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Three Symmetric Positive Solutions for a Second-Order Boundary Value Problem

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Abstract—For the second-order boundary value problem, $y'' + f(y) = 0$, $0 \leq t \leq 1$, $y(0) = 0 = y(1)$, where $f : \mathbb{R} \to [0, \infty)$, growth conditions are imposed on f which yield the existence of at least three symmetric positive solutions.

Keywords—Boundary value problem, Green's function, Multiple solutions.

1. INTRODUCTION

In this paper, we are concerned with the existence of multiple solutions for the second-order boundary value problem

$$y'' + f(y) = 0, \qquad 0 \leq t \leq 1, \tag{1.1}$$

$$y(0) = 0 = y(1), \tag{1.2}$$

where $f : \mathbb{R} \to [0, \infty)$ is continuous. A solution $y \in C^{(2)}[0, 1]$ of (1.1),(1.2) is both nonnegative and concave on $[0, 1]$. We will impose growth conditions on f which ensure the existence of at least three symmetric positive solutions of (1.1),(1.2).

There is much current attention focused on questions of positive solutions of boundary value problems for ordinary differential equations, as well as for for finite difference equations; see [1–11], to name a few. Much of this interest is due to the applicability of certain Krasnosel'skii fixed-point theorems or the Leggett-Williams multiple fixed-point theorem, or a synthesis of both to obtain positive solutions or multiple positive solutions which lie in a cone. The recent book by Agarwal, Wong and O'Regan [12] gives a good overview for much of the work which has been done and the methods used.

In [13], Avery imposed conditions on f to yeild at least three positive solutions to (1.1),(1.2) applying the Leggett-Williams fixed-point theorem. In [14], Henderson and Thompson improved

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these results by using the symmetry of the associated Green's function. Our results extend these results by applying a generalization of the Leggett-Williams fixed-point theorem [15].

In Section 2, we provide some background results, and we state the generalization of the Leggett-Williams fixed-point theorem. Then, in Section 3, we impose growth conditions on f which allow us to apply the generalization of the Leggett-Williams fixed-point theorem in obtaining three symmetric positive solutions of (1.1),(1.2).

2. SOME BACKGROUND DEFINITIONS AND RESULTS

In this section, we provide some background material from the theory of cones in Banach spaces, in order that this paper be self-contained. We also state a fixed-point theorem which is a genereralization of the fixed-point theorem of Leggett and Williams for multiple fixed-points of a cone preserving operator.

DEFINITION 2.1. *Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions:*

(i) $x \in P$, $\lambda \geq 0$ *implies* $\lambda x \in P$;

(ii) $x \in P$, $-x \in P$ *implies* $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$x \leq y, \qquad \text{if and only if } y - x \in P.$$

DEFINITION 2.2. *An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.*

DEFINITION 2.3. *A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if*

$$\alpha : P \to [0, \infty)$$

is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y),$$

for all $x, y \in P$ and $t \in [0,1]$. Similarly, we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if

$$\beta : P \to [0, \infty)$$

is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y),$$

for all $x, y \in P$ and $t \in [0,1]$.

Let γ, β, and θ be nonnegative continuous convex functionals on P and α, ψ be nonnegative continuous concave functionals on P. Then for nonnegative real numbers h, a, b, d, and c we define the following convex sets:

$$\begin{aligned} P(\gamma, c) &= \{x \in P : \gamma(x) < c\}, \\ P(\gamma, \alpha, a, c) &= \{x \in P : a \leq \alpha(x),\ \gamma(x) \leq c\}, \\ Q(\gamma, \beta, d, c) &= \{x \in P : \beta(x) \leq d,\ \gamma(x) \leq c\}, \\ P(\gamma, \theta, \alpha, a, b, c) &= \{x \in P : a \leq \alpha(x),\ \ \theta(x) \leq b,\ \ \gamma(x) \leq c\}, \end{aligned}$$

and

$$Q(\gamma, \beta, \psi, h, d, c) = \{x \in P : h \leq \psi(x),\ \ \beta(x) \leq d,\ \ \gamma(x) \leq c\}.$$

In obtaining multiple symmetric positive solutions of (1.1),(1.2) the following fixed-point theorem due to Avery [15] which is a generalization of the Leggett-Williams fixed-point theorem will be fundamental.

THEOREM 2.4. *Let P be a cone in a real Banach space E. Let α and ψ be nonnegative continuous concave functionals on P and let γ, β, and θ be nonnegative continuous convex functionals on P such that, for some positive numbers c and M,*

$$\alpha(x) \le \beta(x) \quad \text{and} \quad \|x\| \le M\gamma(x),$$

for all $x \in \overline{P(\gamma, c)}$. Suppose

$$A : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$$

is completely continuous and there exists nonnegative numbers h, d, a, b with $0 < d < a$ such that:

(i) $\{x \in P(\gamma, \theta, \alpha, a, b, c) : \alpha(x) > a\} \neq \emptyset$ *and* $\alpha(Ax) > a$ *for* $x \in P(\gamma, \theta, \alpha, a, b, c)$;
(ii) $\{x \in Q(\gamma, \beta, \psi, h, d, c) : \beta(x) < d\} \neq \emptyset$ *and* $\beta(Ax) < d$ *for* $x \in Q(\gamma, \beta, \psi, h, d, c)$;
(iii) $\alpha(Ax) > a$ *for* $x \in P(\gamma, \alpha, a, c)$ *with* $\theta(Ax) > b$;
(iv) $\beta(Ax) < d$ *for* $x \in Q(\gamma, \beta, d, c)$ *with* $\psi(Ax) < h$.

Then A has at least three fixed-points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$\beta(x_1) < d, \quad a < \alpha(x_2), \quad \text{and} \quad d < \beta(x_3), \qquad \text{with } \alpha(x_3) < a.$$

3. MULTIPLE SYMMETRIC POSITIVE SOLUTIONS

In this section, we will impose growth conditions on f which allow us to apply Theorem 2.4 in regard to obtaining three symmetric positive solutions of (1.1),(1.2). We will apply Theorem 2.4 in conjunction with a completely continuous operator whose kernel $G(t, s)$ is the Green's function for

$$-y'' = 0, \tag{3.1}$$

satisfying (1.2). In particular,

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{3.2}$$

We will make use of various properties of $G(t, s)$ which include

$$\int_0^1 G(t,s) = \frac{t(1-t)}{2}, \qquad 0 \le t \le 1, \tag{3.3}$$

$$\int_0^{1/r} G\left(\frac{1}{2}, s\right) ds = \int_{1-(1/r)}^1 G\left(\frac{1}{2}, s\right) ds = \frac{1}{4r^2}, \qquad 2 < r, \tag{3.4}$$

$$\int_{1/r}^{1/2} G\left(\frac{1}{2}, s\right) ds = \int_{1/2}^{1-(1/r)} G\left(\frac{1}{2}, s\right) ds = \frac{r^2-4}{16r^2}, \qquad 2 < r, \tag{3.5}$$

$$\int_{t_1}^{t_2} G(t_1, s)\, ds + \int_{1-t_2}^{1-t_1} G(t_1, s)\, ds = t_1(t_2 - t_1), \qquad 0 < t_1 < t_2 \le \frac{1}{2}, \tag{3.6}$$

$$\max_{0 \le r \le 1} \frac{G((1/2), r)}{G(t, r)} = \frac{1}{2t}, \qquad 0 < t \le \frac{1}{2}, \text{ and} \tag{3.7}$$

$$\min_{0 \le r \le 1} \frac{G(t_1, r)}{G(t_2, r)} = \frac{t_1}{t_2}, \qquad 0 < t_1 < t_2 \le \frac{1}{2}. \tag{3.8}$$

Next, for $0 < t_3 \le 1/2$ let

$$E = C[0, 1]$$

be endowed with the maximum norm,

$$\|y\| = \max_{0 \le t \le 1} |y(t)|,$$

and define the cone $P \subset E$ by

$$P = \left\{ y \in E \;\middle|\; \begin{array}{l} \text{y is concave, symmetric,} \\ \text{nonnegative valued on [0,1],} \\ \text{and} \displaystyle\min_{t \in [t_3, 1-t_3]} y(t) \geq 2t_3 \|y\| \end{array} \right\}.$$

Finally, let the nonnegative continuous concave functionals α, ψ and the nonnegative continuous convex functionals β, θ, and γ be defined on the cone P by

$$\gamma(y) = \max_{t \in [0,t_3] \cup [1-t_3,1]} y(t) = y(t_3), \tag{3.9}$$

$$\psi(y) = \min_{t \in [1/r,(r-1/r)]} y(t) = y\left(\frac{1}{r}\right), \tag{3.10}$$

$$\beta(y) = \max_{t \in [1/r,(r-1/r)]} y(t) = y\left(\frac{1}{2}\right), \tag{3.11}$$

$$\alpha(y) = \min_{t \in [t_1,t_2] \cup [1-t_2,1-t_1]} y(t) = y(t_1), \text{ and} \tag{3.12}$$

$$\theta(y) = \max_{t \in [t_1,t_2] \cup [1-t_2,1-t_1]} y(t) = y(t_2), \tag{3.13}$$

where t_1, t_2, and r are nonnegative numbers such that

$$0 < t_1 < t_2 \leq \frac{1}{2} \qquad \text{and} \qquad \frac{1}{r} \leq t_2.$$

We observe here that, for each $y \in P$,

$$\alpha(y) = y(t_1) \leq y\left(\frac{1}{2}\right) = \beta(y), \tag{3.14}$$

$$\|y\| = y\left(\frac{1}{2}\right) \leq \frac{1}{2t_3} y(t_3) = \frac{1}{2t_3} \gamma(y), \tag{3.15}$$

and also that $y \in P$ is a solution of (1.1),(1.2) if and only if

$$y(t) = \int_0^1 G(t,s) f(y(s))\, ds, \qquad 0 \leq t \leq 1. \tag{3.16}$$

We now present our result of the paper.

THEOREM 3.1. *Suppose there exists nonnegative numbers a, b, and c such that*

$$0 < a < b \leq \frac{ct_1}{t_2},$$

and suppose f satisfies the following conditions:

(i) $f(w) < (8r^2/(r^2-4))(a - (c/(r^2 t_3(1-t_3))))$ *for all* $w \in [(2a/r), a]$,
(ii) $f(w) \geq b/(t_1(t_2 - t_1))$ *for* $w \in [b, (t_2 b)/t_1]$,
(iii) $f(w) \leq (2c)/(t_3(1-t_3))$ *for* $w \in [0, c/(2t_3)]$.

Then, the second-order conjugate boundary value problem (1.1),(1.2) has three symmetric positive solutions y_1, y_2, y_3 such that

$$\max_{t \in [0,t_3] \cup [1-t_3,1]} y_i(t) \leq c, \qquad \text{for } i = 1,2,3,$$

$$\min_{t \in [t_1,t_2] \cup [1-t_2,1-t_1]} y_1(t) > b,$$

$$\max_{t \in [1/r,(r-1)/r]} y_2(t) < a,$$

and

$$\min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} y_3(t) < b, \qquad \text{with} \quad \max_{t\in[1/r,(r-1)/r]} y_3(t) > a.$$

PROOF. Define the completely continuous operator A by

$$Ay(t) = \int_0^1 G(t,s) f(y(s))\,ds.$$

We seek fixed-points of A which satisfy the conclusion of the theorem. We note first, if $y \in P$, then from properties of $G(t,s)$, $Ay(t) \geq 0$ and $(Ay)''(t) = -f(y(t)) \leq 0$, $0 \leq t \leq 1$, $Ay(t_3) \geq 2t_3 Ay(1/2)$, and $Ay(t) = Ay(1-t)$, $0 \leq t \leq 1/2$, and consequently, $Ay \in P$, that is, $A : P \to P$.

Also, for all $y \in P$, by (3.14), we have

$$\alpha(y) \leq \beta(y)$$

and, by (3.15), we have

$$\|y\| \leq \frac{1}{2t_3}\,\gamma(y).$$

If $y \in \overline{P(\gamma,c)}$, then $\|y\| \leq 1/(2t_3)\,\gamma(y) \leq c/(2t_3)$ and by assumption (iii) we have,

$$\begin{aligned}
\gamma(Ay) &= \max_{t\in[0,t_3]\cup[1-t_3,1]} \int_0^1 G(t,s) f(y(s))\,ds \\
&= \int_0^1 G(t_3,s) f(y(s))\,ds \\
&\leq \left(\frac{2c}{t_3(1-t_3)}\right) \int_0^1 G(t_3,s)\,ds \\
&= c.
\end{aligned}$$

Therefore, $A : \overline{P(\gamma,c)} \to \overline{P(\gamma,c)}$. It is immediate that

$$\left\{ y \in P\left(\gamma,\theta,\alpha,b,\frac{bt_2}{t_1},c\right) : \alpha(y) > b \right\} \neq \emptyset \quad \text{and} \quad \left\{ y \in Q\left(\gamma,\beta,\psi,\frac{2a}{r},a,c\right) : \beta(y) < a \right\} \neq \emptyset.$$

In the following claims, we verify the remaining conditions of the generalized Leggett-Williams fixed-point theorem, Theorem 2.4.

CLAIM 1. If $y \in Q(\gamma,\beta,a,c)$ with $\psi(Ay) < (2a)/r$ then $\beta(Ay) < a$.

$$\begin{aligned}
\beta(Ay) &= \max_{t\in[1/r,(r-1)/r]} \int_0^1 G(t,s) f(y(s))\,ds \\
&= \int_0^1 G\left(\frac{1}{2},s\right) f(y(s))\,ds \\
&= \int_0^1 \frac{G(1/2,s)}{G(1/r,s)}\, G\left(\frac{1}{r},s\right) f(y(s))\,ds \\
&\leq \frac{r}{2} \int_0^1 G\left(\frac{1}{r},s\right) f(y(s))\,ds \\
&= \frac{r}{2}\,\psi(Ay) < a.
\end{aligned}$$

CLAIM 2. If $y \in Q(\gamma,\beta,\psi,(2a)/r,a,c)$, then $\beta(Ay) < a$.

$$\begin{aligned}
\beta(Ay) &= \max_{t\in[1/r,(r-1)/r]} \int_0^1 G(t,s) f(y(s))\,ds \\
&= \int_0^1 G\left(\frac{1}{2},s\right) f(y(s))\,ds \\
&= 2\int_0^{1/r} G\left(\frac{1}{2},s\right) f(y(s))\,ds + 2\int_{1/r}^{1/2} G\left(\frac{1}{2},s\right) f(y(s))\,ds \\
&< \frac{c}{r^2 t_3(1-t_3)} + \left(\frac{8r^2}{r^2-4}\right)\left(a - \frac{c}{r^2 t_3(1-t_3)}\right)\left(\frac{r^2-4}{8r^2}\right) = a.
\end{aligned}$$

CLAIM 3. If $y \in P(\gamma, \alpha, b, c)$ with $\theta(Ay) > (bt_2)/t_1$, then $\alpha(Ay) > b$.

$$\begin{aligned}
\alpha(Ay) &= \min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} \int_0^1 G(t,s)f(y(s))\,ds \\
&= \int_0^1 G(t_1,s)\,f(y(s))\,ds \\
&= \int_0^1 \frac{G(t_1,s)}{G(t_2,s)}\,G(t_2,s)f(y(s))\,ds \\
&\ge \frac{t_1}{t_2}\int_0^1 G(t_2,s)f(y(s))\,ds \\
&= \frac{t_1}{t_2}\,\theta(Ay) > b.
\end{aligned}$$

CLAIM 4. If $y \in P(\gamma, \theta, \alpha, b, (bt_2)/t_1, c)$, then $\alpha(Ay) > b$.

$$\begin{aligned}
\alpha(Ay) &= \min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} \int_0^1 G(t,s)f(y(s))\,ds \\
&= \int_0^1 G(t_1,s)f(y(s))\,ds \\
&> \int_{t_1}^{t_2} G(t_1,s)f(y(s))\,ds + \int_{1-t_2}^{1-t_1} G(t_1,s)f(y(s))\,ds \\
&\ge \left(\frac{b}{t_1(t_2-t_1)}\right)\int_{t_1}^{t_2} G(t_1,s)\,ds + \left(\frac{b}{t_1(t_2-t_1)}\right)\int_{1-t_2}^{1-t_1} G(t_1,s)\,ds \\
&= \left(\frac{b}{t_1(t_2-t_1)}\right)\left(\frac{t_1\left[(1-t_1)^2-(1-t_2)^2\right]}{2}\right) + \left(\frac{b}{t_1(t_2-t_1)}\right)\left(\frac{t_1\left(t_2^2-t_1^2\right)}{2}\right) \\
&= b.
\end{aligned}$$

Therefore, the hypotheses of the generalized Leggett-Williams fixed-point theorem are satisfied and there exist three positive solutions $y_1, y_2, y_3 \in \overline{P(\gamma, c)}$ for the second-order conjugate boundary value problem (1.1),(1.2) such that

$$\alpha(y_1) > b,$$

$$\beta(y_2) < a,$$

and

$$\alpha(y_3) < b, \qquad \text{with } \beta(y_3) > a.$$

REMARK. We have chosen to perform the analysis when f is autonomous. However, if $f = f(t, y)$ and in addition, for each fixed y, $f(t, y)$ is symmetric about $t = 1/2$, then an analogous theorem would be valid with respect to the same cone P.

REFERENCES

1. R.P. Agarwal and P.J.Y. Wong, Double positive solutions of (n, p) boundary value problems for higher order difference equations, *Computers Math. Applic.* **32**, 1–21, (1996).
2. R.I. Avery and A.C. Peterson, Multiple positive solutions of a discrete second-order conjugate Problem, *PanAmerican Mathematical Journal* **8**, 1–12, (1998).
3. J.M. Davis, P.W. Eloe and J. Henderson, Triple positive solutions and dependence on higher order derivatives, *Journal of Mathematical Analysis and Applications* (to appear).
4. P.W. Eloe and J. Henderson, Positive solutions for $(n-1, 1)$ conjugate boundary value problems, *Nonlinear Analysis* **28**, 1669–1680, (1997).
5. L.H. Erbe, Boundary value problems for ordinary differential equations, *Rocky Mountain J. Math* **17**, 1–10, (1991).

6. L.H. Erbe, S. Hu and H. Wang, Multiple positive solutions of some boundary value problems, *Journal of Mathematical Analysis and Applications* **184**, 640–648, (1994).
7. L.H. Erbe and H. Wang, On the existence of positive solutions of ordinary differential equations, *Proceedings of the American Mathematical Society* **120**, 743–748, (1994).
8. F. Merdivenci, Two positive solutions of a boundary value problem for difference equations, *Journal of Difference Equations and Applications* **1**, 262–270, (1995).
9. F. Merdivenci, Positive solutions for focal point problems for $2n^{\mathrm{th}}$ order difference equations, *PanAmerican Mathematical Journal* **5**, 71-82, (1995).
10. P.J.Y. Wong, Triple positive solutions of conjugate boundary value problems, *Computers Math. Appl.* **36**, 19–35, (1998).
11. P.J.Y. Wong and R.P. Agarwal, Multiple solutions of difference and partial difference equations with Lidstone conditions, *Mathl. Comput. Modelling* (to appear).
12. R.P. Agarwal, P.J.Y. Wong and D. O'Regan, *Positive Solutions of Differential, Difference, & Integral Equations*, Kluwer Academic, Boston, MA, (1999).
13. R.I. Avery, Existence of multiple positive solutions to a conjugate boundary value problem, *MSR Hot-Line* **2**, 1–6, (1998).
14. J. Henderson and H.B. Thompson, Multiple symmetric positive solutions for a second-order boundary value problem, *Proceedings of the American Mathematical Society* (to appear).
15. R.I. Avery, A generalization of the Leggett-Williams fixed-point theorem, *MSR Hot-Line* (to appear).
16. K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, (1985).
17. D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, (1988).
18. R.W. Leggett and L.R. Williams, Multiple positive fixed-points of nonlinear operators on ordered Banach spaces, *Indiana University Mathematics Journal* **28**, 673–688, (1979).