Linear Algebra and its Applications 431 (2009) 1223–1233

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

On incidence energy of a graph

Ivan Gutman [a,*], Dariush Kiani [b,c], Maryam Mirzakhah [b], Bo Zhou [d]

[a] *Faculty of Science, University of Kragujevac, P.O. Box 60, 34000 Kragujevac, Serbia*
[b] *Faculty of Mathematics and Computer Science, Amirkabir University of Technology, P.O. Box 15875-4413, Tehran, Iran*
[c] *School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran*
[d] *Department of Mathematics, South China Normal University, Guangzhou 510631, PR China*

ARTICLE INFO

Article history:
Received 26 March 2009
Accepted 21 April 2009
Available online 20 May 2009

Submitted by R.A. Brualdi

AMS classification:
05C50
05C90

Keywords:
Graph spectrum
Incidence energy (of graph)
Singular value (of matrix)
Incidence matrix
Laplacian matrix (of graph)
Signless Laplacian matrix (of graph)

ABSTRACT

The Laplacian-energy like invariant $LEL(G)$ and the incidence energy $IE(G)$ of a graph are recently proposed quantities, equal, respectively, to the sum of the square roots of the Laplacian eigenvalues, and the sum of the singular values of the incidence matrix of the graph G. However, $IE(G)$ is closely related with the eigenvalues of the Laplacian and signless Laplacian matrices of G. For bipartite graphs, $IE = LEL$. We now point out some further relations for IE and LEL: IE can be expressed in terms of eigenvalues of the line graph, whereas LEL in terms of singular values of the incidence matrix of a directed graph. Several lower and upper bounds for IE are obtained, including those that pertain to the line graph of G. In addition, Nordhaus–Gaddum-type results for IE are established.

1. Introduction

Let G be a simple graph on n vertices. The eigenvalues of G are the eigenvalues of its adjacency matrix $\mathbf{A}(G)$ [2]. These eigenvalues, arranged in a non-increasing order, will be denoted as $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$. Then the *energy* of the graph G is defined as

* Corresponding author. Fax: +381 34 335040.
E-mail addresses: gutman@kg.ac.rs (I. Gutman), dkiani@aut.ac.ir (D. Kiani), mirzakhah@aut.ac.ir (M. Mirzakhah), zhoubo@scnu.edu.cn (B. Zhou).

0024-3795/$ - see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2009.04.019

$$E(G) = \sum_{i=1}^{n} |\lambda_i(G)|.$$

Various properties of graph energy may be found in [4,5,8].

The concept of graph energy was extended to any matrix by Nikiforov [16] in the following manner. The singular values of a real (not necessarily square) matrix $\mathbf{M}$ are the square roots of the eigenvalues of the (square) matrix $\mathbf{M}\,\mathbf{M}^t$, where $\mathbf{M}^t$ denotes the transpose of $\mathbf{M}$. The energy $E(\mathbf{M})$ of the matrix $\mathbf{M}$ is then defined [16] as the sum of its singular values. Obviously, $E(G) = E(\mathbf{A}(G))$.

Let $\mathbf{I}(G)$ be the (vertex–edge) incidence matrix of the graph G. For a graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{e_1, e_2, \ldots, e_m\}$, the (i,j)-entry of $\mathbf{I}(G)$ is 1 if v_i is incident with e_j and 0 otherwise. (In what follows, the unit matrix of order p will be denoted by $\mathbf{I}_p$, and it should not be confused with the incidence matrix.)

Motivated by Nikiforov's idea, Jooyandeh et al. [10] introduced the concept of *incidence energy* $IE(G)$ of a graph G, defining it as the sum of the singular values of the incidence matrix $\mathbf{I}(G)$. Some basic properties of this quantity were established in [7,10].

If the singular values of $\mathbf{I}(G)$ are $\sigma_1, \sigma_2, \ldots, \sigma_n$, then, by definition [10],

$$IE(G) := \sum_{i=1}^{n} \sigma_i.$$

Let $\mathbf{D}(G)$ be the diagonal matrix of order n whose (i,i)-entry is the degree of the vertex v_i of the graph G. Then the matrix $\mathbf{L}(G) = \mathbf{D}(G) - \mathbf{A}(G)$ is the *Laplacian matrix* of the graph G, for details see [13,14]. The matrix $\mathbf{L}^+(G) = \mathbf{D}(G) + \mathbf{A}(G)$ is the *signless Laplacian matrix*, for details see [3].

Denote by $\mu_1, \mu_2, \ldots, \mu_n$ the eigenvalues of the Laplacian matrix $\mathbf{L}(G)$ and by $\mu_1^+, \mu_2^+, \ldots, \mu_n^+$ the eigenvalues of the signless Laplacian matrix $\mathbf{L}^+(G)$. All eigenvalues of both $\mathbf{L}(G)$ and $\mathbf{L}^+(G)$ are real and non-negative. In what follows it is assumed that both eigenvalues are arranged in a non-increasing order.

If the graph G is connected, then $\mu_i > 0$ for $i = 1, 2, \ldots, n-1$ and $\mu_n = 0$ [13,14]. If G is a connected non-bipartite graph, then $\mu_i^+ > 0$ for $i = 1, 2, \ldots, n$ [3].

The following result is well known [3,13,14]:

Lemma 1.1. *The spectra of* $\mathbf{L}(G)$ *and* $\mathbf{L}^+(G)$ *coincide if and only if the graph G is bipartite.*

Another well known fact is the identity [13,14]:

$$\mathbf{I}(G)\,\mathbf{I}(G)^t = \mathbf{A}(G) + \mathbf{D}(G) \quad \text{i.e.,} \quad \mathbf{I}(G)\,\mathbf{I}(G)^t = \mathbf{L}^+(G). \tag{1}$$

Its immediate consequence is that $\sigma_i = \sqrt{\mu_i^+}$ and therefore,

$$IE(G) = \sum_{i=1}^{n} \sqrt{\mu_i^+}. \tag{2}$$

Short time ago, Liu and Liu [11] introduced the so-called *Laplacian-energy like invariant*, $LEL(G)$ of a graph G, as the sum of the square roots of the eigenvalues of the Laplacian matrix of G, i.e.,

$$LEL(G) := \sum_{i=1}^{n} \sqrt{\mu_i}. \tag{3}$$

In [11] and in the subsequent papers [12,19,20] a number of properties of LEL were established.

Comparing Eqs. (2) and (3), we see that there is an intimate relation between incidence energy and the Laplacian-energy like invariant. In particular, in view of Lemma 1.1, if the graph G is bipartite, then

$$IE(G) = LEL(G). \tag{4}$$

2. Some more relations for *IE* and *LEL*

In addition to the identity (1), for the incidence matrix of a graph G with n vertices and $m \geqslant 1$ edges we have

$$\mathbf{I}(G)^t\,\mathbf{I}(\mathbf{G}) = 2\,\mathbf{I}_m + \mathbf{A}(\mathcal{L}(G)),$$

where $\mathcal{L}(G)$ is the line graph of G, and where $\mathbf{I}_m$ stands for the unit matrix of order m. From this identity we immediately get

$$IE(G) = \sum_{i=1}^{m} \sqrt{2 + \lambda_i(\mathcal{L}(G))}. \tag{5}$$

The following result holds for any graph G [13,14]. If any edge of G is given an orientation (in arbitrary direction), then an oriented graph $\overrightarrow{G}$ is obtained. The (i,j)-entry of the incidence matrix $\mathbf{I}(\overrightarrow{G})$ of $\overrightarrow{G}$ is +1 if the vertex v_i is the head of the oriented edge e_j, -1 if the vertex v_i is the tail of the oriented edge e_j, and 0 otherwise. Then, no matter how the edges are oriented, $\mathbf{I}(\overrightarrow{G})$ satisfies the identity

$$\mathbf{I}(\overrightarrow{G})\,\mathbf{I}(\overrightarrow{G})^t = \mathbf{D}(G) - \mathbf{A}(G) \quad \text{that is } \mathbf{I}(\overrightarrow{G})\,\mathbf{I}(\overrightarrow{G})^t = \mathbf{L}(G).$$

As a consequence of the above identity, the energy of the the matrix $\mathbf{I}(\overrightarrow{G})$ is equal to the sum of the square roots of the ordinary Laplacian eigenvalues of the (undirected) graph G, i.e.,

$$IE(\overrightarrow{G}) = \sum_{i=1}^{n} \sqrt{\mu_i},$$

i.e.,

$$LEL(G) = IE(\overrightarrow{G}). \tag{6}$$

Eq. (6) provides a new interpretation of the Laplacian–energy like invariant of Liu and Liu [11], and offers a new insight into its possible physical or chemical meaning.

As a final relation for the incidence energy we mention:

$$IE(G) = \frac{1}{2}\,E(S(G)), \tag{7}$$

where $S(G)$ be the subdivision graph of the graph G, obtained by inserting an additional vertex into each edge of G. Eq. (7) was first reported in [10], and represents a direct extension of a result from [25].

3. Bounds for incidence energy

In [10] the following fundamental properties of the incidence energy were established:

Theorem 3.1. *(i)* $IE(G) \geqslant 0$, *and equality holds if and only if* $m = 0$.

(ii) If the graph G *has components* $G_1, \ldots, G_p$, *then* $IE(G) = \sum_{i=1}^{p} IE(G_i)$.

(iii) Let G *be a graph of order* n *with* m *edges. Then* $\sqrt{2m} \leqslant IE(G) \leqslant \sqrt{2mn}$. *The left equality holds if and only if* $m \leqslant 1$, *whereas the right equality holds if and only if* $m = 0$.

A similar upper bound is known for *LEL* [11], namely $LEL(G) \leqslant \sqrt{2m(n-p)}$, where p is the number of components of the graph G. In the general case, *IE* does not satisfy the analogous inequality. For instance, the inequality $IE(G) \leqslant \sqrt{2m(n-p)}$ is violated by the complete graph K_n if $n \geqslant 3$. On the other hand, in view of Eq. (4), it is satisfied by all bipartite graphs.

Theorem 3.2. *Let* G *be a graph with* m *edges. Then*

$$IE(G) \leqslant \sqrt{2}\,m$$

with equality if and only if G *consists of* m *copies of* K_2 *and possibly isolated vertices.*

Proof. The case $m = 0$ is trivial. Suppose that $m \geqslant 1$. By Eq. (5) and the Cauchy–Schwarz inequality,

$$IE(G) = \sum_{i=1}^{m} \sqrt{2 + \lambda_i(\mathcal{L}(G))} \leqslant \sqrt{m \sum_{i=1}^{m} [2 + \lambda_i(\mathcal{L}(G))]} = \sqrt{m \sum_{i=1}^{m} 2} = \sqrt{2}\, m,$$

with equality if and only if $2 + \lambda_i(\mathcal{L}(G))$ is a constant for all i, i.e., $\mathcal{L}(G) = \overline{K_m}$, i.e., G consists of m copies of K_2 and possibly isolated vertices. $\square$

Let P_n be the path with n vertices and K_n the complete graph with n vertices. Let $K_{r,s}$ be the bipartite graph with r and s vertices in its two partite sets. Let $\overline{G}$ be the complement of the graph G.

Note that

$$IE(K_n) = \sqrt{2n-2} + (n-1)\sqrt{n-2} \geqslant IE(\overrightarrow{K_n}) = (n-1)\sqrt{n},$$

with equality if and only if $n = 1, 2$.

For a graph G with at least one edge, $G - e$ denotes the graph obtained from G by deleting the edge e of G. In [10] it was shown that $IE(G - e) < IE(G)$. This immediately implies that for a graph G with n vertices,

$$0 \leqslant IE(G) \leqslant \sqrt{2(n-1)} + (n-1)\sqrt{n-2},$$

with left (resp. right) equality if and only if $G \cong \overline{K_n}$ (resp. $G \cong K_n$). Moreover, if G is a bipartite graph with r and s vertices in its two partite sets, then

$$0 \leqslant IE(G) \leqslant \sqrt{r+s} + (r-1)\sqrt{s} + (s-1)\sqrt{r},$$

with left (resp. right) equality if and only if $G \cong \overline{K_n}$ (resp. $G \cong K_{r,s}$). In particular, if G is a bipartite graph with $n \geqslant 2$ vertices, then as a function of r with $1 \leqslant r \leqslant \lfloor n/2 \rfloor$, $(r-1)\sqrt{n-r} + (n-r-1)\sqrt{r}$ is increasing for r, and thus

$$IE(G) \leqslant \sqrt{n} + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\sqrt{\left\lceil \frac{n}{2} \right\rceil} + \left(\left\lceil \frac{n}{2} \right\rceil - 1\right)\sqrt{\left\lfloor \frac{n}{2} \right\rfloor},$$

with equality if and only if $G \cong K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

In what follows we establish further bounds for the incidence energy. First we consider upper bounds.

Theorem 3.3. *Let G be a connected graph with n vertices and m edges. Then*

$$IE(G) \leqslant IE(P_n) + \sqrt{2}(m - n + 1),$$

with equality if and only if $G \cong P_n$.

Proof. In [10], it was shown that $IE(G) \leqslant IE(G - e) + \sqrt{2}$ for any edge e of G. A repeated application of this inequality yields

$$IE(G) \leqslant IE(T) + \sqrt{2}(m - n + 1)$$

for a spanning tree T of G. In [7] it was shown that $IE(T) \leqslant IE(P_n)$, with equality if and only if $T \cong P_n$. This implies the result. $\square$

Denote by d_i the degree of the vertex v_i of the graph G. We now introduce an auxiliary quantity α, defined as

$$\alpha = 2\sqrt{\frac{1}{n} \sum_{i=1}^{n} d_i^2}.$$

Recall that the so-called *first Zagreb index* $Zg(G)$ of a graph G is defined as the sum of squares of vertex degrees of the graph G. This quantity found many applications in chemistry [6,21]. Thus, in the above notation, $Zg = n\,\alpha^2/4$.

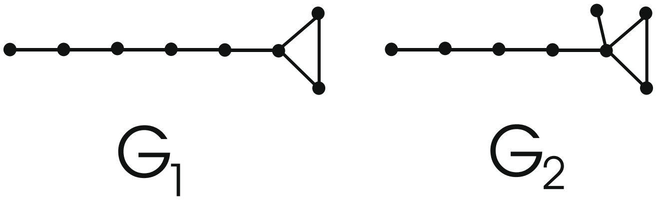

Fig. 1. Examples showing that inequalities (8) and (10) are incompatible.

In order to demonstrate the validity of the next Theorem 3.5 we need:

Lemma 3.4 [9]. *Let G be a connected graph on n vertices, with vertex degrees* $d_1, d_2, \ldots, d_n$. *Then* $\mu_1^+ \geqslant \alpha$, *with equality if and only if G is a connected regular graph.*

Theorem 3.5. *Let G be a connected graph of order n*, $n \geqslant 3$, *with m edges. Then*

$$IE(G) \leqslant \sqrt{\alpha} + \sqrt{(n-1)(2m-\alpha)}. \tag{8}$$

Moreover, equality holds if and only if $G \cong K_n$.

Proof. We first observe that $\sum_{i=2}^{n} \mu_i^+ = 2m - \mu_1^+$. By using the Cauchy–Schwarz inequality we obtain from Eq. (2)

$$IE(G) \leqslant \sqrt{\mu_1^+} + \sqrt{(n-1)(2m-\mu_1^+)}. \tag{9}$$

Inequality (8) follows from the fact that the function $f(x) = \sqrt{x} + \sqrt{(n-1)(2m-x)}$ decreases on $x > 2m/n$, and that by Lemma 3.4 and by using the Cauchy–Schwarz inequality, $2m/n < 4m/n \leqslant \alpha \leqslant \mu_1^+$.

It is easy to show that if $G \cong K_n$, then equality holds. Conversely, if the equality holds in (8), then $\mu_1^+ = \alpha$ and $\mu_2^+ = \cdots = \mu_n^+$. Therefore Lemma 3.4 implies that G is an r-regular graph and that $\mathbf{L}^+(G)$ has two distinct eigenvalues. Because $\mathbf{L}^+(G) = r\,\mathbf{I}_n + \mathbf{A}(G)$, it follows that G is a regular graph with two distinct eigenvalues (of the adjacency matrix), equal to λ and r with multiplicities $n-1$ and 1, respectively, where $\lambda = \mu_2^+ - r$. Then by Smith's theorem [18], G must be the complete graph. □

Denote by Δ the greatest vertex degree of the graph G. Let S_n be the n-vertex star.

Theorem 3.6. *Let G be a connected graph of order n*, $n \geqslant 3$, *with m edges. Then*

$$IE(G) < \sqrt{1+\Delta} + \sqrt{(n-1)(2m-1-\Delta)}. \tag{10}$$

Proof. We use the recently obtained results [22] that if G has at least one edge, then $\mu_1^+(G) \geqslant 1 + \Delta$, and that if G is connected, $\mu_1^+(G) = 1 + \Delta$ if and only if $G \cong S_n$. Recall that for S_n, $\mu_n^+ = 0 \neq \mu_2^+$. The rest of the proof is then fully analogous to the proof of Theorem 3.5, bearing in mind that $1 + \Delta > \Delta \geqslant 2m/n$. □

Note that the inequalities (8) and (10) are incomparable. Let G_1 and G_2 be the two graphs shown in Fig. 1. For G_1 the upper bounds (8) is better than (10), whereas for G_2 the upper bounds (10) is better than (8).

Theorem 3.5 can be slightly improved. For the graph G and its vertex u, let t_u be the sum of the degrees of the first neighbors of u in G. If G is graph that is not empty, then define

$$T(G) := \sqrt{\frac{\sum_{u \in V(G)} (d_u^2 + t_u)^2}{\sum_{u \in V(G)} d_u^2}}.$$

Theorem 3.7. *Let G be a graph with n vertices and $m > 0$ edges. Then*

$$IE(G) \leqslant \sqrt{T(G)} + \sqrt{(n-1)[2m - T(G)]} \tag{11}$$

with equality if and only if either $G \cong K_2 \cup \overline{K_{n-2}}$ *or* $G \cong K_n$.

Proof. By (9), $IE(G) \leqslant f(\mu_1^+)$, where $f(x) = \sqrt{x} + \sqrt{(n-1)(2m-x)}$.

Recall that for an $n \times n$ nonnegative matrix $\mathbf{M}$, its largest eigenvalue is greater than or equal to

$$\frac{\mathbf{x}^t\, \mathbf{M}\, \mathbf{x}}{\mathbf{x}^t\, \mathbf{x}}$$

for $\mathbf{x}$ being any non-zero n-dimensional real column vector (see, e.g. [15]). Because $(\mu_1^+)^2$ is equal to the largest eigenvalue of the matrix $\mathbf{L}^+(G)^2$, for $\mathbf{x} = (d_1, d_2, \ldots, d_n)^t$ we get $\mathbf{L}^+(G)\, \mathbf{x} = (d_1^2 + t_1, d_2^2 + t_2, \ldots, d_n^2 + t_n)^t$, and therefore

$$\mu_1^+ \geqslant \sqrt{\frac{\mathbf{x}^t\ (\mathbf{L}^+(G)^2)\ \mathbf{x}}{\mathbf{x}^t\, \mathbf{x}}} = \sqrt{\frac{(\mathbf{L}^+(G)\, \mathbf{x})^t\ (\mathbf{L}^+(G)\, \mathbf{x})}{\mathbf{x}^t\, \mathbf{x}}} \equiv T(G).$$

Note that [6]

$$\sum_{u \in V(G)} t_u = \sum_{u \in V(G)} d_u^2 = Zg(G).$$

By the Cauchy–Schwarz inequality,

$$\mu_1^+ \geqslant T(G) \geqslant \sqrt{\left[\sum_{u \in V(G)} (d_u^2 + t_u)\right]^2 \left[n \sum_{u \in V(G)} d_u^2\right]^{-1}} = 2\sqrt{\frac{Zg(G)}{n}} \equiv \alpha \geqslant \frac{4m}{n}.$$

Thus, $IE(G) \leqslant f(T(G))$, from which (11) follows.

Suppose that equality holds in (11). Then either $\mu_1^+ = \mu_2^+ = \cdots = \mu_n^+$ or $\mu_1^+ > \mu_2^+ = \cdots = \mu_n^+$. The former case is impossible, because then it would be $\mu_1^+ = 2m/n < 4m/n \leqslant \alpha$, which is a contradiction. Consider the latter case. The number of distinct signless Laplacian eigenvalues of a connected graph with diameter d is at least $d + 1$ [3]. This implies that if G has exactly two distinct signless Laplacian eigenvalues and $\mu_2^+ = 0$, then G consists of K_2 and $n - 2$ isolated vertices. If $\mu_2^+ > 0$, then by the Perron–Frobenius theorem, $\mathbf{L}^+(G)$ is irreducible, i.e., G is a connected graph, and thus $G \cong K_n$.

Conversely, if G consists of one K_2 and $n - 2$ isolated vertices or $G \cong K_n$, then it is evident that (11) is an equality. □

By Theorem 3.7, for any graph with $n \geqslant 3$ vertices, m edges, and the first Zagreb index Zg, we also have (8) with equality if and only if $G \cong \overline{K_n}$ or $G \cong K_n$. If we would use finer lower bounds for μ_1^+, then we would arrive at additionally improved (but still more complicated) upper bounds for $IE(G)$.

Now we consider lower bounds for the incidence energy.

Theorem 3.8. *Let G be a connected graph on n vertices. Then* $IE(G) \geqslant \sqrt{n} + n - 2$. *Equality holds if and only if* $G \cong S_n$.

Proof. The graph G has at least one spanning tree T as its subgraph. In [10] it was shown that if H is any proper subgraph of the graph G, then $IE(G) > IE(H)$. Therefore, $IE(G) \geqslant IE(T)$, and equality holds if and only if $G \cong T$. In [7] it was shown that among n-vertex trees, the star (and only the star) has minimal incidence energy. Thus, $IE(T) \geqslant IE(S_n) = \sqrt{n} + n - 2$, and equality holds if and only if $G \cong S_n$. □

Let $a_1, a_2, \ldots, a_s$ be positive integers. By Hölder's inequality, we have

$$\sum_{i=1}^{s} a_i^2 = \sum_{i=1}^{s} a_i^{4/3} a_i^{2/3}$$
$$\leqslant \left[\sum_{i=1}^{s} \left(a_i^{4/3}\right)^3\right]^{1/3} \left[\sum_{i=1}^{s} \left(a_i^{2/3}\right)^{3/2}\right]^{2/3} = \left(\sum_{i=1}^{s} a_i^4\right)^{1/3} \left(\sum_{i=1}^{s} a_i\right)^{2/3}$$

i.e.,

$$\sum_{i=1}^{s} a_i \geqslant \sqrt{\frac{\left(\sum_{i=1}^{s} a_i^2\right)^3}{\sum_{i=1}^{s} a_i^4}}, \tag{12}$$

with equality if and only if $a_1 = a_2 = \cdots = a_s$. Note that (12) is a particular case of an inequality in [26].

Theorem 3.9. *Let G be a graph with n vertices and m edges. Then*

$$IE(G) \geqslant \frac{2m}{\sqrt{n}}, \tag{13}$$

with equality if and only if $G \cong \overline{K_n}$ *or* $G \cong K_2$.

Proof. If $m = 0$, then $IE(G) = 0$. Suppose that $m \geqslant 1$. Note that $Zg(G) \leqslant (n-1)\sum_{u \in V(G)} d_u = 2m(n-1)$, with equality if and only if $d_u = n-1$ for all $u \in V(G)$, i.e., $G \cong K_n$. (The other case, namely that $d_u = 0$ for all $u \in V(G)$ cannot happen since $m \geqslant 1$.) By (12) and using $\sum_{i=1}^{n} \mu_i^+ = 2m$ and $\sum_{i=1}^{n} (\mu_i^+)^2 = Zg(G) + 2m$, we have

$$IE(G) = \sum_{i=1}^{n} \sqrt{\mu_i^+} \geqslant \sqrt{\frac{(2m)^3}{Zg(G) + 2m}} \geqslant \sqrt{\frac{(2m)^3}{2m(n-1) + 2m}} = \frac{2m}{\sqrt{n}},$$

with equalities if and only if all nonzero signless Laplacian eigenvalues are equal, and $G \cong K_n$, i.e., $G \cong K_2$. Inequality (13) follows, with equality if and only if $G \cong \overline{K_n}$ or $G \cong K_2$. □

Theorem 3.10. *Let G be a* K_{r+1}*-free graph with n vertices and m edges, where* $2 \leqslant r \leqslant n$. *Then*

$$IE(G) \geqslant \frac{2m}{\sqrt{\frac{r-1}{r} n + 1}},$$

with equality if and only if $G \cong \overline{K_n}$ *or* $r = 2$ *and* $G \cong K_2$.

Proof. If $m = 0$, then $IE(G) = 0$. Suppose that $m \geqslant 1$. Note that $Zg(G) \leqslant \frac{2r-2}{r} nm$ with equality if and only if G is a complete bipartite graph for $r = 2$ and a regular complete r-partite graph for $r \geqslant 3$ [23]. As above,

$$IE(G) \geqslant \sqrt{\frac{(2m)^3}{Zg(G) + 2m}} \geqslant \sqrt{\frac{(2m)^3}{\frac{2r-2}{r} nm + 2m}} = \frac{2m}{\sqrt{\frac{r-1}{r} n + 1}},$$

with equalities if and only if all nonzero signless Laplacian eigenvalues are equal, G is a complete bipartite graph for $r = 2$ and a regular complete r-partite graph for $r \geqslant 3$, i.e., $r = 2$ and $G \cong K_2$. This is because the number of distinct signless Laplacian eigenvalues of a connected graph with diameter d is at least $d + 1$ [3]. It follows that

$$IE(G) \geqslant \frac{2m}{\sqrt{\frac{r-1}{r}\, n+1}},$$

with equality if and only if $G \cong \overline{K_n}$ or $r = 2$ and $G \cong K_2$. $\square$

By Theorem 3.10 ($r = n$), we have Theorem 3.9. If G is a bipartite graph with n vertices and m edges, then by Theorem 3.10 ($r = 2$), we have

$$IE(G) \geqslant \frac{2\sqrt{2}\, m}{\sqrt{n+2}},$$

with equality if and only if $G \cong \overline{K_n}$ or $G \cong K_2$. An equivalent result for $E(S(G))$ was noted in [24].

Let G be a graph with n vertices and $m \geqslant 1$ edges. Note that $\sum_{i=1}^{n-1} \mu_i = 2m$ and $\sum_{i=1}^{n-1} \mu_i^2 = Zg(G) + 2m$. Then by (12),

$$LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i} \geqslant \sqrt{\frac{(2m)^3}{Zg(G)+2m}}$$

with equality if and only if all nonzero Laplacian eigenvalues are equal. We note that by similar arguments as above, the lower bounds in Theorems 3.9 and 3.10 are also lower bounds for LEL, the former is attained if and only if $G \cong \overline{K_n}$ or $G \cong K_n$, while the latter is attained if and only if $G \cong \overline{K_n}$, or $r = n$ and $G \cong K_n$.

4. On incidence energy of line graphs

If G is a graph and $\mathcal{L}(G) = \mathcal{L}^1(G)$ is its line graph, then $\mathcal{L}^k(G)$, $k = 2, 3, \ldots$, defined recursively via $\mathcal{L}^k(G) = \mathcal{L}(\mathcal{L}^{k-1}(G))$, are the iterated line graphs of G. It is both consistent and convenient to set $G = \mathcal{L}^0(G)$.

If G is regular, then its line graph is also regular. In particular, the line graph of a regular graph G of order n_0 and of degree r_0 is a regular graph of order $n_1 = r_0\, n_0/2$ and of degree $r_1 = 2\, r_0 - 2$. Therefore, the order and degree of $\mathcal{L}^k(G)$ are $n_k = r_{k-1}\, n_{k-1}/2$ and $r_k = 2\, r_{k-1} - 2$, where n_{k-1} and r_{k-1} are, respectively, the order and degree of $\mathcal{L}^{k-1}(G)$.

If the eigenvalues of the adjacency matrix of the graph G are $\lambda_i(G)$, $i = 1, \ldots, n_0$ (arranged in non-increasing order), then the respective eigenvalues of $\mathcal{L}(G)$ are -2 with multiplicity $n_1 - n_0$ and $\lambda_i(G) + r_0 - 2$ for $i = 1, \ldots, n_0$ [2]. Since $\mathbf{L}^+(\mathcal{L}(G)) = \mathbf{A}(\mathcal{L}(G)) + r_1\, \mathbf{I}_{n_1}$ and $\mu_i^+(G) = \lambda_i(G) + r_0$, it follows that the eigenvalues of the matrix $\mathbf{L}^+(\mathcal{L}(G))$ are:

$$\begin{pmatrix} 2r_0 - 4 & \mu_1^+(G) + 2r_0 - 4 & \cdots & \mu_n^+(G) + 2r_0 - 4 \\ n_1 - n_0 & 1 & \cdots & 1 \end{pmatrix}.$$

Analogously, the eigenvalues of the matrix $\mathbf{L}^+(\mathcal{L}^{k+1}(G))$ are:

$$\begin{pmatrix} 2r_k - 4 & \mu_1^+(\mathcal{L}^k(G)) + 2r_k - 4 & \cdots & \mu_n^+(\mathcal{L}^k(G)) + 2r_k - 4 \\ n_{k+1} - n_k & 1 & \cdots & 1 \end{pmatrix}.$$

Theorem 4.1. *Let G be a regular graph on n vertices and of degree r. Then*

$$IE(\mathcal{L}(G)) \leqslant \frac{n(r-2)}{2}\sqrt{2r-4} + \sqrt{4r-4} + \sqrt{(2r-4)(n-1)^2 + r(n-1)(n-2)}. \tag{14}$$

Equality holds if and only if $G \cong K_n$.

Proof. Recalling that [1,2] $\lambda_1(G) = r$, we have

$$\begin{aligned} IE(\mathcal{L}(G)) &= \frac{n(r-2)}{2}\sqrt{2r-4} + \sum_{i=1}^{n}\sqrt{\lambda_i(G)+3r-4} \\ &= \frac{n(r-2)}{2}\sqrt{2r-4} + \sqrt{4r-4} + \sum_{i=2}^{n}\sqrt{\lambda_i(G)+3r-4} \\ &\leqslant \frac{n(r-2)}{2}\sqrt{2r-4} + \sqrt{4r-4} + \sqrt{(n-1)\sum_{i=2}^{n}[\lambda_i(G)+3r-4]}. \end{aligned}$$

Inequality (14) follows now by observing that $\sum_{i=2}^{n}\lambda_i(G) = -r$.

Equality in (14) holds if and only if $\lambda_2(G) = \cdots = \lambda_n(G)$. Because a connected graph with n vertices and diameter d has at least $d+1$ distinct eigenvalues [1,2], G must be the complete graph. □

Corollary 4.2. *Let G be same as in Theorem 4.1. Then*

$$IE(\mathcal{L}^{k+1}(G)) \leqslant \frac{n_k(r_k-2)}{2}\sqrt{2r_k-4} + \sqrt{4r_k-4} + \sqrt{(n_k-1)[(3r_k-4)(n_k-1)-r_k]}.$$

Equality holds if and only if $\mathcal{L}^k(G) \cong K_n$.

Theorem 4.3. *Let G be a non-bipartite connected regular graph on n vertices and of degree $r \geqslant 2$. Then*

$$IE(\mathcal{L}(G)) > \frac{n(r-2)}{2}\sqrt{2r-4} + \sqrt{4r-4} + (n-1)\sqrt{2r-4}.$$

Proof

$$IE(\mathcal{L}(G)) = \frac{n(r-2)}{2}\sqrt{2r-4} + \sqrt{4r-4} + \sum_{i=2}^{n}\sqrt{\mu_i^+(G)+2r-4}.$$

Since for connected non-bipartite graphs $\mu_n^+(G) > 0$ (see [3]), we have

$$\sum_{i=2}^{n}\sqrt{\mu_i^+(G)+2r-4} \geqslant \sum_{i=2}^{n}\sqrt{0+2r-4} = (n-1)\sqrt{2r-4}. \qquad \square$$

5. Nordhaus–Gaddum-type results for incidence energy

Nordhaus and Gaddum [17] gave bounds for the sum of the chromatic numbers of a graph and its complement. Nordhaus–Gaddum-type results for many graph invariants are known. Here we give Nordhaus–Gaddum-type results for the incidence energy.

Theorem 5.1. *Let G be a graph with $n \geqslant 2$ vertices. Then*

$$\sqrt{n}(n-1) \leqslant IE(G) + IE(\overline{G}) < 2\sqrt{n-1} + (n-1)\sqrt{2(n-2)},$$

with left equality if and only if $n = 2$.

Proof. Let m and $\overline{m}$ be, respectively, the number of edges of G and $\overline{G}$. By Theorem 3.9,

$$IE(G) + IE(\overline{G}) \geqslant \frac{2m+2\overline{m}}{\sqrt{n}} = \sqrt{n}(n-1), \tag{15}$$

with equality if and only if $m, \overline{m} = 0, 1$, i.e., $n = 2$ for $n \geqslant 2$.

Let $\overline{\mu_1^+}$ be the largest signless Laplacian eigenvalue of $\overline{G}$. By the Cauchy–Schwarz inequality,

$$\begin{aligned} IE(G)+IE(\overline{G}) &\leqslant \sqrt{\mu_1^+}+\sqrt{\overline{\mu_1^+}}+\sqrt{(n-1)(2m-\mu_1^+)}+\sqrt{(n-1)(2\overline{m}-\overline{\mu_1^+})} \\ &\leqslant \sqrt{2(\mu_1^+ + \overline{\mu_1^+})}+\sqrt{2(n-1)\left[n(n-1)-(\mu_1^+ + \overline{\mu_1^+})\right]} \end{aligned}$$

and if equalities are attained, then $\mu_1^+ = \overline{\mu_1^+}$ and $\mu_2^+ = \cdots = \mu_n^+$. Consider the function $g(x) = \sqrt{2x} + \sqrt{2(n-1)[n(n-1)-x]}$. It is decreasing for $x \geqslant n-1$. Note that (from the proof of Theorem 3.7)

$$\mu_1^+ + \overline{\mu_1^+} \geqslant \frac{4m}{n} + \frac{4\overline{m}}{n} = 2(n-1),$$

with equality if and only if G is regular. Now

$$IE(G)+IE(\overline{G}) \leqslant g\,(2(n-1)) = 2\sqrt{n-1}+(n-1)\sqrt{2(n-2)}$$

and the equality can not be attained, otherwise, $\lambda_2(G) = \cdots = \lambda_n(G) = -\frac{1}{2}$, which is impossible, because by the interlacing theorem, $\lambda_n(G) = 0$ or $\lambda_n(G) \leqslant -1$. $\square$

We now give two examples. For the complete graph K_n,

$$IE\,(K_n) + IE\,(\overline{K_n}) = IE\,(K_n) = \sqrt{2n-2} + (n-1)\sqrt{n-2}.$$

For the complete bipartite graph $K_{n/2,/2}$, with n even,

$$IE\,(K_{n/2,/2}) = \sqrt{n} + \frac{\sqrt{2}}{2}(n-1)\sqrt{n}$$

and

$$IE\,(\overline{K_{n/2,n/2}}) = 2\sqrt{n-2} + \frac{\sqrt{2}}{2}(n-2)\sqrt{n-4}.$$

Thus,

$$IE\,(K_{n/2,/2}) + IE\,(\overline{K_{n/2,/2}}) = \sqrt{n} + 2\sqrt{n-2} + \frac{\sqrt{2}}{2}(n-1)\sqrt{n} + \frac{\sqrt{2}}{2}(n-2)\sqrt{n-4}.$$

These examples and the Theorem 5.1 imply:

Theorem 5.2. *Let* $\min IE_{NG}(n)$ *and* $\max IE_{NG}(n)$ *be respectively the minimum and maximum values of* $IE(G)+IE(\overline{G})$ *over all graphs with n vertices. Then*

$$\lim_{n\to\infty} \frac{\min IE_{NG}(n)}{n^{3/2}} = 1 \quad \textit{and} \quad \lim_{n\to\infty} \frac{\max IE_{NG}(n)}{n^{3/2}} = \sqrt{2}.$$

By using structural parameters other than the number of vertices, the upper bound in Theorem 5.1 can be improved as follows. Let

$$\kappa = \frac{2}{\sqrt{n}}\left[\sqrt{Zg(G)} + \sqrt{n(n-1)^2 - 4m(n-1) + Zg(G)}\right].$$

Theorem 5.3. *Under the same conditions as in Theorem 5.1,*

$$IE(G)+IE(\overline{G}) < \sqrt{2\kappa} + \sqrt{2(n-1)[n(n-1)-\kappa]}. \tag{16}$$

Proof. Repeat the reasoning in the proof of Theorem 5.1 until (15). From the proof of Theorem 3.9) we get

$$\begin{aligned}\mu_1^+ + \overline{\mu_1^+} &\geqslant \alpha(G) + \alpha(\overline{G}) \\ &= \frac{2}{\sqrt{n}}\left[\sqrt{\sum_{i=1}^{n} d_i(G)^2} + \sqrt{\sum_{i=1}^{n} d_i(\overline{G})^2}\,\right] \\ &= \frac{2}{\sqrt{n}}\left[\sqrt{\sum_{i=1}^{n} d_i(G)^2} + \sqrt{\sum_{i=1}^{n}(n-1-d_i(G))^2}\,\right] \\ &= \frac{2}{\sqrt{n}}\left[\sqrt{Zg(G)} + \sqrt{n(n-1)^2 - 4m(n-1) + Zg(G)}\right],\end{aligned}$$

with equality if and only if G is regular. As explained in the proof of Theorem 5.1 we now have $IE(G) + IE(\overline{G}) \leqslant g(\kappa)$ which immediately implies (16). $\square$

Acknowledgements

The research of the second author was in part supported by IPM (Grant No. 87200116). I.G. thanks for support by the Serbian Ministry of Science (Grant No. 144015G). B.Z. thanks for support by the Guangdong Provincial Natural Science Foundation of China (No. 8151063101000026).

References

[1] N. Biggs, Algebraic Graph Theory, Cambridge Univ. Press, Cambridge, 1993.
[2] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs – Theory and Application, Academic Press, New York, 1980.
[3] D. Cvetković, P. Rowlinson, S. Simić, Signless Laplacians of finite graphs, Linear Algebra Appl. 423 (2007) 155–171.
[4] I. Gutman, The energy of a graph: old and new results, in: A. Betten, A. Kohnert, R. Laue (Eds.), Algebraic Combinatorics and Applications, Springer-Verlag, Berlin, 2001, pp. 196–211.
[5] I. Gutman, On graphs whose energy exceeds the number of vertices, Linear Algebra Appl. 429 (2008) 2670–2677.
[6] I. Gutman, K.C. Das, The first Zagreb index 30 years after, MATCH Commun. Math. Comput. Chem. 50 (2004) 83–92.
[7] I. Gutman, D. Kiani, M. Mirzakhah, On incidence energy of graphs, MATCH Commun. Math. Comput. Chem. 62 (2009) 573–580.
[8] I. Gutman, X. Li, J. Zhang, Graph energy, in: M. Dehmer, F. Emmert-Streib (Eds.), Analysis of Complex Networks. From Biology to Linguistics, Wiley-VCH, Weinheim, 2009, pp. 145–174.
[9] Y. Hong, X.D. Zhang, Sharp upper and lower bounds for largest eigenvalue of the Laplacian matrices of trees, Discrete Math. 296 (2005) 187–197.
[10] M.R. Jooyandeh, D. Kiani, M. Mirzakhah, Incidence energy of a graph, MATCH Commun. Math. Comput. Chem. 62 (2009) 561–572.
[11] J. Liu, B. Liu, A Laplacian-energy like invariant of a graph, MATCH Commun. Math. Comput. Chem. 59 (2008) 355–372.
[12] J. Liu, B. Liu, S. Radenković, I. Gutman, Minimal LEL-equienergetic graphs, MATCH Commun. Math. Comput. Chem. 61 (2009) 471–478.
[13] R. Merris, Laplacian matrices of graphs: a survey, Linear Algebra Appl. 197–198 (1994) 143–176.
[14] R. Merris, A survey of graph Laplacians, Linear and Multilinear Algebra 39 (1995) 19–31.
[15] C.D. Meyer, Matrix Analysis and Applied Linear Algebra, SIAM, Philadelphia, 2000.
[16] V. Nikiforov, The energy of graphs and matrices, J. Math. Anal. Appl. 326 (2007) 1472–1475.
[17] E.A. Nordhaus, J.W. Gaddum, On complementary graphs, Am. Math. Monthly 63 (1956) 175–177.
[18] J.H. Smith, Some properties of the spectrum of a graph, in: R. Guy, H. Hanani, N. Sauer, J. Schönheim (Eds.), Combinatorial Structures and Their Applications, Gordon and Breach, New York, 1970, pp. 403–406.
[19] D. Stevanović, Laplacian-like energy of trees, MATCH Commun. Math. Comput. Chem. 61 (2009) 407–417.
[20] D. Stevanović, A. Ilić, On the Laplacian coefficients of unicyclic graphs, Linear Algebra Appl. 430 (2009) 2290–2300.
[21] R. Todeschini, V. Consonni, Handbook of Molecular Descriptors, Wiley-VCH, Weinheim, 2000.
[22] Y. Zhang, X. Liu, B. Zhang, X. Yong, The lollipop graph is determined by its Q-spectrum, Discrete Math. 309 (2009) 3364–3369.
[23] B. Zhou, Remarks on Zagreb indices, MATCH Commun. Math. Comput. Chem. 57 (2007) 591–596.
[24] B. Zhou, On sum of powers of the Laplacian eigenvalues of graphs, Linear Algebra Appl. 429 (2008) 2239–2246.
[25] B. Zhou, I. Gutman, A connection between ordinary and Laplacian spectra of bipartite graphs, Linear and Multilinear Algebra 56 (2008) 305–310.
[26] B. Zhou, I. Gutman, J.A. de la Peña, J. Rada, L. Mendoza, On spectral moments and energy of graphs, MATCH Commun. Math. Comput. Chem. 57 (2007) 183–191.