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An algorithm for the number of path homomorphisms

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ABSTRACT

A homomorphism of a graph $G_1 = (V_1, E_1)$ to a graph $G_2 = (V_2, E_2)$ is a mapping from the vertex set V_1 of G_1 to the vertex set V_2 of G_2 which preserves edges. In this paper we provide an algorithm to determine the number of homomorphisms from an arbitrary finite undirected path to another arbitrary finite undirected path.

1. Introduction

We use in our paper standard notations and terminology from graph theory, see [3]. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be simple undirected graphs. A *graph homomorphism* (or *homomorphism*) of the graph G_1 into the graph G_2 is a mapping $f : V_1 \to V_2$ (we sometimes write $f : G_1 \to G_2$) from the vertex set V_1 into the vertex set V_2 which preserves edges, i. e. $\{x, y\} \in E_1$ implies $\{f(x), f(y)\} \in E_2$, for all $x, y \in V_1$. We denote by $\mathrm{Hom}(G_1, G_2)$, the set of all homomorphisms from the graph G_1 into the graph G_2.

A *path of length* n, P_n, is a graph with the vertex set $V_n = \{0, 1, \ldots, n\}$ and edge set $E_n = \{\{x, y\} \subseteq V_n : |x - y| = 1\}$. By P_∞ we denote an *infinite path* defined on the set of integers $\mathbb{Z}$ in which $\{x, y\}$ is an edge iff $|x - y| = 1$. Let $\mathrm{Hom}^i_j(P_m, P_n)$ and $\mathrm{Hom}^i_j(P_m, P_\infty)$ be the set of homomorphisms f from P_m to P_n or P_∞, respectively, which map 0 to i and m to j (that is $f(0) = i$ and $f(m) = j$, where i and j are integers). One may say that these (path) homomorphisms start with i and end with j. We also write $\mathrm{Hom}^i(P_m, P_n)$ and $\mathrm{Hom}^i(P_m, P_\infty)$ for the set of those homomorphisms which start with i.

Our aim is the structure analysis of graph homomorphisms, see [2] and [4]. The present paper is a sequel of [1]. We provide recursive formulas for the size of $\mathrm{Hom}^i_j(P_m, P_n)$ and $\mathrm{Hom}^i_j(P_m, P_\infty)$. These can be used to design an algorithm for computing $|\mathrm{Hom}^i_j(P_m, P_n)|$ in $O\,(mn)$ time using dynamic programming, see [5]. Further, we show that the formulas yield that $|\mathrm{Hom}^i_j(P_m, P_\infty)| = \binom{m}{\frac{m-i+j}{2}}$ and $|\mathrm{Hom}^i(P_m, P_\infty)| = 2^m$.

Clearly,

$$\mathrm{Hom}^i(P_m, P_n) = \bigcup_{j=0}^{n} \mathrm{Hom}^i_j(P_m, P_n) \qquad \mathrm{Hom}(P_m, P_n) = \bigcup_{i=0}^{n} \mathrm{Hom}^i(P_m, P_n).$$

Since the sets $\{\mathrm{Hom}^i_j(P_m, P_n)\}_{i,j}$ are pairwise disjoint, we can compute the number of homomorphisms $\mathrm{Hom}(P_m, P_n)$, using the following

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Proposition 1.1.

$$|\mathrm{Hom}(P_m, P_n)| = \sum_{i=0}^{n} |\mathrm{Hom}^i(P_m, P_n)| = \sum_{i=0}^{n}\sum_{j=0}^{n} |\mathrm{Hom}^i_j(P_m, P_n)|$$

where the symbol $|A|$ stands for the cardinality of (i.e. the number of elements in) a (finite) set A.

2. Recursive formulas

In this section we compute the numbers $|\mathrm{Hom}(P_m, P_\infty)|$ and give recursive formulas for $|\mathrm{Hom}(P_m, P_n)|$ with arbitrary (m, n)'s. It is clear that

Proposition 2.1. *For each $m \in \mathbb{N}$ and $i, j \in \mathbb{Z}$*

$$|\mathrm{Hom}^i_j(P_0, P_\infty)| = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$|\mathrm{Hom}^i_j(P_{m+1}, P_\infty)| = |\mathrm{Hom}^i_{j-1}(P_m, P_\infty)| + |\mathrm{Hom}^i_{j+1}(P_m, P_\infty)|.$$

Using the above recursive formulas we easily get the following table with the numbers $|\mathrm{Hom}^0_j(P_m, P_\infty)|$ for $m \le 7$:

$P_m \setminus j$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0
2	0	0	0	0	0	1	0	2	0	1	0	0	0	0	0
3	0	0	0	0	1	0	3	0	3	0	1	0	0	0	0
4	0	0	0	1	0	4	0	6	0	4	0	1	0	0	0
5	0	0	1	0	5	0	10	0	10	0	5	0	1	0	0
6	0	1	0	6	0	15	0	20	0	15	0	6	0	1	0
7	1	0	7	0	21	0	35	0	35	0	21	0	7	0	1
or	$\binom{7}{0}$	0	$\binom{7}{1}$	0	$\binom{7}{2}$	0	$\binom{7}{3}$	0	$\binom{7}{4}$	0	$\binom{7}{5}$	0	$\binom{7}{6}$	0	$\binom{7}{7}$

One should immediately recognize there Pascal's Triangle by use of which one can compute Newton's symbols

$$\binom{m}{r} = \begin{cases} \dfrac{m!}{r!(m-r)!} & \text{if } r = 0, 1, \ldots, m \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we easily get to the following general conclusion

Theorem 2.2. *For each $m \in \mathbb{N}$ and $i, j \in \mathbb{Z}$,*

$$|\mathrm{Hom}^i_j(P_m, P_\infty)| = \binom{m}{\frac{m-i+j}{2}} \quad \text{and} \quad |\mathrm{Hom}^i(P_m, P_\infty)| = 2^m.$$

Proof. We use mathematical induction. Note that $|\mathrm{Hom}^i_j(P_0, P_\infty)| = \binom{0}{\frac{j-i}{2}}$. Then, by Proposition 2.1, we also get

$$|\mathrm{Hom}^i_j(P_{m+1}, P_\infty)| = |\mathrm{Hom}^i_{j-1}(P_m, P_\infty)| + |\mathrm{Hom}^i_{j+1}(P_m, P_\infty)| = \binom{m}{\frac{m-i+j-1}{2}} + \binom{m}{\frac{m-i+j+1}{2}} = \binom{m+1}{\frac{m+1-i+j}{2}}.$$

Moreover,

$$|\mathrm{Hom}^i(P_m, P_\infty)| = \sum_{j=-\infty}^{\infty} |\mathrm{Hom}^i_j(P_m, P_\infty)| = \sum_{r=0}^{m} \binom{m}{r} = 2^m \quad \blacksquare$$

The computation of the numbers $|\mathrm{Hom}^i_j(P_m, P_n)|$ is not so easy. Clearly, without problems we get the following recursive formulas for them.

Proposition 2.3. *For each* $m, n \in \mathbb{N}$ *and* $i, j \in P_n$

$$|\mathrm{Hom}_j^i(P_0, P_n)| = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise}. \end{cases}$$

$$|\mathrm{Hom}_j^i(P_{m+1}, P_n)| = \begin{cases} |\mathrm{Hom}_1^i(P_m, P_n)|, & j = 0 \\ |\mathrm{Hom}_{j-1}^i(P_m, P_n)| + |\mathrm{Hom}_{j+1}^i(P_m, P_n)|, & 0 < j < n \\ |\mathrm{Hom}_{n-1}^i(P_m, P_n)|, & j = n. \end{cases}$$

They are sufficient for any practical computation of the numbers. In particular, we get the following table with $|\mathrm{Hom}_j^0(P_m, P_3)|$ for $m \le 7$.

Example 2.4.

$P_m \setminus j$	0	1	2	3
0	1	0	0	0
1	0	1	0	0
2	1	0	1	0
3	0	2	0	1
4	2	0	3	0
5	0	5	0	3
6	5	0	8	0
7	0	13	0	8

There remains, however, to unwind the recursive formulas to get an explicit presentation of $|\mathrm{Hom}(P_m, P_n)|$.

3. The number of path homomorphisms

In this section we define an explicit formula for the computation of $|\mathrm{Hom}(P_m, P_n)|$, for arbitrary numbers m and n, by a combinatorial argument. The main step in our reasoning is the following

Theorem 3.1. *If* $0 \le i \le n$ *and* $0 \le j \le n$*, then*

$$|\mathrm{Hom}_j^i(P_m, P_n)| = \sum_{t=-m}^{m} \left(|\mathrm{Hom}_{2t(n+2)+j}^i(P_m, P_\infty)| - |\mathrm{Hom}_{2t(n+2)-2-j}^i(P_m, P_\infty)| \right).$$

Proof. Let $i, j \in P_n$. For each $f \in \mathrm{Hom}_j^i(P_m, P_\infty)$, we obviously have

$$f \in \mathrm{Hom}_j^i(P_m, P_n) \Leftrightarrow \forall_x \, (f(x) \notin \{-1, n+1\}).$$

Thus, to compute $|\mathrm{Hom}_j^i(P_m, P_n)|$ it is sufficient to know $|\mathrm{Hom}_j^i(P_m, P_\infty)|$ and subtract from this the number of those path homomorphisms which take as (one of) its values -1 or $n + 1$. Note that if m is large relatively to n then there may exist such homomorphisms which take both numbers as their values. To make these two types of homomorphisms disjoint, let us define

$$L_j^i = \left\{ f \in \mathrm{Hom}_j^i(P_m, P_\infty) : \exists_{x_0} \left[f(x_0) = -1 \ \wedge \ \forall_{x > x_0} \, (0 \le f(x) \le n) \right] \right\}$$

$$R_j^i = \left\{ f \in \mathrm{Hom}_j^i(P_m, P_\infty) : \exists_{x_0} \left[f(x_0) = n + 1 \ \wedge \ \forall_{x > x_0} \, (0 \le f(x) \le n) \right] \right\}.$$

Then it is clear that

$$\mathrm{Hom}_j^i(P_m, P_\infty) = \mathrm{Hom}_j^i(P_m, P_n) \ \cup \ L_j^i \ \cup \ R_j^i \tag{1}$$

and the three sets on the right-hand side of the equation are disjoint. We need to compute $|R_j^i|$ and $|L_j^i|$. To perform this task we need, however, a recursive formula. So, for each integer t and each j' such that $t \cdot (n+2) \le j' \le t \cdot (n+2) + n$, we define subsets of $\mathrm{Hom}_{j'}^i(P_m, P_\infty)$ (assuming that $f \in \mathrm{Hom}_{j'}^i(P_m, P_\infty)$)

$$L_{j'}^i(t) = \{ f : \exists_{x_0} \left[f(x_0) = t \cdot (n+2) - 1 \ \wedge \ \forall_{x > x_0} \, (0 \le f(x) - t \cdot (n+2) \le n) \right] \}$$

$$R_{j'}^i(t) = \{ f : \exists_{x_0} \left[f(x_0) = t \cdot (n+2) + n + 1 \ \wedge \ \forall_{x > x_0} \, (0 \le f(x) - t \cdot (n+2) \le n) \right] \}.$$

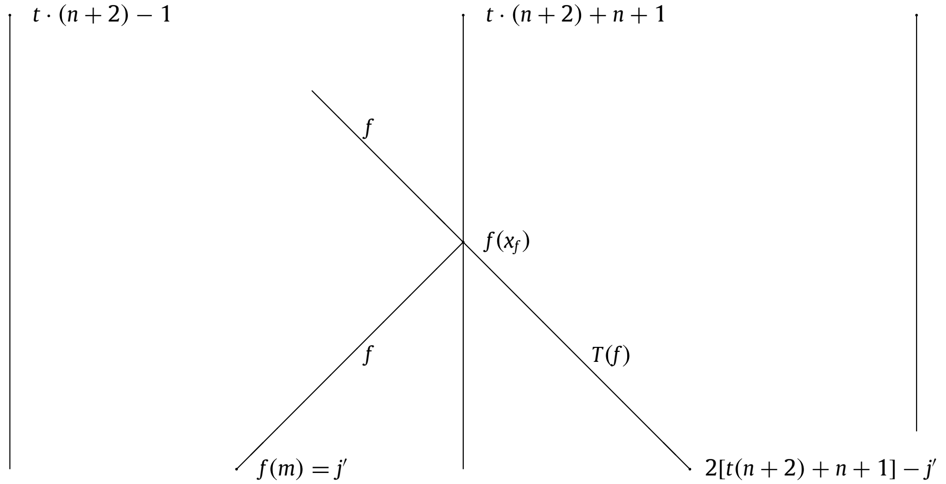
Note that $R_j^i(0) = R_j^i$ and $L_j^i(0) = L_j^i$. Let us prove

$$|R_{j'}^i(t)| = |L_{2[t \cdot (n+2)+n+1]-j'}^i(t+1)| \quad \text{for each } t \ge 0. \tag{2}$$

Let $f \in \mathrm{Hom}(P_m, P_\infty)$ and $f^{-1}(\{t \cdot (n+2) + n + 1\}) \neq \emptyset$. Assume that x_f is the greatest number in $\{0, \dots, m\}$ such that $f(x_f) = t \cdot (n+2) + n + 1$. We define $T(f) \in \mathrm{Hom}(P_m, P_\infty)$ setting

$$T(f)(x) = \begin{cases} f(x) & \text{if } x \leq x_f \\ 2[t \cdot (n+2) + n + 1] - f(x) & \text{if } x \geq x_f. \end{cases} \tag{3}$$

We may assume that $T(f) = f$ if $f^{-1}(\{t \cdot (n+2) + n + 1\}) = \emptyset$. Thus, we define a transformation T on the set of path homomorphisms $\mathrm{Hom}(P_m, P_\infty)$. Geometrically, $T(f)$ results from f by reflecting its final part (for $x > x_f$) along the axis $y = t \cdot (n+2) + n + 1$:



One should easily notice that $TT(f) = f$ and hence the transformation T is one-to-one. If one restricts the transformation to the set $R^i_{j'}(t)$, assuming that $0 \leq j' - t \cdot (n+2) \leq n$, one gets a one-to-one correspondence

$$T : R^i_{j'}(t) \leftrightarrow L^i_{2[t \cdot (n+2)+n+1]-j'}(t+1).$$

Indeed, if $f \in R^i_{j'}(t)$, then $f(x_f) = t \cdot (n+2) + n + 1$ and $0 \leq f(x) - t \cdot (n+2) \leq n$ for each $x > x_f$. Thus, by (3), we get $T(f)(x_f) = (t+1) \cdot (n+2) - 1$ and

$$0 \leq T(f)(x) - (t+1) \cdot (n+2) \leq n \quad \text{for each } x > x_f$$

which means that $T(f) \in L^i_{2[t \cdot (n+2)+n+1]-j'}(t+1)$. On the other hand, if we have $g \in L^i_{2[t \cdot (n+2)+n+1]-j'}(t+1)$, then $T(g) \in R^i_{j'}(t)$ and hence by $TT(g) = g$ one concludes that T establishes a one-to-one correspondence between elements of the set $R^i_{j'}(t)$ and $L^i_{2[t \cdot (n+2)+n+1]-j'}(t+1)$. This completes our proof of (2).

Since $0 \leq i \leq n$, then for each $t \neq 0$ and $t \cdot (n+2) \leq j' \leq t \cdot (n+2) + n$, we have

$$\mathrm{Hom}^i_{j'}(P_m, P_\infty) = L^i_{j'}(t) \cup R^i_{j'}(t) \tag{4}$$

and the sets on the right-hand side of the equation are disjoint. Now, by (2), we get $|R^i_j| = |R^i_j(0)| = |L^i_{2n+2-j}(1)|$. Hence, using (4),

$$|R^i_j| = |L^i_{2n+2-j}(1)| = \mathrm{Hom}^i_{2n+2-j}(P_m, P_\infty) - |R^i_{2n+2-j}(1)|.$$

Again, using (2), we get $|R^i_{2n+2-j}(1)| = |L^i_{2[(n+2)+n+1]-2n-2+j}(2)| = |L^i_{2(n+2)+j}(2)|$, and hence, by (4),

$$|R^i_j| = \mathrm{Hom}^i_{2n+2-j}(P_m, P_\infty) - |\mathrm{Hom}^i_{2(n+2)+j}(P_m, P_\infty)| + |R^i_{2(n+2)+j}(2)|.$$

Repeating the above argument at infinity, we obtain (easy induction steps are omitted)

$$\begin{aligned} |R^i_j| = {} & |\mathrm{Hom}^i_{2n+2-j}(P_m, P_\infty)| - |\mathrm{Hom}^i_{2(n+2)+j}(P_m, P_\infty)| + |\mathrm{Hom}^i_{4n+6-j}(P_m, P_\infty)| \\ & - |\mathrm{Hom}^i_{4(n+2)+j}(P_m, P_\infty)| + |\mathrm{Hom}^i_{6n+10-j}(P_m, P_\infty)| - |\mathrm{Hom}^i_{6(n+2)+j}(P_m, P_\infty)| + \cdots. \end{aligned}$$

In a similar way one can compute $|L^i_j|$. First, we get (by symmetry)

$$|L^i_{j'}(t)| = |R^i_{2[t \cdot (n+2)-1]-j'}(t-1)|$$

for each $t \leq 0$ and $t \cdot (n+2) \leq j' \leq t \cdot (n+2) + n$. Then, by (4),

$$\begin{aligned} |L^i_j| = {} & |\mathrm{Hom}^i_{-2-j}(P_m, P_\infty)| - |\mathrm{Hom}^i_{-2(n+2)+j}(P_m, P_\infty)| + |\mathrm{Hom}^i_{-2n-6-j}(P_m, P_\infty)| \\ & - |\mathrm{Hom}^i_{-4(n+2)+j}(P_m, P_\infty)| + |\mathrm{Hom}^i_{-4n-10-j}(P_m, P_\infty)| - |\mathrm{Hom}^i_{-6(n+2)+j}(P_m, P_\infty)| + \cdots. \end{aligned}$$

Thus, by (1),

$$|\mathrm{Hom}^i_j(P_m, P_n)| = \sum_{t=-\infty}^{\infty} \left(|\mathrm{Hom}^i_{2t(n+2)+j}(P_m, P_\infty)| - |\mathrm{Hom}^i_{2t(n+2)-2-j}(P_m, P_\infty)|\right).$$

To complete our proof it suffices to notice that the above sum is finite as it contains only finitely many non-zero elements. The very rough bound is $|t| \le m$ as $|f(m)| \le i + m \le m + n$ for each $f \in \mathrm{Hom}^i(P_m, P_\infty)$, and hence $\mathrm{Hom}^i_{2t(n+2)+j}(P_m, P_\infty) = \mathrm{Hom}^i_{2t(n+2)-2-j}(P_m, P_\infty) = \emptyset$ if $|t| > m$. ■

Since, for each integer t, $\sum_{j=0}^{n} |\mathrm{Hom}^i_{2t(n+2)-2-j}| = \sum_{j=0}^{n} |\mathrm{Hom}^i_{(2t-1)(n+2)+j}|$, by the above theorem and Proposition 1.1, we obtain

Corollary 3.2. *If* $0 \le i \le n$, *then*

$$|\mathrm{Hom}^i(P_m, P_n)| = \sum_{t=-m}^{m} (-1)^t \sum_{j=0}^{n} |\mathrm{Hom}^i_{t\cdot(n+2)+j}(P_m, P_\infty)|.$$

The advantage to be gained from our characterization of $|\mathrm{Hom}^i(P_m, P_n)|$ is that the computation of the numbers $|\mathrm{Hom}^i_j(P_m, P_\infty)|$ is easy. It suffices to match the numbers $|\mathrm{Hom}^i_j(P_m, P_\infty)|$ with the corresponding Newton's symbols, see Theorem 2.2. We obviously have

Theorem 3.3. 1. *If* $m - i$ *is even, then*

$$|\mathrm{Hom}^i(P_m, P_n)| = \sum_{t=-m}^{m} (-1)^t \sum_{u=0}^{\lfloor \frac{n}{2} \rfloor} \binom{m}{\frac{m-i}{2} + u + \lceil \frac{(n+2)t}{2} \rceil}.$$

2. *If* $m - i$ *is odd, then*

$$|\mathrm{Hom}^i(P_m, P_n)| = \sum_{t=-m}^{m} (-1)^t \sum_{u=1}^{\lceil \frac{n}{2} \rceil} \binom{m}{\lfloor \frac{m-i}{2} \rfloor + u + \lfloor \frac{(n+2)t}{2} \rfloor}.$$

As it has been already mentioned one can reduce the number of elements in the first sum, if one improves the estimation $|t| \le m$. It suffices to take into account the obvious fact that if t is big enough then Newton's symbols occurring in the formula take the value 0. However, if there are no limits on the number m then the 'double-sum' presentation of $|\mathrm{Hom}^i(P_m, P_n)|$ seems to be unavoidable.

The situation changes if m is restricted with respect to n. The above formula reduces, for instance, if we count the number of endomorphisms on P_n, that is $End(P_n) = \mathrm{Hom}(P_n, P_n)$. Since $m = n$ in this case, we get $|t| \le 1$ and hence

Corollary 3.4. *If* $0 \le i \le n$, *then*

$$|\mathrm{End}^i(P_n)| = \sum_{u=\lceil \frac{n-i}{2} \rceil}^{n-\lceil \frac{i}{2} \rceil} \binom{n}{u} - \sum_{u=0}^{\lfloor \frac{n-i}{2} \rfloor - 1} \binom{n}{u} - \sum_{u=n-\lfloor \frac{i}{2} \rfloor + 1}^{n} \binom{n}{u}.$$

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