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The Liouville Equation in L^1 Spaces

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Abstract—We consider the first order equation $\frac{\partial u}{\partial t} = \mathfrak{a} \cdot \nabla u$ in the Banach lattice $L^1(\mathbf{R}^N)$. By requiring a minimal amount of Sobolev regularity on the vector-field $\mathfrak{a}$, we show that $\mathfrak{a} \cdot \nabla$ generates a C_0-group, thereby generalizing a result of [1]. From there, we conclude the well-posedness of Liouville equation $\frac{\partial u}{\partial t} = -\xi \cdot \nabla_x u + \nabla_x V \cdot \nabla_\xi u$, for a given potential V. The comparison between the general and force-free Liouville evolution yields the existence of the wave and scattering operators, which in turn are used to prove that the spectrum of the Liouville operator is purely residual in $L^1(\mathbf{R}^6)$.

Keywords—Liouville equation, Dunford-Pettis property, Mild solution, Scattering operator, Residual spectrum.

1. GENERAL RESULTS

Let $\mathfrak{a} = (a_1, \dots, a_N)$ be a vector-field on $\mathbf{R}^N$ and

$$\begin{aligned} \dot{X} &= \mathfrak{a}(X), \\ X(0) &= x \in \mathbf{R}^N \end{aligned} \tag{P}$$

the corresponding dynamical system. If we impose enough regularity on $\mathfrak{a}(X)$ (e.g., $\mathfrak{a} \in [C_b^1(\mathbf{R}^N)]^N$), then by the Cauchy-Lipschitz Theorem, there exists a unique *continuous* flow $\Phi(t)$ on $\mathbf{R}^N$ such that $X(x,t) = \Phi(-t)x$ is the solution of (P). Now, if $f \in L^p(\mathbf{R}^N)$ $(1 \le p < \infty)$, one can define the family of operators on $L^p(\mathbf{R}^N)$, $\{U(t)\}_{t\in\mathbf{R}}$, by

$$U(t)f(x) = f(X(x,t)). \tag{1.1}$$

The following result, which is an L^p-version of "Koopmanism" (see [2]), can be found in [3].

LEMMA 1.1 [3]. *If $\mathfrak{a} \in [C_b^1(\mathbf{R}^N)]^N$, the family $\{U(t)\}_{t\in\mathbf{R}}$ is a C_0-group and $A = -\mathfrak{a}(x)\cdot\nabla$, with $D(A) = \{u \in L^p(\mathbf{R}^N) \mid \mathfrak{a}(x)\cdot\nabla u \in L^p(\mathbf{R}^N)\}$ is its infinitesimal generator in $L^p(\mathbf{R}^N)$. Moreover,*

$$\|U(t)f\|_p \le e^{t\omega/p}\|f\|_p, \tag{1.2}$$

where $\omega = \|\operatorname{div}\mathfrak{a}\|_\infty$.

In [1], DiPerna and Lions have shown that if we do not require a classical solution for $\frac{du}{dt} = Au$, the vector-field $\mathfrak{a}$ needs only to belong to a Sobolev space instead of being Lipschitz continuous.

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By slightly weakening their assumptions (we remove their assumptions on the asymptotic behaviour of $\mathfrak{a}$), we will show an analogue of Lemma 1.1, for $p = 1$ (see Theorem 1.3 below).

In order to prove that result, we need to define a weak solution of an abstract Cauchy problem

$$\frac{du}{dt} = Au + g(t), \qquad u(0) = f \in \mathcal{X}, \tag{1.3}$$

where $g \in C([0,T];\mathcal{X})$, in a Banach space $\mathcal{X}$, in the sense of Ball [4].

A function $u \in C([0,T];\mathcal{X})$ is a *weak solution* for (1.3) if for any $v \in D(A^*)$ the function $\langle u(t), v\rangle$ is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\langle u(t), v\rangle = \langle u(t), A^* v\rangle + \langle g(t), v\rangle \tag{1.4}$$

for almost all $t \in [0,T]$. In (1.4), $\langle .,.\rangle$ denotes the pairing between $\mathcal{X}$ and its dual. Related to this definition we have the following lemma.

LEMMA 1.2 [4]. *A densely defined closed linear operator A is the generator of a C_0-semigroup on $\mathcal{X}$ iff for any $f \in \mathcal{X}$, (1.2) has a unique weak solution in $\mathcal{X}$, which is also a mild solution.*

THEOREM 1.3. *Suppose $\mathfrak{a} \in [W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N)]^N$ and $\operatorname{div}\mathfrak{a} \in L^\infty(\mathbf{R}^N)$; then $A = -\mathfrak{a}(x)\cdot\nabla$ generates a C_0-group $U(t)$ on $L^1(\mathbf{R}^N)$ which satisfies (1.2) for $p = 1$.*

PROOF. As in [1], we start by regularizing the vector-field $\mathfrak{a}$ by setting $\mathfrak{a}_\epsilon = \mathfrak{a} * \varphi_\epsilon$, where $\varphi_\epsilon = \epsilon^{-N}\varphi(\cdot/\epsilon)$, $\varphi \in [C_0^\infty(\mathbf{R}^N)]_+$, and $\int \varphi(x)\,dx = 1$.

Applying Lemmas 1.1 and 1.2 for $A_\epsilon = -\mathfrak{a}_\epsilon\cdot\nabla$, we conclude that, for any $\epsilon > 0$, there exists a weak solution u_ϵ for

$$\begin{aligned} \frac{dw}{dt} &= A_\epsilon w, \\ w(0) &= f \in L^1\left(\mathbf{R}^N\right), \end{aligned} \tag{P_ϵ}$$

which satisfies for each $v \in D(A^*)$

$$\frac{d}{dt}\langle u_\epsilon(t), v\rangle = \langle u_\epsilon(t), A_\epsilon^* v\rangle, \tag{1.5}$$

where $D(A^*) = \{v \in L^\infty(\mathbf{R}^N) \mid \operatorname{div}(v\mathfrak{a}) \in L^\infty(\mathbf{R}^N)$ and $-\int(\mathfrak{a}\cdot\nabla u)v\,dx = \int \operatorname{div}(v\mathfrak{a})u\,dx$, for all $u \in D(A)\}$. This makes sense since $D(A^*) \subseteq D(A_\epsilon^*)$. Let us denote by $J_t^\epsilon(y)$ the Jacobian of $X_\epsilon(\cdot,t)$ at y.

Since $\|\operatorname{div}\mathfrak{a}_\epsilon\|_\infty \le \|\operatorname{div}\mathfrak{a}\|_\infty = \omega$,

$$\int_{\mathbf{R}^N} |u_\epsilon(x,t)|\,dx = \int_{\mathbf{R}^N} |f(X_\epsilon(x,t)|\,dx = \int_{\mathbf{R}^N} |f(y)|\,J_t^\epsilon(y)\,dy \le e^{t\omega}\|f\|_1,$$

and since X_ϵ is Lipschitz continuous, for any measurable subset E of $\mathbf{R}^N$ and any Lebesgue integrable function f, we have

$$\int_E |u_\epsilon(x,t)|\,dx = \int_{X_\epsilon(E)} |f(y)|\,J_t^\epsilon(y)\,dy \le e^{t\omega}\int_{X_\epsilon(E)} |f(y)|\,dy \to 0$$

as the Lebesgue measure $\mu(E) \to 0$. For, $\mu(X_\epsilon(E)) \le e^{t\omega}\mu(E)$ is ϵ-independent. This proves that the family $\{u_\epsilon\}$ is a bounded uniformly integrable subset of $L^1(\mathbf{R}^N)$ and, consequently, relatively weakly compact. Let $u \in \mathcal{C}^1(L^1(\mathbf{R}^N))$ for which (1.4) with $g \equiv 0$ is the limit of (1.5). In fact, for each $\epsilon > 0$, the map $t \to \langle u_\epsilon(t), v\rangle$ is absolutely continuous and converges (extracting a subsequence if necessary) to $\langle u(t), v\rangle$ as $\epsilon \to 0$. We shall prove that the uniform convergence of $\frac{d}{dt}\langle u_\epsilon(t), v\rangle \to \frac{d}{dt}\langle u(t), v\rangle$ as $\epsilon \to 0$, on $[-T,T]$. The local integrability of a_j and $\frac{\partial a_j}{\partial x_j}$ implies that

$$\int u \operatorname{div}(v\mathfrak{a}_\epsilon)\,dx \to \int u \operatorname{div}(v\mathfrak{a})\,dx \qquad \text{as } \epsilon \to 0,$$

which means that $\langle u, A_\epsilon^* v\rangle \to \langle u, A^* v\rangle$, for any $u \in L^1(\mathbf{R}^N)$. Now due to the Dunford-Pettis property of L^1, $\langle \cdot, A_\epsilon^* v\rangle$ converges uniformly on each weakly compact subset of $L^1(\mathbf{R}^N)$ (see [5, Chapter II, Theorem 9.7]). Since the set $\{u_\epsilon(t) \mid t \in [-T,T]\}$ is weakly compact, we conclude the theorem. ∎

REMARK 1.4. In the previous theorem, the weak limit of the sequence $u_\epsilon(x,t)$ in $L^1(\mathbf{R}^N)$ defines $U(t)f$. Thus we are not allowed to obtain the properties of the C_0-group $U(t)$ directly from the expression (1.1). One of the consequences of our result is: $\int_{\mathbf{R}^N} |U(t)f(x)|\,dx = \lim_{\epsilon\to 0}\int_{\mathbf{R}^N} |f(X_\epsilon(x,t))|\,dx$ which is due to AL property [5] of L^1 spaces.

REMARK 1.5. The argument used to prove Theorem 1.3 holds only for $p=1$, since L^p does not have the Dunford-Pettis property for $p \neq 1$.

COROLLARY 1.6. *If the potential $V \in W^{2,1}_{\mathrm{loc}}(\mathbf{R}^3)$, then the Liouville operator $L = -\xi\cdot\nabla_x + \nabla_x V \cdot \nabla_\xi$ generates a C_0-group of isometries on $L^1(\mathbf{R}^3_x \times \mathbf{R}^3_\xi)$.*

PROOF. Note that $\mathfrak{a}(x,\xi) = (\xi, -\nabla_x V(x))$ and $\mathfrak{a}_\epsilon(x,\xi) = [\mathfrak{a}*\varphi_\epsilon](x,\xi)$ are divergence free vectorfields on $\mathbf{R}^3_x \times \mathbf{R}^3_\xi$. Hence by applying Remark 1.4, the operator L generates a C_0-group $S(t)$, which satisfies

$$\|U(t)f\|_1 = \lim_{\epsilon\to 0}\int_{\mathbf{R}^6} |f(X_\epsilon(x,\xi,t)|\,dx\,d\xi = \lim_{\epsilon\to 0}\int_{\mathbf{R}^6} |f(y,\eta)|J_t^\epsilon(y,\eta)\,dy\,d\eta = \|f\|_1,$$

for $\operatorname{div}\mathfrak{a}_\epsilon = 0$ implies that $J_t^\epsilon(y,\eta) = 1$. ∎

2. SCATTERING OPERATOR FOR LIOUVILLE EQUATION AND THE SPECTRUM OF LIOUVILLE OPERATOR

In classical mechanics, the motion of a simple particle in an external force field F is described by the Newton equation $\ddot{x} = F(x)$. For $F \equiv 0$, we can write this equation as the following system:

$$\begin{aligned} \dot{x} &= \xi, & \dot{\xi} &= 0, \\ x(0) &= x_0, & \xi(0) &= \xi_0. \end{aligned} \tag{P_0}$$

Let us denote by X_0 the solution of (P_0), which is given by the global flow Φ_0 as $X_0(x_0,\xi_0,t) = \Phi_0(-t)(x_0,\xi_0) = (x_0 - t\xi_0, \xi_0)$.

If we assume that the force is conservative, that means there exists a potential V such that $F(x) = -\nabla V(x)$. If $F \in [C_b^1(\mathbf{R}^3)]^3$, this implies that the system

$$\begin{aligned} \dot{x} &= \xi, & \dot{\xi} &= -\nabla V(x), \\ x(0) &= x_0, & \xi(0) &= \xi_0 \end{aligned} \tag{P_1}$$

has a unique solution $X(x_0,\xi_0,t) = \Phi(-t)(x_0,\xi_0)$ for all time given by the flow $\Phi(t)(x_0,\xi_0) = (x(t),\xi(t))$.

Now let us denote $\Omega(t,s) \equiv \Phi_0(-t)\Phi(t-s)\Phi_0(s)$; then the property of asymptotic completeness [6] is equivalent with the existence of the *scattering transformation* defined by

$$\mathcal{S}(x,\xi) \equiv \lim_{\min\{t,-s\}\to+\infty} \Omega(t,s)(x,\xi)$$

on some subset of $\mathbf{R}^6$.

To establish the existence of this limit, further restrictions on the potential V are needed (see [7]). Namely, if we denote $\langle x \rangle \equiv (1+|x|^2)^{1/2}$, we shall assume

$$V \in \mathcal{C}^2(\mathbf{R}^3) \quad \text{and} \quad F = -\nabla V \in \left[\mathcal{C}_b^1(\mathbf{R}^3)\right]^3; \tag{H1}$$

$$|F(x)| \le C\langle x \rangle^{-2-\epsilon} \qquad \text{for all } x \text{ and some } \epsilon > 0; \tag{H2}$$

and

$$\left|\frac{\partial F(x)}{\partial x_i}\right| \le C\langle x \rangle^{-3-\epsilon} \qquad \text{for all } x, \quad i = 1,2,3 \text{ and some } \epsilon > 0. \tag{H3}$$

Under these hypotheses the scattering transformation $\mathcal{S}$ exists [6, Theorem XI.2] and we have

$$\mathcal{S}(x,\xi) \equiv (\Omega^+)^{-1}\,\Omega^-(x,\xi)$$

where

$$\Omega^\pm(x,\xi) \equiv \Omega(0,\mp\infty) = \lim_{t\to\pm\infty} \Phi(t)\Phi_0(-t)(x,\xi).$$

Based on Corollary 1.6, the Liouville operators $L_0 \equiv -\xi\cdot\nabla_x$ and $L \equiv -\xi\cdot\nabla_x + \nabla_x V\cdot\nabla_\xi$ are the infinitesimal generators of the C_0-groups e^{tL_0} and e^{tL} of isometries on $L^1(\mathbf{R}^6)$, defined by $\left[e^{tL_0}f\right](x,\xi) = f(X_0(x,\xi,t))$ and $\left[e^{tL}f\right](x,\xi) = f(X(x,\xi,t))$, respectively.

The intertwining between the two evolutions is realized by the wave operators

$$W_\pm(L,L_0) \equiv s - \lim_{t\to\pm\infty} e^{-tL}e^{tL_0} \qquad \text{on } L^1(\mathbf{R}^6) \tag{2.1}$$

and

$$W_\pm(L_0,L) \equiv s - \lim_{t\to\pm\infty} e^{-tL_0}e^{tL} \qquad \text{on } R_\mp \equiv \operatorname{Im}(W_\mp(L,L_0)). \tag{2.2}$$

If the wave operators $W_- \equiv W_-(L,L_0)$ and $W_+ \equiv W_+(L_0,L)$ exist, respectively, on $L^1(\mathbf{R}^6)$ and R_-, then the scattering operator is defined by $S \equiv W_+W_-$ on $L^1(\mathbf{R}^6)$.

In [8], it is proved that if the flow $\Phi(t)$ exists globally in time and the hypothesis (H1) holds true, one can define the scattering operator S as the limit of the propagator $W(s,t) \equiv e^{-sL_0} \cdot e^{(s-t)L}\, e^{tL_0}$, as $\min\{s,-t\} \to +\infty$, and S is induced by the scattering transformation $\mathcal{S}$; i.e., $\left[Sf\right](x,\xi) = f(\mathcal{S}(x,\xi))$. This also yields the existence of the wave operators $W_\pm$.

Furthermore, the range spaces $R_\pm$ are characterized by $R_+ = R_- = L^p(\Sigma_{\text{in}}) = L^p(\Sigma_{\text{out}})$ for any $1 \le p < \infty$, where $\Sigma_{\text{in}} \equiv \operatorname{Im}(\Omega^-)$ and $\Sigma_{\text{out}} \equiv \operatorname{Im}(\Omega^+)$ and $\Sigma_{\text{out}}, \Sigma_{\text{in}}$ and $\mathbf{R}^6\backslash\Sigma_b$ agree up to sets of measure 0, where $\Sigma_b \equiv \{(x,\xi) \mid \sup_t |x(t)| < \infty\}$.

LEMMA 2.1. *The existence of $W_+(L_0,L)$ implies that the Liouville system e^{tL} is locally decaying in Σ_{in}. Namely, for each compact subset K of Σ_{in},*

$$\lim_{t\to\infty} \|U(t)f\|_{1,K} = 0 \qquad \text{for all } f \in \mathcal{X},$$

where $\|f\|_{1,K} = \int_K |f(x,\xi)|\,dx\,d\xi$.

For the proof of this lemma, we can proceed as a similar result in [9], but instead of $L^1(\mathbf{R}^6)$, we use $L^1(\Sigma_{\text{in}})$.

We will use this lemma to characterize the spectrum of the Liouville operator L. Let us denote by $\sigma(L)$ (respectively, $\sigma_r(L)$, $\sigma_p(L)$) the spectrum (respectively, residual spectrum, point spectrum) of the operator L.

The following proposition is proved independently by [10–12] in the case when $V \equiv 0$.

PROPOSITION 2.2. *Under hypotheses* (H1)–(H3), $\sigma(L) = \sigma_r(L) = i\mathbf{R}$ *in* $L^1(\mathbf{R}^6)$.

PROOF. Let us construct a function $\psi(x,\xi)$ on $\mathbf{R}^6$ satisfying the inhomogeneous Liouville equation

$$-L\psi = \xi\nabla_x\psi - \nabla_x V(x)\cdot\nabla_\xi\psi = 1. \tag{2.3}$$

For a fixed $(x_0,\xi_0) \in \mathbf{R}^6$ and a given value of $\psi(x_0,\xi_0)$, let us define ψ on the whole trajectory $(x(t),\xi(t))$ of (P_1) by

$$\psi\left(\Phi(t)\left(x_0,\xi_0\right)\right) = \psi\left(x_0,\xi_0\right) + t. \tag{2.4}$$

In fact, if $u(x,\xi,t) \equiv e^{tL}\psi(x,\xi) = \psi(\Phi(-t)(x,\xi)) = \psi(x,\xi) - t$, then $L\psi = L(\psi - t) = Lu = \frac{\partial u}{\partial t} = -1$. The group property of the flow Φ implies that the trajectories cannot intersect each other; hence if we ascribe the value of ψ in one point of each trajectory, the function ψ can be defined in whole $\mathbf{R}^6$ according to (2.4) and verifies (2.3).

According to Corollary 1.6, $\|e^{tL}\| = 1$, and therefore $\sigma(L) \subset i\mathbf{R}$. It is known that if $\overline{\lambda} \in \sigma_p(L^*)$, then λ belongs either to $\sigma_p(L)$ or to $\sigma_r(L)$. Since for any real β, u_β given by $u_\beta(x,\xi) = e^{i\beta\psi(x,\xi)}$ belongs to $D(L^*)$ and satisfies $L^*u_\beta = -i\beta u_\beta$, then $i\beta$ belongs either to $\sigma_p(L)$ or to $\sigma_r(L)$. To conclude the proposition, it is enough to show that $\sigma_p(L) = \emptyset$. In fact, the converse yields to the existence of $f \in L^1(\mathbf{R}^6)$ such that $Lf = i\beta f$ or by spectral mapping theorem

$$e^{tL}f = e^{i\beta t}f. \tag{2.5}$$

But due to Lemma 2.1, we have $\|e^{tL}f\|_{1,K} \to 0$ as $t \to \infty$, which contradicts (2.5), since (2.5) implies

$$\left\|e^{tL}f\right\|_{1,K} = \|f\|_{1,K} \qquad \text{for all } t \in \mathbf{R}. \qquad \blacksquare$$

REFERENCES

1. R.J. DiPerna and P.L. Lions, Ordinary differential equation, transport theory and Sobolev spaces, *Invent. Math.* **98**, 511–547 (1989).
2. R. Abraham and J. Marsden, *Foundations of Mechanics*, 2^{nd} edition, Benjamin Cummings, London, (1978).
3. C. Bardos, Problèmes aux limites pour les equation aux derivées partielles du premier ordre à coefficients réels; Théorèmes d'approximation; Application à l'équation de transport, *Ann. Sci. Ec. Norm. Sup.* **3**, 185–233 (1970).
4. J.M. Ball, Strongly continuous semigroup, weak solutions and the variation of constants formula, *Proc. Amer. Math. Soc.* **63**, 370–373 (1977).
5. H. Schaefer, *Banach Lattices and Positive Operators*, Springer-Verlag, New York, (1974).
6. M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. 3, Acad. Press, New York, (1979).
7. B. Simon, Wave operators for classical particle scattering, *Commun. Math. Phys.* **23**, 37–48 (1971).
8. R. Prosser, On the asymptotic behavior of certain dynamical systems, *J. Math. Phys.* **13**, 186–196 (1972).
9. J. Voigt, On the existence of the scattering operators for the linear Boltzmann equation, *J. Math. Anal. Appl.* **58**, 541–558 (1977).
10. H. Emamirad, On the Lax and Phillips scattering theory for transport equation, *J. Funct. Analysis* **62**, 276-303 (1985).
11. J. Hejtmanek, Dynamics and spectrum of the linear multiple scattering operator in the Banach lattice $L^1(\mathbf{R}^3 \times \mathbf{R}^3)$, *Transport Theory Statis. Phys.* **1**, 29–44 (1979).
12. T. Umeda, Scattering and spectral theory for the linear Boltzmann equation, *J. Math. Kyoto Univ.* **24**, 205–218 (1984).