac{2\pi - 2\eta}{T\sqrt{2(b-a)}} \right)^2 + (1+\underline{\delta})(b-a) V_\eta$$

$$\geqslant \frac{4(\pi-\eta)}{T} \sqrt{\left(\frac{1+\underline{\delta}}{2}\right) V_\eta} \equiv \frac{\underline{c}}{T} \tag{10}$$

which proves (a).

(b) follows from the fact that f_T is a decreasing function of T and the min–max procedure does not depend on $T \in (2\pi N, 2\pi(N+1))$.

To prove (c) we notice that the fact that $c(T)$ is a.e. differentiable is a consequence of (b). Then for $[\alpha, \beta] \subset (0, 2\pi)$ we have that

$$-c(2\pi N + \beta) + c(2\pi N + \alpha) \geqslant \int_\alpha^\beta |c'(2\pi N + \varphi)| \, d\varphi$$

so we can find $\varphi_N \in (\alpha, \beta)$ such that c is differentiable at $2\pi N + \varphi_N$ and

$$|c'(2\pi N + \varphi_N)|(\beta - \alpha) \leqslant c(2\pi N + \alpha) - c(2\pi N + \beta) \leqslant \frac{\bar{c}}{2\pi N + \alpha} - \frac{\underline{c}}{2\pi N + \beta}. \qquad \square$$

Proposition 8. *Assume V and δ satisfy assumptions* (V1)–(V3), (δ1) *and* (δ2).

Let $T = 2\pi N + \varphi$, $N \in \mathbb{N}$, $\varphi \in (0, 2\pi)$ and $c(T)$ be defined as in (8). *Then there is a critical point for f_T at level $c(T)$, to which corresponds a solution of problem* (PT).

Proof. It is an immediate consequence of Lemmas 6 and 7. $\square$

In order to pass to the limit as $T \to \infty$ we will need some more information on (at least some of) the critical points at level $c(T)$.

We want to show that one can find, at least for some T's, critical points at level $c(T)$ such that

(1) $q(t) \in [0, 2\pi]$ for all $t \in [0, 1]$;
(2) $\int_0^1 \frac{\dot{x}^2 + \dot{q}^2}{2T^2} \leqslant B$ for some B not depending on T.

To prove the first claim, we introduce

$$\Gamma^* = \{ q \in \Gamma \mid q(t) \in [0, 2\pi] \quad \forall t \in [0,1] \},$$

$$\mathcal{H}_N^* = \{ h \in \mathcal{H}_N \mid h(x) \in E \times \Gamma^* \quad \forall x \in E_N^- \},$$

$$c^*(T) = \inf_{h \in \mathcal{H}_N^*} \sup_{x \in E_N^-} f_T(h(x)).$$

Lemma 9. *For all* $T \neq 2\pi N$,

$$c^*(T) = c(T).$$

Proof. $\mathcal{H}_N^* \subset \mathcal{H}_N$ implies that $c^*(T) \geqslant c(T)$.

To prove the other inequality take $h \in \mathcal{H}_N$ and let $h^* \in \mathcal{H}_N^*$ be defined as $h^*(x) = h(x)^*$ where $(x,q)^* = (x,q^*)$ and

$$q^*(t) = \begin{cases} q(t) & \text{if } 0 \leqslant q(t) \leqslant 2\pi, \\ 2\pi & \text{if } q(t) > 2\pi, \\ 0 & \text{if } q(t) < 0. \end{cases}$$

Then, since

$$(1 + \delta(x(t)))V(q^*(t)) \leqslant (1 + \delta(x(t)))V(q(t)) \quad \forall t \in [0,1],$$

we immediately have that

$$f_T(h^*(x)) \leqslant f_T(h(x)) \quad \forall x \in E_N^-, \ \forall h \in \mathcal{H}_N$$

and

$$c^*(T) \leqslant c(T)$$

follows. □

Proposition 10. *Assume* V *and* δ *satisfy assumptions* (V1)–(V3), (δ1) *and* (δ2).

Let $[\alpha,\beta] \subset (0,2\pi)$ *and* $\varphi_N \in [\alpha,\beta]$ *be as in* (c) *of Lemma* 7.

Then there is a critical point $(x,q) \in E \times \Gamma^*$ *of* $f_{2\pi N + \varphi_N}$ *at level* $c(2\pi N + \varphi_N)$ *such that*

$$\int_0^1 \frac{\dot{x}^2 + \dot{q}^2}{2T^2} \leqslant \frac{1}{\beta - \alpha}(2\bar{c} - \underline{c}) + \frac{1}{3} \equiv B. \tag{11}$$

Proof. Fix N and φ_N as in Lemma 7, let $T = 2\pi N + \varphi_N$. Suppose there are no points in the closed set

$$A = \left\{ (x,q) \in E \times \Gamma^* \,\middle|\, \int_0^1 \frac{\dot{x}^2 + \dot{q}^2}{2T^2} \leqslant B \right\}$$

at level $c(T) = c^*(T)$ which are critical points for f_T. Let K_T be the set of critical points at level $c(T)$. Let $N_\delta(K_T)$ be the neighborhood of K_T of radius δ. Since (PS) holds we can find $\delta_0 > 0$ such that

$$A \cap N_{\delta_0}(K_T) = \emptyset.$$

Moreover, there exists a deformation $\eta : [0,1] \times E \times \Gamma \to E \times \Gamma$ and $\varepsilon_0 > 0$ such that

- $s \mapsto f_T(\eta(s,x,q))$ is not increasing;
- for all $0 < \varepsilon < \varepsilon_0$, we have that

$$\eta(1,x,q) \in f_T^{c(T)-\varepsilon} \quad \text{for all } (x,q) \in f_T^{c(T)+\varepsilon} \backslash N_{\delta_0}(K_T),$$

  where $f_T^{\alpha} = \{(x,q) \in E \times \Gamma : f_T(x,q) \leqslant \alpha\}$;
- $\eta(s,x,q) = (x,q)$ if $f_T(x,q) < 0$.

Let us now consider a sequence $T_n \nearrow T$, $T_n > 2\pi N$ for all n. Since $(c(T_n) - c(T))/(T_n - T) \to c'(T)$, we can find $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ we have

$$c(T_n) - c(T) \leqslant \left(|c'(T)| + \frac{1}{10T}\right)(T - T_n), \tag{12}$$

$$\left(|c'(T)| + \frac{1}{10}\right)(T - T_n) < \varepsilon_0. \tag{13}$$

Then, for all $n \geqslant n_0$, let $h_n \in \mathcal{H}_N^*$ be such that

$$\sup_{E_N^-} f_{T_n}(h_n(x)) \leqslant c(T_n) + \frac{1}{10T}(T - T_n)$$

(recall that $c(T_n) = c^*(T_n)$). Then, using (12), we have for all $n \geqslant n_0$

$$\begin{aligned}
\sup_{x \in E_N^-} f_T(h_n(x)) \leqslant \sup_{x \in E_N^-} f_{T_n}(h_n(x)) &\leqslant c(T_n) + \frac{1}{10T}(T - T_n) \\
&\leqslant c(T) + \left(|c'(T)| + \frac{1}{10T}\right)(T - T_n) + \frac{1}{10T}(T - T_n) \\
&< c(T) + \left(|c'(T)| + \frac{1}{5T}\right)(T - T_n) \\
&< c(T) + \left(|c'(T)| + \frac{1}{10}\right)(T - T_n) < c(T) + \varepsilon_0,
\end{aligned}$$

that is

$$h_n(x) \in f_T^{c(T)+\varepsilon_0} \quad \text{for all } x \in E_N^-, \quad n \geqslant n_0. \tag{14}$$

Take now $n \geqslant n_0$ and $(\bar{x}, \bar{q}) \in h_n(E_N^-)$ such that

$$f_T(\bar{x}, \bar{q}) \geqslant c(T) - \frac{1}{10T}(T - T_n).$$

We have that

$$\begin{aligned} f_{T_n}(\bar{x},\bar{q}) - f_T(\bar{x},\bar{q}) &\leqslant c(T_n) + \frac{1}{10T}(T-T_n) - c(T) + \frac{1}{10T}(T-T_n) \\ &\leqslant \left(|c'(T)| + \frac{1}{10T}\right)(T-T_n) + \frac{1}{5T}(T-T_n) \\ &\leqslant \left(|c'(T)| + \frac{1}{2T}\right)(T-T_n). \end{aligned}$$

On the other hand, we have

$$f_{T_n}(\bar{x},\bar{q}) - f_T(\bar{x},\bar{q}) = \left(\frac{1}{T_n^2} - \frac{1}{T^2}\right)\int_0^1 \frac{\dot{\bar{x}}^2 + \dot{\bar{q}}^2}{2}$$

which implies

$$\frac{T^2 - T_n^2}{T_n^2 T^2}\int_0^1 \frac{\dot{\bar{x}}^2 + \dot{\bar{q}}^2}{2} \leqslant \left(|c'(T)| + \frac{1}{2T}\right)(T-T_n),$$

that is

$$\frac{T + T_n}{T_n^2 T^2}\int_0^1 \frac{\dot{\bar{x}}^2 + \dot{\bar{q}}^2}{2} \leqslant |c'(T)| + \frac{1}{2T} \quad \forall n \geqslant n_0.$$

Finally, we have that

$$\begin{aligned} \frac{1}{T^2}\int_0^1 \frac{\dot{\bar{x}}^2 + \dot{\bar{q}}^2}{2} &\leqslant \frac{T_n^2}{T+T_n}\left(|c'(T)| + \frac{1}{2T}\right) \leqslant \frac{2}{3}\,T\left(|c'(T)| + \frac{1}{2T}\right) \\ &\leqslant \frac{2}{3}\,T\,\frac{1}{\beta-\alpha}\left(\frac{\bar{c}}{2\pi N + \alpha} - \frac{\underline{c}}{2\pi N + \beta}\right) + \frac{1}{3} \\ &\leqslant \frac{1}{\beta-\alpha}(2\bar{c} - \underline{c}) + \frac{1}{3} = B, \end{aligned}$$

from which we deduce that, for all $n \geqslant n_0$ and $x \in E_N^-$,

$$h_n(x) \in f_T^{c(T)-(T-T_n)/10} \bigcup A. \tag{15}$$

Using now (14) and (15) and applying the deformation η we find that, for all $n \geqslant n_0$ and $x \in E_N^-$

$$\eta(1, h_n(x)) \in f_T^{c(T)-(T-T_n)/10}.$$

Since $x \mapsto \eta(1, h_n(x))$ is a map in $\mathcal{H}_N$, we have reached a contradiction. □

Lemma 11. *Assume V and δ satisfy assumptions* (V1)–(V4), *and* (δ1)–(δ3).

Let $T \neq 2\pi N$. *Let* $(x, q) \in E \times \Gamma^*$ *be a critical point of* f_T *such that*

$$f_T(x, q) = c(T).$$

Then for all $0 < \eta \leqslant \eta_0$ (η_0 *given by* (3)) *one can find* $0 < \tau_1 < \tau_2 < 1$ *such that*

$$\begin{aligned} &0 \leqslant q(t) \leqslant \eta \quad \forall t \in [0, \tau_1], \\ &q(t) \geqslant \eta \quad \forall t \in [\tau_1, 1], \\ &q(t) \leqslant 2\pi - \eta \quad \forall t \in [0, \tau_2], \\ &2\pi - \eta \leqslant q(t) \leqslant 2\pi \quad \forall t \in [\tau_2, 1]. \end{aligned}$$

Proof. Recall that (x, q) is a solution of

$$\begin{cases} \ddot{q}(s) = T^2(1 + \delta(x(s)))V'(q(s)), \\ \ddot{x}(s) + T^2 x(s) = T^2 \delta'(x(s))V(q(s)), \end{cases}$$

in $[0, 1]$ such that $q(0) = 0$, $q(s) \in [0, 2\pi]$ for all $s \in [0, 1]$, $q(1) = 2\pi$.

Take $\eta \leqslant \eta_0$ and let $\tau_1 = \inf\{s \in [0, 1] \mid q(s) > \eta\}$ and $\tau_2 = \sup\{s \in [0, 1] \mid q(s) < 2\pi - \eta\}$. If the lemma does not hold, then there is $\tau_1' \in (\tau_1, 1]$ such that $q(\tau_1') = \eta$ (or there is $\tau_2' \in [0, \tau_2)$ such that $q(\tau_2') = 2\pi - \eta$; we will only discuss the first case). Then $q(t)$ has a max $\tau_1'' \in (\tau_1, \tau_1')$, hence $\ddot{q}(\tau_1'') \leqslant 0$. But

$$\ddot{q}(\tau_1'') = T^2(1 + \delta(x(\tau_1'')))V'(q(\tau_1''))$$

implies, by (V4), that $q(\tau_1'') \geqslant \bar{\eta}$. An estimate very similar to the (10) in Lemma 7 shows then that

$$\int_{\tau_1}^{\tau_1'} \left[\frac{1}{2T^2}\dot{q}^2 + (1 + \underset{\sim}{\delta} - \delta^*)V(q)\right] dt \geqslant \frac{2\bar{\eta}}{T}\sqrt{\frac{1 + \underset{\sim}{\delta} - \delta^*}{2}}\, V_{\bar{\eta}/2}.$$

Now we define a new function $\bar{q} \in \Gamma^*$, setting

$$\bar{q}(t) = \begin{cases} 0, & 0 \leqslant t \leqslant \tau_1' - \tau_1, \\ q(t - \tau_1' + \tau_1), & \tau_1' - \tau_1 \leqslant t \leqslant \tau_1', \\ q(t), & \tau_1' \leqslant t \leqslant 1. \end{cases}$$

We also let $\bar{h}(\bar{x}) = (\bar{x}, \bar{q})$ for all $\bar{x} \in E_N^-$. Clearly $\bar{h} \in \mathcal{H}_N^*$, so that, since $f_T(x, q) = c(T)$

$$0 \leqslant \sup_{\bar{x} \in E_N^-} f_T(\bar{h}(\bar{x})) - f_T(x, q).$$

We will now estimate, for all $\bar{x} \in E_N^-$

$$\begin{aligned} f_T(x,q) - f_T(\bar{x},\bar{q}) &= \int_0^1 \left[\frac{\dot{x}^2 + \dot{q}^2}{2T^2} - \frac{x^2}{2} + (1+\delta(x))V(q) \right] \\ &\quad - \left\{ \frac{1}{2} \int_0^1 \left[\frac{1}{T^2} \dot{\bar{x}}^2 - \bar{x}^2 \right] + \int_0^1 \left[\frac{1}{2T^2} \dot{\bar{q}}^2 + (1+\delta(\bar{x}))V(\bar{q}) \right] \right\} \\ &\geqslant \int_0^1 \left[\frac{1}{2T^2} \dot{q}^2 + \left(1 + \delta(x) - \frac{1}{2}\delta'(x)x\right) V(q) \right] \\ &\quad - \int_{\tau_1' - \tau_1}^1 \left[\frac{1}{2T^2} \dot{\bar{q}}^2 + (1+\delta(\bar{x}))V(\bar{q}) \right] \\ &\geqslant \int_0^{\tau_1} (\underline{\delta} - \delta^* - \bar{\delta}) V(q) + \int_{\tau_1}^{\tau_1'} \left[\frac{1}{2T^2} \dot{q}^2 + (1 + \underline{\delta} - \delta^*) V(q) \right] \\ &\quad + \int_{\tau_1'}^1 (\underline{\delta} - \delta^* - \bar{\delta}) V(q) \\ &\geqslant \int_{\tau_1}^{\tau_1'} \left[\frac{1}{2T^2} \dot{q}^2 + (1 + \underline{\delta} - \delta^*) V(q) \right] + \int_0^1 (\underline{\delta} - \delta^* - \bar{\delta}) V(q) \\ &\geqslant \frac{2\bar{\eta}}{T} \sqrt{\frac{1}{2}(1 + \underline{\delta} - \delta^*) V_{\bar{\eta}/2}} - \frac{\bar{\delta} + \delta^* - \underline{\delta}}{1 + \underline{\delta} - \delta^*} \frac{\bar{c} + 1}{T}. \end{aligned}$$

We have used here the fact that, being x a 1-periodic solution of

$$\ddot{x} + T^2 x = T^2 \delta'(x) V(q),$$

one has

$$\frac{1}{2} \int_0^1 \left[\frac{1}{T^2} \dot{x}^2 - x^2 \right] = -\frac{1}{2} \int_0^1 \delta'(x) x V(q)$$

and also the estimate on $\int_0^1 V(q)$ of Lemma 6 and that of $c(T)$ given by Lemma 7.

From estimate (9) and assumption ($\delta 3$) one deduces that

$$f_T(x,q) - f_T(\bar{x},\bar{q}) \geqslant \frac{\bar{\eta}}{T} \sqrt{\frac{1}{2}(1 + \underline{\delta} - \delta^*) V_{\bar{\eta}/2}}$$

and the lemma follows. □

4. The limiting procedure

Recall that we want to pass to the limit as $T \to +\infty$. To do this, we go back to the interval $[0, T]$. As a consequence of the results of the previous sections (see in particular, Proposition 10 and Lemma 11) we have

Proposition 12. *Assume V and δ satisfy assumptions* (V1)–(V4), *and* (δ1)–(δ3).

Then for all $N \in \mathbb{N}$ and $[\alpha, \beta] \subset (0, 2\pi)$ we can find $\varphi_N \in [\alpha, \beta]$ and a solution (x_N, q_N) of

$$\begin{cases} \ddot{q}_N = (1 + \delta(x_N))V'(q_N), \\ \ddot{x}_N + x_N = \delta'(x_N)V(q_N) \end{cases}$$

in $[0, 2\pi N + \varphi_N]$ such that (letting $T_N = 2\pi N + \varphi_N$)

(a) $q_N(0) = 0$, $q_N(T_N) = 2\pi$, $q_N(t) \in [0, 2\pi]$ *for all* $t \in [0, T_N]$;
(b) $x_N(0) = x_N(T_N)$, $\dot{x}_N(0) = \dot{x}_N(T_N)$;
(c) $\frac{1}{2}\int_0^{T_N} \dot{q}_N^2 + \int_0^{T_N} (1 + \underline{\delta} - \delta^*)V(q_N)) \leqslant \bar{c} + 1$;
(d) $\frac{1}{2}\int_0^{T_N} (\dot{x}_N^2 + \dot{q}_N^2) \leqslant BT_N$;
(e) *for all $0 < \eta \leqslant \eta_0$ one can find τ_N^1, $\tau_N^2 \in [0, T_N]$ such that*

$$0 \leqslant q_N(t) \leqslant \eta \quad \text{for all } t \in [0, \tau_N^1],$$

$$q_N(t) \geqslant \eta \quad \text{for all } t \in [\tau_N^1, T_N],$$

$$q_N(t) \leqslant 2\pi - \eta \quad \text{for all } t \in [0, \tau_N^2],$$

$$2\pi - \eta \leqslant q_N(t) \leqslant 2\pi \quad \text{for all } t \in [\tau_N^2, T_N];$$

(f) $\dot{q}_N(0) = \dot{q}_N(T_N) > 0$.

Proof. The only point which needs some comment is (f). This is a consequence of the fact that the total energy is conserved, that is

$$\frac{1}{2}(\dot{x}_N^2(t) + x_N^2(t)) + \frac{1}{2}\dot{q}_N^2(t) - (1 + \delta(x_N(t)))V(q_N(t)) = E_N$$

for some $E_N \in \mathbb{R}$ and for all $t \in [0, T_N]$. Since $V(q_N(0)) = V(0) = 0$, we have that $E_N > 0$ (it cannot be $E_N = 0$ otherwise $q(t) \equiv 0$, $x(t) \equiv 0$). Since x_N is T_N periodic, we also deduce that

$$\dot{q}_N^2(0) = \dot{q}_N^2(T_N)$$

and, since $0 = q_N(0) \leqslant q_N(t) \leqslant q_N(T_N) = 2\pi$ we have that

$$\dot{q}_N(0) = \dot{q}_N(T_N) > 0$$

and q_N can be seen as a periodic function on S^1. $\square$

Lemma 13. *Assume V and δ satisfy assumptions* (V1)–(V4), *and* (δ1)–(δ3).

Let (x_N, q_N) *be as in Proposition* 12. *Let, for* $0 < \eta \leqslant \eta_0$, τ_N^1 *and* τ_N^2 *be given by* (e) *of Proposition* 12. *Then*

$$\tau_N^2 - \tau_N^1 \leqslant \frac{\bar{c}+1}{(1+\underline{\delta}-\delta^*)V_\eta}, \tag{16}$$

where $V_\eta = \min\{V(s), s \in [\eta, 2\pi - \eta]\}$;

$$\eta \frac{e^{\sqrt{\bar{a}}t} - e^{-\sqrt{\bar{a}}t}}{e^{\sqrt{\bar{a}}\tau_N^1} - e^{-\sqrt{\bar{a}}\tau_N^1}} \leqslant q_N(t) \leqslant \eta \frac{e^{\sqrt{\underline{a}}t} - e^{-\sqrt{\underline{a}}t}}{e^{\sqrt{\underline{a}}\tau_N^1} - e^{-\sqrt{\underline{a}}\tau_N^1}} \equiv y(t)$$

for all $t \in [0, \tau_N^1]$, *where* $\bar{a} = 2\mu(1+\bar{\delta})$ *and* $\underline{a} = \frac{\mu}{2}(1+\underline{\delta})$.

Proof. From

$$\bar{c} + 1 \geqslant \int_{\tau_N^1}^{\tau_N^2} (1+\underline{\delta}-\delta^*)V(q_N) \geqslant (1+\underline{\delta}-\delta^*)V_\eta(\tau_N^2 - \tau_N^1)$$

we immediately deduce that (16) holds.

Recall that $V'(q) \geqslant \frac{\mu}{2} q$ for all $0 \leqslant q \leqslant \eta_0$. Let y be the solution of

$$\begin{cases} \ddot{y} - \underline{a}y = 0, \\ y(0) = 0, \\ y(\tau_N^1) = q(\tau_N^1) = \eta, \end{cases}$$

that is

$$y(t) = \eta \frac{e^{\sqrt{\underline{a}}t} - e^{-\sqrt{\underline{a}}t}}{e^{\sqrt{\underline{a}}\tau_N^1} - e^{-\sqrt{\underline{a}}\tau_N^1}}.$$

Then, for all $t \in [0, \tau_N^1]$

$$\begin{aligned} -(\ddot{y} - \ddot{q}_N) + \underline{a}(y - q_N) &= \ddot{q}_N - \underline{a}q_N \\ &= (1+\delta(x_N(t)))V'(q_N(t)) - \underline{a}q_N(t) \\ &\geqslant \underline{a}q_N(t) - \underline{a}q_N(t) = 0. \end{aligned}$$

By maximum principle, $y(t) - q_N(t) \geqslant 0$, that is

$$0 \leqslant q_N(t) \leqslant y(t) \quad \forall t \in [0, \tau_N^1].$$

Similarly, if $z(t) = \eta \frac{e^{\sqrt{\bar{a}}t} - e^{-\sqrt{\bar{a}}t}}{e^{\sqrt{\bar{a}}\tau_N^1} - e^{-\sqrt{\bar{a}}\tau_N^1}}$, then

$$-(\ddot{q}_N - \ddot{z}) + \bar{a}(q_N - z) \geqslant 0$$

and we deduce that

$$0 \leqslant z(t) \leqslant q_N(t) \quad \forall t \in [0, \tau_N^1]. \qquad \square$$

Remark 14. Clearly we also have for all $t \in [\tau_N^2, T_N]$

$$\begin{aligned}\tilde{z}(t) &\equiv \eta \frac{e^{\sqrt{\bar{a}}(T_N - t)} - e^{-\sqrt{\bar{a}}(T_N - t)}}{e^{\sqrt{\bar{a}}(T_N - \tau_N^2)} - e^{-\sqrt{\bar{a}}(T_N - \tau_N^2)}} \leqslant 2\pi - q_N(t)\\ &\leqslant \eta \frac{e^{\sqrt{\underline{a}}(T_N - t)} - e^{-\sqrt{\underline{a}}(T_N - t)}}{e^{\sqrt{\underline{a}}(T_N - \tau_N^2)} - e^{-\sqrt{\underline{a}}(T_N - \tau_N^2)}} \equiv \tilde{y}(t).\end{aligned}$$

Lemma 15. *Under the same assumptions as in Lemma* 13 *there exists K such that*

$$\frac{1}{K}\tau_N^1 \leqslant T_N - \tau_N^2 \leqslant K\tau_N^1$$

Proof. Since $T_N \to +\infty$, as a consequence of Lemma 13, either $\tau_N^1 \to +\infty$ or $T_N - \tau_N^2 \to +\infty$. We want to show that both diverge. Suppose $\tau_N^1 \to +\infty$. Then, using the results of Lemma 13 and Remark 14 we deduce that,

$$\begin{aligned}\dot{q}(0) &= \lim_{t \to 0^+} \frac{q(t) - q(0)}{t} = \lim_{t \to 0^+} \frac{q(t)}{t}\\ &\leqslant \lim_{t \to 0^+} \frac{y(t)}{t} = \dot{y}(0)\\ &= \frac{2\eta\sqrt{\underline{a}}}{e^{\sqrt{\underline{a}}\tau_N^1} - e^{-\sqrt{\underline{a}}\tau_N^1}}.\end{aligned}$$

As a consequence

$$\begin{aligned}\dot{q}(0) = \dot{q}(T_N) &= \lim_{t \to 0^+} \frac{q(T_N) - q(T_N - t)}{t}\\ &= \lim_{t \to 0^+} \frac{2\pi - q(T_N - t)}{t} \geqslant \lim_{t \to 0^+} \frac{\tilde{z}(T_N - t)}{t}\\ &= -\dot{\tilde{z}}(T_N) = \frac{2\eta\sqrt{\bar{a}}}{e^{\sqrt{\bar{a}}(T_N - \tau_N^2)} - e^{-\sqrt{\bar{a}}(T_N - \tau_N^2)}}.\end{aligned}$$

Using the above estimates, we get

$$\begin{aligned}&\frac{2\eta\sqrt{\underline{a}}}{e^{\sqrt{\underline{a}}\tau_N^1} - e^{-\sqrt{\underline{a}}\tau_N^1}} \geqslant \frac{2\eta\sqrt{\bar{a}}}{e^{\sqrt{\bar{a}}(T_N - \tau_N^2)} - e^{-\sqrt{\bar{a}}(T_N - \tau_N^2)}},\\ &\sinh(\sqrt{\bar{a}}(T_N - \tau_N^2)) \geqslant \sqrt{\frac{\bar{a}}{\underline{a}}}\sinh(\sqrt{\underline{a}}\tau_N^1) \geqslant \sinh(\sqrt{\underline{a}}\tau_N^1).\end{aligned}$$

The conclusion follows by monotonicity of the function sinh. $\square$

Theorem 16. *Assume V and δ satisfy assumptions* (V1)–(V4), *and* (δ1)–(δ3). *Choose* $[\alpha, \beta] \subset (0, 2\pi)$ *and let* (x_N, q_N) *be as in Proposition* 12.

Then for all $N \in \mathbb{N}$ there is $\tau_N \in [\tau_N^1, \tau_N^2]$ such that, up to a subsequence,

$$q_N(\cdot - \tau_N) \to \bar{q}, \quad x_N(\cdot - \tau_N) \to \bar{x},$$

where $(\bar{x}, \bar{q})$ *is a solution of problem* (1) *satisfying* (2).

Proof. We know that

$$0 \leqslant q_N(t) \leqslant 2\pi, \quad \int_0^{T_N} \dot{q}_N^2 \leqslant 2(\bar{c} + 1),$$

hence $q_N \in H^1_{loc}(\mathbb{R}, \mathbb{R})$. Let us fix $\eta \leqslant \eta_0$ and sufficiently small. Then we can find τ_N^1, τ_N^2 such that

$$\begin{aligned}
&|q_N(\tau_N^1)| = \eta, \\
&|q_N(t)| < \eta \quad \forall t \in [0, \tau_N^1], \\
&|q_N(t) - 2\pi| < \eta \quad \forall t \in [\tau_N^2, T_N], \\
&|\tau_N^2 - \tau_N^1| \leqslant \frac{\bar{c} + 1}{(1 + \underset{\sim}{\delta} - \delta^*) V_\eta}.
\end{aligned}$$

Let τ_N be the τ_N^0 corresponding to η_0, and

$$\tilde{q}_N(t) = q_N(t + \tau_N), \quad t \in [-\tau_N, T_N - \tau_N].$$

Then $\tilde{q}_N(0) = \eta_0$ for all N and

$$\begin{aligned}
&\int_{-\tau_N}^{T_N - \tau_N} |\dot{\tilde{q}}_N|^2 \leqslant 2(\bar{c} + 1), \\
&\tilde{q}_N(t) \in [0, 2\pi] \quad \forall t \in [-\tau_N, T_N - \tau_N].
\end{aligned}$$

We also have that

$$\tau_N^1 \leqslant \tau_N \leqslant \tau_N^2.$$

Fix $a < b \in \mathbb{R}$. Since $-\tau_N \to -\infty$, $T_N - \tau_N \to +\infty$, we have that $\tilde{q}_N \in H^1(a, b)$ for all N large and $||\tilde{q}_N||^2_{H^1(a,b)} \leqslant 2(\bar{c} + 1) + (b - a)4\pi^2$ so that, up to a subsequence,

$$\begin{aligned}
&\tilde{q}_N \rightharpoonup q \in H^1(a, b), \\
&\tilde{q}_N \to q \quad \text{uniformly in } [a, b].
\end{aligned}$$

Then $||q||_\infty \leqslant 2\pi$ and

$$\int_{\mathbb{R}} |\dot{q}|^2 = \sup_{a<b} \int_a^b |\dot{q}|^2 \leqslant \sup_{a<b} \liminf_{N\to\infty} \int_a^b |\dot{\tilde{q}}_N|^2 \leqslant 2(\bar{c}+1),$$

$$\int_{\mathbb{R}} V(q) = \sup_{a<b} \int_a^b V(q) \leqslant \sup_{a<b} \liminf_{N\to\infty} \int_a^b V(\tilde{q}_N) \leqslant \frac{\bar{c}+1}{1+\underline{\delta}-\delta^*}.$$

Moreover,

$$|q(t)| \leqslant \eta \quad \forall t < -\frac{\bar{c}+1}{(1+\underline{\delta}-\delta^*)V_\eta} < \tau_N^1 - \tau_N^2 < \tau_N^1 - \tau_N,$$

$$|q(t)-2\pi| \leqslant \eta \quad \forall t > \frac{\bar{c}+1}{(1+\underline{\delta}-\delta^*)V_\eta} \geqslant \tau_N^2 - \tau_N^1 \geqslant \tau_N^2 - \tau_N.$$

We immediately deduce that

$$q(t) \to 0 \text{ as } t \to -\infty, \quad q(t) \to 2\pi \text{ as } t \to +\infty.$$

Since $(\tilde{x}_N, \tilde{q}_N)$ is a solution we have that for some constant $E_N > 0$ and for all $t \in [-\tau_N, T_N - \tau_N]$

$$E_N = \frac{\dot{\tilde{q}}_N^2(t)}{2} - (1+\delta(\tilde{x}_N(t)))V(\tilde{q}_N(t)) + \frac{\dot{\tilde{x}}_N^2(t) + \tilde{x}_N^2(t)}{2}.$$

We deduce

$$\frac{\dot{\tilde{x}}_N^2(t) - \tilde{x}_N^2(t)}{2} + \frac{\dot{\tilde{q}}_N^2(t)}{2} + (1+\delta(\tilde{x}_N(t)))V(\tilde{q}_N(t)) = \dot{\tilde{x}}_N^2(t) + \dot{\tilde{q}}_N^2(t) - E_N.$$

Since

$$T_N c(T_N) = \int_{-\tau_N}^{T_N-\tau_N} \left[\frac{\dot{\tilde{x}}_N^2 - \tilde{x}_N^2}{2} + \frac{\dot{\tilde{q}}_N^2}{2} + (1+\delta(\tilde{x}_N))V(\tilde{q}_N)\right]$$

$$= \int_{-\tau_N}^{T_N-\tau_N} (\dot{\tilde{x}}_N^2 + \dot{\tilde{q}}_N^2) - E_N T_N,$$

we have that

$$E_N = \frac{1}{T_N}\int_0^{T_N} (\dot{\tilde{x}}_N^2 + \dot{\tilde{q}}_N^2) - c(T_N)$$

is bounded by (d) and $0 \leqslant c(T_N)T_N \leqslant \bar{c}$. This immediately implies that

$$\int_a^b \frac{\dot{\tilde{x}}_N^2 + \tilde{x}_N^2}{2} = (b-a)E_N - \int_a^b \frac{\dot{\tilde{q}}_N^2}{2} + \int_a^b (1+\delta(\tilde{x}_N))V(\tilde{q}_N)$$
$$\leqslant (b-a)E_N + \frac{(\bar{c}+1)(1+\bar{\delta})}{1+\underline{\delta}-\delta^*}$$

so that $||\tilde{x}_N||^2_{H^1(a,b)}$ is bounded. This implies that $\tilde{x}_N \rightharpoonup x$ in $H^1(a,b)$ and $\tilde{x}_N \to x$ uniformly. We can now pass to the limit in the equations

$$\ddot{\tilde{q}}_N = (1+\delta(\tilde{x}_N))V'(\tilde{q}_N),$$
$$\ddot{\tilde{x}}_N + \tilde{x}_N = \delta'(\tilde{x}_N)V(\tilde{q}_N),$$

to deduce that (x,q) is a solution of

$$\ddot{q} = (1+\delta(x))V'(q),$$
$$\ddot{x} + x = \delta'(x)V(q),$$

in the interval $[a,b]$ and hence also in $\mathbb{R}$. From what we have seen above we also have that

$$\lim_{t\to+\infty} q(t) = 2\pi \qquad \lim_{t\to-\infty} q(t) = 0;$$

which also implies that $\dot{q}(t) \to 0$ for $t \to \pm\infty$.

Let E_∞ be the energy of such a solution. We know that $E_\infty \geqslant 0$. To show that there exists $R \geqslant 0$ and φ_1, $\varphi_2 \in [0,2\pi)$ such that

$$\lim_{t\to-\infty} |x(t) - R\cos(t+\varphi_1)| = 0,$$

we simply remark that, for $k \to +\infty$, $x_k(t) = x(\cdot - 2\pi k)$ converges uniformly on compact set to a solution of

$$\ddot{x} + x = 0,$$
$$\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^2 = E_\infty.$$

Since the solutions of such a limit problem are

$$y(t) = \sqrt{E_\infty}\cos(t+\varphi_1),$$

one can find φ_1, $\varphi_2 \in [0,2\pi)$ and, for all $\varepsilon > 0$, t_0, $t_1 \in \mathbb{R}$ such that

$$|x(t) - \sqrt{E_\infty}\cos(t+\varphi_1)| < \varepsilon \quad \text{for all } t \leqslant t_0,$$

$$|x(t) - \sqrt{E_\infty}\cos(t+\varphi_2)| < \varepsilon \quad \text{for all } t \geqslant t_1. \qquad \square \tag{17}$$

Theorem 17. *Assume V and δ satisfy assumptions* (V1)–(V4), *and* (δ1)–(δ3).
Suppose, moreover, that system (1) *has no solutions satisfying*

$$\lim_{t\to\pm\infty} |x(t)| = 0,$$
$$\lim_{t\to-\infty} q(t) = 0,$$
$$\lim_{t\to+\infty} q(t) = 2\pi, \tag{18}$$

that is having zero energy.

Choose $[\alpha,\beta]\subset(0,2\pi)$ and let (x_N,q_N) and $\varphi_N\in[\alpha,\beta]$ be as in Proposition 12. *By Theorem* 16 *there is a subsequence such that $(\tilde{x}_N,\tilde{q}_N)=(q_N(\cdot-\tau_N),x_N(\cdot-\tau_N))$ converges to a solution of* (1) *satisfying, for suitable φ_1 and φ_2, the boundary condition* (2).

Let, up to subsequence, $\varphi=\lim_{N\to+\infty}\varphi_N\in[\alpha,\beta]$. Then

$$\varphi_2-\varphi_1\equiv\varphi\ (\mathrm{mod}\ 2\pi).$$

Proof. Fix $\varepsilon>0$. Let $\eta\leqslant\eta_0$ be such that

$$||\delta'||_\infty\frac{4\mu\eta^2}{\sqrt{\underline{a}}}<\frac{\varepsilon}{4}$$

and consider the corresponding τ_N^1, τ_N^2. By Theorem 16 (see in particular (17)) we can find $t_0<0$ and $t_1>\tau_N^2-\tau_N^1$ such that

$$|x(t)-\sqrt{E_\infty}\cos(t+\varphi_1)|<\varepsilon \quad \text{for all } t\leqslant t_0,$$
$$|x(t)-\sqrt{E_\infty}\cos(t+\varphi_2)|<\varepsilon \quad \text{for all } t\geqslant t_1,$$
$$t_0+\varphi_1\in 2\pi\mathbb{Z}. \tag{19}$$

Choose N_0 such that for all $N\geqslant N_0$ one has

$$|x(t)-\tilde{x}_N(t)|<\varepsilon \quad \text{for all } t\in[t_0-2\pi,t_0+2\pi],$$
$$|x(t)-\tilde{x}_N(t)|<\varepsilon \quad \text{for all } t\in[t_1-2\pi,t_1+2\pi]. \tag{20}$$

Let k_1 be such that $2\pi(k_1+1)\leqslant\tau_N^1$ and $\theta\in[0,2\pi)$. Then, using the exponential estimates of q (see Lemma 13), we have

$$\begin{aligned}|-x_N(2\pi k_1+\theta)+x_N(\theta)| &= \left|\int_\theta^{2\pi k_1+\theta}(\ddot{x}_N(t)+x_N(t))\sin(t-\theta)\,dt\right|\\ &= \left|\int_\theta^{2\pi k_1+\theta}\delta'(x_N(t))V(q_N(t))\sin(t-\theta)\,dt\right|\end{aligned}$$

$$\leqslant \|\delta'\|_\infty \int_\theta^{2\pi k_1+\theta} V(q_N(t))\,dt$$

$$\leqslant \|\delta'\|_\infty \frac{4\mu\eta^2}{\sqrt{\underline{a}}} < \frac{\varepsilon}{4} \tag{21a}$$

and

$$|\dot{x}_N(2\pi k_1 + \theta) - \dot{x}_N(\theta)| \leqslant \|\delta'\|_\infty \frac{4\mu\eta^2}{\sqrt{\underline{a}}} < \frac{\varepsilon}{4}. \tag{21b}$$

Similarly, if $2\pi k_2 \leqslant T_N - \tau_N^2$ and $\theta \in [0, 2\pi]$

$$|x_N(T_N - 2\pi + \theta) - x_N(T_N + \theta - 2\pi k_2)| \leqslant \|\delta'\|_\infty \frac{4\mu\eta^2}{\sqrt{\underline{a}}} < \frac{\varepsilon}{4}, \tag{21c}$$

$$|\dot{x}_N(T_N - 2\pi + \theta) - \dot{x}_N(T_N + \theta - 2\pi k_2)| \leqslant \|\delta'\|_\infty \frac{4\mu\eta^2}{\sqrt{\underline{a}}} < \frac{\varepsilon}{4}. \tag{21d}$$

We also recall that

(1) $x_N(t)$ solves $\ddot{x}_N + x_N = \delta'(x_N)V(q_N)$;
(2) $q_N(t)$ is exponentially small in $[0, 2\pi]$;
(3) $q_N(t) - 2\pi$ is exponentially small in $[T_N - 2\pi, T_N]$;
(4) $\frac{1}{2}(\dot{x}_N^2(t) + x_N^2(t)) + \frac{1}{2}\dot{q}_N^2(t) - (1 + \delta(x_N(t)))V(q_N(t)) = E_N$ for some $E_N > 0$ and for all $t \in [0, T_N]$;
(5) $x_N(0) = x_N(T_N)$, $\dot{x}_N(0) = \dot{x}_N(T_N)$.

Then there is φ_1^N such that for all $\varepsilon > 0$ we can find $N_1 \geqslant N_0$ such that, for all $N \geqslant N_1$

$$\sup_{\theta \in [0,2\pi]} |x_N(\theta) - \sqrt{E_N}\cos(\theta + \varphi_1^N)| < \varepsilon/4,$$

$$\sup_{\theta \in [T_N - 2\pi, T_N]} |x_N(\theta) - \sqrt{E_N}\cos(\theta + \varphi_1^N)| < \varepsilon/4.$$

As a consequence,

$$\sup_{\theta \in [0,2\pi]} |x_N(\theta) - x_N(T_N - 2\pi + \theta)| < \varepsilon/2 \quad \forall N \geqslant N_1. \tag{22}$$

Using (21) and (22) we deduce that for all $N \geqslant N_1$, for all $k_1 \leqslant \tau_N^1/(2\pi) - 1$ and $k_2 \leqslant (T_N - \tau_N^2)/(2\pi)$

$$\sup_{\theta \in [0,2\pi]} |x_N(\theta + 2\pi k_1) - x_N(T_N - 2\pi k_2 + \theta)| < \varepsilon$$

i.e.

$$\sup_{\theta \in [0,2\pi]} |\tilde{x}_N(\theta + 2\pi k_1 - \tau_N^1) - \tilde{x}_N(T_N - 2\pi k_2 + \theta - \tau_N^1)| < \varepsilon. \tag{23}$$

Since $t_0 < 0$ and $t_1 > \tau_N^2 - \tau_N^1$ we can write $t_0 = \theta_1 + 2\pi k_1 - \tau_N^1$ with $\theta_1 \in [0, 2\pi)$, $k_1 \in \mathbb{N}$, $k_1 \leqslant \tau_N^1/(2\pi) - 1$ and $t_1 = T_N - 2\pi k_2 + \theta_2 - \tau_N^1$ with $\theta_2 \in (0, 2\pi]$, $k_2 \in \mathbb{N}$, $k_2 \leqslant (T_N - \tau_N^2)/(2\pi)$. Then

$$T_N - 2\pi k_2 + \theta_1 - \tau_N^1 = t_1 - \theta_2 + \theta_1,$$

so that, using (23)

$$|\tilde{x}_N(t_0) - \tilde{x}_N(t_1 - \theta_2 + \theta_1)| < \varepsilon$$

and (using (19) and (20))

$$|\sqrt{E_\infty}\cos(t_0 + \varphi_1) - \sqrt{E_\infty}\cos(t_1 - \theta_2 + \theta_1 + \varphi_2)| < 5\varepsilon. \tag{24}$$

Recall that $\theta_1 + 2\pi k_1 - \tau_N^1 + \varphi_1 = t_0 + \varphi_1 = 2\pi\ell$ (see (19)) and that $t_1 = T_N - 2\pi k_2 + \theta_2 - \tau_N^1$, $k_2 \leqslant (T_N - \tau_N^2)/2\pi$, $\theta_2 \in (0, 2\pi]$, so that

$$\begin{aligned} t_1 - \theta_2 + \theta_1 + \varphi_2 &= T_N - 2\pi k_2 + \theta_2 - \tau_N^1 - \theta_2 \\ &\quad + (-\varphi_1 + 2\pi\ell - 2\pi k_1 + \tau_N^1) + \varphi_2 \\ &= 2\pi N + \varphi_N - 2\pi k_2 - \varphi_1 + \varphi_2 + 2\pi\ell - 2\pi k_1 \end{aligned}$$

and hence

$$\cos(t_1 - \theta_2 + \theta_1 + \varphi_2) = \cos(\varphi_N - \varphi_1 + \varphi_2).$$

From this and (24) we deduce that

$$|1 - \cos(\varphi_N - (\varphi_1 - \varphi_2))| < \frac{5\varepsilon}{\sqrt{E_\infty}}$$

so that, passing to the limit we have that

$$|1 - \cos(\varphi - (\varphi_1 - \varphi_2))| < \frac{5\varepsilon}{\sqrt{E_\infty}}$$

and since ε can be chosen arbitrarily small, the theorem follows. □

Corollary 18. *Under the same assumptions as in Theorem* 17 *there are infinitely many solutions of* (1) *satisfying condition* (2).

Proof. An application of Theorem 17 with $[\alpha, \beta] \subset (0, 2\pi)$ gives a solution (x, q) characterized by a phase shift $\varphi \in [\alpha, \beta]$. Choose $[\alpha_1, \beta_1] \subset (0, 2\pi)\backslash\{\varphi\}$. Then applying again Theorem 17 gives a second solution, different from the first one. A repeated application of the theorem gives the result. □

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