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On the periodic Wigner–Poisson–Fokker–Planck system

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Abstract

This paper is concerned with the existence and uniqueness analysis of global classical solutions of a diffusive quantum evolution equation with nonlinear coupling to the Poisson equation. The main technical difficulty in the existence proof is to show that the quantum Fokker–Planck term is a semigroup-generator in a weighted L^2-space. The potential term is then a Lipschitz perturbation of it.

Keywords: Wigner equation; Fokker–Planck equation; Diffusive quantum systems; Semigroup theory

1. Introduction

The object of this paper is the analysis of the coupled Wigner–Poisson–Fokker–Planck (WPFP) system in one dimension with periodic boundary conditions in the spatial direction. We focus on the existence and uniqueness of global-in-time solutions to this system.

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Wigner functions provide a kinetic description of quantum mechanics (cf. [14]) and have recently become a valuable modeling and simulation tool in fields like semiconductor device modeling (cf. [9] and references therein), quantum Brownian motion, and quantum optics [4,6]. The real-valued Wigner function $w(x,v,t)$ is a probabilistic quasi-distribution function in the position–velocity (x,v) phase space for the considered quantum system at time t.

Its temporal evolution is governed by the Wigner–Fokker–Planck (WFP) equation

$$w_t + v w_x + \Theta[V]w = \beta(vw)_v + \sigma w_{vv} + 2\gamma w_{xv} + \alpha w_{xx}, \quad t > 0, \quad (1.1)$$

on the phase space slab $x \in (0, 2\pi)$, $v \in \mathbb{R}$ with periodic boundary conditions in x

$$w(0, v, t) = w(2\pi, v, t),$$

and the initial condition

$$w(x, v, t = 0) = w^I(x, v).$$

With a vanishing right-hand side Eq. (1.1) would be the (diffusion-free) Wigner equation. It describes the reversible evolution of a quantum system under the action of a (possibly time-dependent) electrostatic potential $V = V(x,t)$. Its effect enters in the equation via the pseudo-differential operator $\Theta[V]$:

$$\begin{aligned}\big(\Theta[V]w\big)(x,v,t) &= i\left[V\left(x + \frac{1}{2i}\nabla_v, t\right) - V\left(x - \frac{1}{2i}\nabla_v, t\right)\right]w(x,v,t) \\ &= \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \delta V(x,\eta,t)\mathcal{F}_v w(x,\eta,t)e^{iv\eta}\,d\eta \\ &= \frac{i}{2\pi} \int_{\mathbb{R}}\int_{\mathbb{R}} \delta V(x,\eta,t) w(x,v',t)e^{i(v-v')\eta}\,dv'\,d\eta, \quad (1.2)\end{aligned}$$

where $\delta V(x,\eta,t) = V(x+\eta/2,t) - V(x-\eta/2,t)$ and $\mathcal{F}_v w$ denotes the Fourier transform of w with respect to v:

$$\mathcal{F}_v w(x,\eta,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} w(x,v',t)e^{-iv'\eta}\,dv'.$$

For simplicity of the notation we have here set the Planck constant, particle mass and charge equal to unity.

The right-hand side of (1.1) is a Fokker–Planck type model for the nonreversible interaction of this quantum system with an environment, e.g., the interaction of electrons with a phonon bath (cf. [7,13]). In (1.1), $\beta \geqslant 0$ is the friction parameter and the parameters α, $\gamma \geqslant 0$, $\sigma > 0$ constitute the phase-space diffusion matrix of the system. In the kinetic Fokker–Planck equation of classical mechanics (cf. [5,12]) one would have $\alpha = \gamma = 0$. For the WFP equation (1.1) we have to assume

$$\begin{pmatrix} \alpha & \gamma + \frac{i}{4}\beta \\ \gamma - \frac{i}{4}\beta & \sigma \end{pmatrix} \geqslant 0,$$

which guarantees that the system is *quantum mechanically correct*. More precisely, it guarantees that the corresponding von Neumann equation is in Lindblad form and that the density matrix of the quantum system stays a positive operator under temporal evolution (see [2] for details).

In the sequel we shall hence assume

$$\alpha\sigma \geqslant \gamma^2 + \frac{\beta^2}{16}. \tag{1.3}$$

However, the subsequent mathematical analysis will even hold for

$$\alpha\sigma \geqslant \gamma^2.$$

The WFP equation (1.1) is self-consistently coupled with the Poisson equation for the (real-valued) potential $V[w](x,t)$:

$$V_{xx} = n[w] - D, \quad x \in (x, 2\pi),\ t > 0,$$
$$V(0,t) = V(2\pi,t),$$

with the particle density

$$n[w](x,t) = \int_{\mathbb{R}} w(x,v,t)\,dv. \tag{1.4}$$

$D = D(x)$ denotes the density of some fixed charges ("doping profile" in the context of semiconductor modeling), which is assumed to be given.

Mathematical properties of the Wigner–Poisson equation and dissipative Wigner systems have been intensively studied in the last decade (see [1,9] and references therein). The (friction-free) WPFP equation in 3 dimensions was first analyzed in [2], where unique local-in-time solutions were constructed. The main analytical challenge of Wigner–Poisson systems lies in controlling the particle density (1.4) in appropriate L^p-spaces. Usually this is achieved by either reformulating the Wigner equation as a Schrödinger system or a von Neumann equation [1,9] or by exploiting the dissipative structure of the system [2]. The 1-dimensional Wigner–Poisson equation, however, allows for a "direct" analysis (cf. [3, §5]). Hence our interest in this analytical framework.

2. Existence and uniqueness of global-in-time solution

In this section we shall establish existence and uniqueness of global mild and classical solutions to the WPFP system (1.1)–(1.4). This system will be considered as an evolution problem in the weighted (real-valued) L^2-space

$$X = L^2\big((0, 2\pi) \times \mathbb{R}; (1 + v^2)\, dx\, dv\big),$$

endowed with the scalar product

$$\langle u, w\rangle_X = \int_0^{2\pi}\int_{\mathbb{R}} uw(1 + v^2)\, dv\, dx.$$

This choice of the space X allows to define the particle density $n[w]$ of a Wigner function $w \in X$: a simple estimate (using Cauchy–Schwartz) yields

$$\big\|n[w]\big\|_{L^2(0,2\pi)} \leqslant C\|w\|_X. \tag{2.1}$$

Here and in the sequel C denotes generic, but not necessarily equal constants.

We shall use semigroup techniques to prove existence and uniqueness of a solution to the semilinear WPFP system (1.1)–(1.4). To this end the quadratically nonlinear potential term $\Theta[V]w$ will be considered as a bounded perturbation in the kinetic Fokker–Planck equation $w_t + vw_x = \beta(vw)_v + \sigma w_{vv} + 2\gamma w_{xv} + \alpha w_{xx}$.

We first consider the unbounded linear operator $A : D(A) \to X$,

$$Au = -v\partial_x u + \beta\partial_v(vu) + \sigma\partial_v^2 u + 2\gamma\partial_v\partial_x u + \alpha\partial_x^2 u, \tag{2.2}$$

defined on

$$D(A) = \big\{u \in X \mid vu_x, u_{vv}, vu_v, u_{xx}, u_{xv} \in X;\\ u(0, v) = u(2\pi, v),\ u_x(0, v) = u_x(2\pi, v)\ \forall v \in \mathbb{R}\big\}.$$

Clearly, the restriction (to $(0, 2\pi) \times \mathbb{R}$) of $C^\infty(\mathbb{R}^2)$-functions that are 2π-periodic in x and have a compact support in v are included in $D(A)$. Hence, $D(A)$ is dense in X. A simple calculation shows that for $u \in D(A)$, u_v is also in X.

A straightforward calculation using the periodicity in x and integrations by part yields

$$\langle Au, w\rangle_X = \langle u, A_1^* w\rangle_X + \langle u, A_2^* w\rangle_X, \quad \forall u, w \in D(A),$$

with

$$A_1^* w = v\partial_x w - \beta v\partial_v w + \sigma\partial_v^2 w + 2\gamma\partial_v\partial_x w + \alpha\partial_x^2 w,$$
$$A_2^* w = \frac{1}{1 + v^2}\big[2\sigma(w + 2v\partial_v w) - 2\beta v^2 w + 4\gamma v w_x\big].$$

Hence, $A^*|_{D(A)}$—the restriction of the adjoint of the operator A to $D(A)$—is given by $A^* w = A_1^* w + A_2^* w$, $w \in D(A)$. A^* is densely defined on $D(A^*) \supseteq$

$D(A)$, and hence A is a closable operator (cf. [11, Theorem VIII.1.b]). Its closure $\overline{A}$ satisfies $(\overline{A})^* = A^*$ (cf. [11, Theorem VIII.1.c]).

Next we study the dissipation property of the operator A, which is defined on the Hilbert space X (over $\mathbb{R}$) by

$$\langle Au, u\rangle_X \leqslant 0, \quad \forall u \in D(A).$$

Lemma 2.1. *Let the coefficients of the operator A satisfy $\alpha\sigma \geqslant \gamma^2$. Then $A - (\sigma + \beta/2)I$ and its closure are dissipative.*

Proof. Using integrations by part we have for $u \in D(A)$

$$\begin{aligned}
\langle Au, u\rangle_X = &- \iint vu_x u + \beta \iint (vu)_v u + \sigma \iint u_{vv} u + 2\gamma \iint u_{xv} u \\
&+ \alpha \iint u_{xx} u - \iint v^3 u_x u + \beta \iint v^2 (vu)_v u \\
&+ \sigma \iint v^2 u_{vv} u + 2\gamma \iint v^2 u_{xv} u + \alpha \iint v^2 u_{xx} u \\
= &-\beta \iint uvu_v - \sigma \iint u_v u_v + 2\gamma \iint u_{xv} u - \alpha \iint u_x u_x \\
&- \beta \iint (v^2 u)_v vu - \sigma \iint (v^2 u)_v u_v + 2\gamma \iint (vu)_{xv} vu \\
&- 2\gamma \iint u_x vu - \alpha \iint v^2 u_x u_x,
\end{aligned}$$

where $\iint f$ denotes the integral $\int_0^{2\pi}\int_{\mathbb{R}} f(x, v)\, dv\, dx$.

For the two integrals of the right side that involve mixed x–v derivatives we shall now use the interpolation inequality

$$\iint u_{xv} u \leqslant \frac{\epsilon}{2} \|u_x\|_2^2 + \frac{1}{2\epsilon} \|u_v\|_2^2, \quad \epsilon > 0, \tag{2.3}$$

which is immediately obtained by an integration by parts (in v) and Young's inequality. With $\epsilon = \gamma/\sigma$ we then obtain

$$\begin{aligned}
\langle Au, u\rangle_X \leqslant &\frac{\beta}{2}\|u\|_2^2 - \sigma\|u_v\|_2^2 + \epsilon\gamma\|u_x\|_2^2 + \frac{1}{\epsilon}\gamma\|u_v\|_2^2 \\
&- \alpha\|u_x\|_2^2 - 2\beta\|vu\|_2^2 - \beta \iint v^3 u_v u - 2\sigma \iint vuu_v \\
&-\sigma\|vu_v\|_2^2 + \epsilon\gamma\|vu_x\|_2^2 + \frac{1}{\epsilon}\gamma\big\|(vu)_v\big\|_2^2 - \alpha\|vu_x\|_2^2 \\
= &\frac{\beta}{2}\|u\|_2^2 + \frac{\gamma^2}{\sigma}\|u_x\|_2^2 - \alpha\|u_x\|_2^2 - 2\beta\|vu\|_2^2 + \frac{3}{2}\beta\|vu\|_2^2 \\
&+ \sigma\|u\|_2^2 + \frac{\gamma^2}{\sigma}\|vu_x\|_2^2 - \alpha\|vu_x\|_2^2
\end{aligned}$$

$$\leqslant \left(\sigma + \frac{\beta}{2}\right)\|u\|_2^2.$$

Thus

$$\left\langle \left[A - \left(\sigma + \frac{\beta}{2}\right)I\right]u, u\right\rangle_X \leqslant -\sigma\|vu\|_2^2 - \frac{\beta}{2}\|vu\|_2^2 \leqslant 0 \tag{2.4}$$

and the operator $A - (\sigma + \beta/2)I$ is dissipative. By Theorem 1.4.5b of [10] its closure,

$$\overline{A - \left(\sigma + \frac{\beta}{2}\right)I} = \overline{A} - \left(\sigma + \frac{\beta}{2}\right)I,$$

is also dissipative. □

It is easy to see that the operator $A - (\beta/2)I$ defined on $\widetilde{D(A)} = \{u \in L^2((0, 2\pi) \times \mathbb{R}) \mid vu_x, u_{vv}, vu_v, u_{xx} \in L^2((0, 2\pi) \times \mathbb{R});\ u(0, v) = u(2\pi, v), u_x(0, v) = u_x(2\pi, v),\ \forall v \in \mathbb{R}\}$ is dissipative in $L^2((0, 2\pi) \times \mathbb{R})$ and the L^2-adjoint of A is $A^* = A_1^*$ on $\widetilde{D(A)}$.

Let us now study the dissipativity of the operator A^* restricted to $D(A)$. Analogously to Lemma 2.1 we have

$$\langle A^*u, u\rangle_X \leqslant \left(\sigma + \frac{\beta}{2}\right)\|u\|_2^2, \quad \forall u \in D(A).$$

Hence the restriction of the operator $A^* - (\sigma + \beta/2)I = [A - (\sigma + \beta/2)I]^*$ to $D(A)$ is dissipative.

Next we consider the dissipativity of this operator on its proper domain $D(A^*)$, which, however, is not known explicitly. To this end we shall use the following technical lemma whose proof is deferred to Appendix A. Here we shall denote by $\tilde{u}$ the (in x) 2π-periodic extension of a function $u \in X$ to $\mathbb{R}^2$.

Lemma 2.2. *Let $P := p(v, \partial_x, \partial_v)$ be a linear operator in X, where p is a quadratic polynomial and*

$$D(P) := \left\{u \in X \mid \tilde{u} \in C^\infty(\mathbb{R}^2) \text{ with compact support in } v\right\} \subset X.$$

Then $\overline{P}$ is the maximum extension of P in the sense that

$$D(\overline{P}) := \left\{u \in X \mid \text{the distribution } Pu \in X\right\}.$$

We now apply Lemma 2.2 to $P = A^* - (\sigma + \beta/2)I$, which is dissipative on $D(P) \subset D(A)$. Since A^* is closed, we have $D(A^*) = D(\overline{P}) = \{u \in X \mid A^*u \in X\}$ and $A^* - (\sigma + \beta/2)I$ is dissipative on all of $D(A^*)$.

Applying Corollary 1.4.4 of [10] to $\overline{A} - (\sigma + \beta/2)I$ (with $(\overline{A})^* = A^*$) then implies that $\overline{A} - (\sigma + \beta/2)I$ generates a C_0 semigroup of contractions on X, and the C_0 semigroup generated by $\overline{A}$ satisfies

$$\left\|e^{t\overline{A}}u\right\|_X \leqslant e^{(\sigma+\beta/2)t}\|u\|_X, \quad u \in X,\ t \geqslant 0.$$

By the same arguments $\overline{A} - (\beta/2)I$ generates a C_0 semigroup of contractions on the space $L^2((0, 2\pi) \times \mathbb{R})$.

Next we shall analyze the properties of the quadratically nonlinear term $\Theta[V]w$, which will later be considered as a perturbation of the generator $\overline{A}$.

For $V \in L^\infty(\mathbb{R})$ the pseudo-differential operator $\Theta[V]$ from (1.2) is defined by

$$\big(\mathcal{F}_v \Theta[V]u\big)(x, \eta) = i\delta V(x, \eta)\mathcal{F}_v u(x, \eta), \quad u \in L^2\big((0, 2\pi) \times \mathbb{R}_v\big).$$

Since $\delta V(x, \eta) \in \mathbb{R}$, the operator $\Theta[V]$ is skew-symmetric on $L^2((0, 2\pi) \times \mathbb{R}_v)$ and it satisfies (cf. [3,8])

$$\big\|\Theta[V]\big\|_{\mathcal{B}(L^2((0,2\pi)\times\mathbb{R}_v))} \leqslant 2\|V\|_\infty.$$

For $V \in L^\infty(\mathbb{R})$ we define the pseudo-differential operator $\Omega[V]$ on $L^2((0, 2\pi) \times \mathbb{R}_v)$ by

$$\begin{aligned}
&\big(\Omega[V]u\big)(x, v) \\
&\quad = \frac{1}{2}\left[V\left(x + \frac{1}{2i}\nabla_v\right) + V\left(x - \frac{1}{2i}\nabla_v\right)\right]u(x, v) \\
&\quad = \frac{1}{2\sqrt{2\pi}} \int\limits_{\mathbb{R}} \left[V\left(x + \frac{\eta}{2}\right) + V\left(x - \frac{\eta}{2}\right)\right]\mathcal{F}_v u(x, \eta)e^{iv\eta}\, d\eta.
\end{aligned} \tag{2.5}$$

As for the operator $\Theta[V]$ we obtain

$$\big\|\Omega[V]\big\|_{\mathcal{B}(L^2((0,2\pi)\times\mathbb{R}_v))} \leqslant \|V\|_\infty. \tag{2.6}$$

Proposition 2.3. *Let $V \in W^{1,\infty}(\mathbb{R})$. Then,*

$$\Theta[V](vw) = v\Theta[V]w + \Omega[V_x]w \tag{2.7}$$

holds for $w \in X$.

Proof. By partial integration we obtain

$$\begin{aligned}
\Theta[V](vw) &= \frac{i}{2\pi} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \left(V\left(x + \frac{\eta}{2}\right) - V\left(x - \frac{\eta}{2}\right)\right) \\
&\qquad \times v' w(x, v')e^{i(v-v')\eta}\, dv'\, d\eta \\
&= \frac{i}{2\pi} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \left[\left(V\left(x + \frac{\eta}{2}\right) - V\left(x - \frac{\eta}{2}\right)\right)w(x, v')e^{iv\eta}\right] \\
&\qquad \times \left[v' e^{-iv'\eta}\right] d\eta\, dv' \\
&= \frac{1}{4\pi} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \left(V_x\left(x + \frac{\eta}{2}\right) + V_x\left(x - \frac{\eta}{2}\right)\right)
\end{aligned}$$

$$\begin{aligned}
&\times w(x,v')e^{i(v-v')\eta}\,d\eta\,dv' \\
&+\frac{i}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} v\left(V\left(x+\frac{\eta}{2}\right)-V\left(x-\frac{\eta}{2}\right)\right) \\
&\times w(x,v')e^{i(v-v')\eta}\,d\eta\,dv' \\
={}&\Omega[V_x]w+v\Theta[V]w. \qquad \square
\end{aligned}$$

Now, let us consider the nonlinear operator B defined on X by

$$u\mapsto Bu:=-\Theta\big[V[u]\big]u,$$

where $V[u]$ is the 2π-periodically extended solution of the Poisson equation

$$\begin{aligned}
&V_{xx}=n[u]-D, \quad x\in(0,2\pi),\\
&V(0)=V(2\pi),
\end{aligned} \tag{2.8}$$

with $n[u](x)=\int_{\mathbb{R}}u(x,v)\,dv$.

Lemma 2.4. *Let $D\in L^1(0,2\pi)$. Then*

(a) *B maps X into itself.*
(b) *Moreover, the operator B is of class C^∞ in X, and satisfies*

$$\|Bu_1-Bu_2\|_X\leqslant C\big(\|u_1\|_X+\|u_2\|_X+\|D\|_{L^1(0,2\pi)}\big)\|u_1-u_2\|_X,$$

for $u_1,u_2\in X$.

For the simple proof we refer the reader to [3].

Remark 2.5. In the proof of Lemma 2.4 it is essential that $\|u\|_X$ controls $n[u]$ in $L^1(0,2\pi)$ (see (2.1)). Hence the solution of the Poisson equation (2.8) satisfies $V[u]\in W^{1,\infty}(0,2\pi)$ and $\|\Theta[V[u]]\|_{\mathcal{B}(X)}\leqslant C\|V[u]\|_{W^{1,\infty}(\mathbb{R})}$.

We rewrite the WPFP system as

$$\begin{aligned}
&w_t=\overline{A}w+Bw, \quad t>0,\\
&w(t=0)=w^I\in X.
\end{aligned} \tag{2.9}$$

The main result of this paper is

Theorem 2.6. *Let $D\in L^1(0,2\pi)$.*

(a) *For every $w^I\in X$, the WPFP problem* (2.9) *has a unique mild solution $w\in C([0,\infty),X)$.*
(b) *If $w^I\in D(\overline{A})$, w is a classical solution, i.e., $w\in C^1([0,\infty),X)$ and $w(t)\in D(\overline{A})$ for $t\geqslant 0$.*

Proof. We consider B as a bounded perturbation of the generator $\overline{A}$. Since B is locally Lipschitz continuous, Theorem 6.1.4 of [10] shows that (2.9) has a unique mild solution for every $w^I \in X$ on some time interval $[0, t_{\max})$. Moreover, if $t_{\max} = t_{\max}(w^I) < \infty$ then $\lim_{t \nearrow t_{\max}} \|w\|_X = \infty$. Since B is of class C^∞ in X, Theorem 6.1.5 in [10] proves that w is a classical solution on $[0, t_{\max})$ for $w^I \in D(\overline{A})$.

To prove $t_{\max} = \infty$ we shall now derive an a priori estimate for $\|w(t)\|_X$.

Step 1. Here we shall derive this a priori estimate under the assumption $w^I \in D(\overline{A})$. To this end we consider the evolution equation for $\|w\|_X^2$. By computing its time derivative and taking into account (2.9), we deduce

$$\frac{1}{2}\frac{d}{dt}\|w\|_X^2 = \langle \overline{A}w, w\rangle_X + \langle Bw, w\rangle_X.$$

Using the dissipativity of $\overline{A} - (\sigma + \beta/2)I$ (cf. (2.4)) we conclude that

$$\frac{1}{2}\frac{d}{dt}\|w\|_X^2 \leqslant \left(\sigma + \frac{\beta}{2}\right)\|w\|_X^2 + \langle Bw, w\rangle_X.$$

The skew-symmetry of the operator $\Theta[V]$ implies finally that

$$\frac{1}{2}\frac{d}{dt}\|w\|_X^2 \leqslant \left(\sigma + \frac{\beta}{2}\right)\|w\|_X^2 + \iint vw\Omega\big[V_x(t)\big]w. \tag{2.10}$$

On the other hand, since $\overline{A} - (\beta/2)I$ is dissipative on the space $L^2((0, 2\pi) \times \mathbb{R})$, the estimates

$$\frac{d}{dt}\|w\|_2^2 \leqslant \beta\|w\|_2^2 \quad \text{and} \quad \|w\|_2^2 \leqslant \|w^I\|_2^2 e^{\beta t} \tag{2.11}$$

follow. From the proof of Lemma 2.4 in [3] we have for the solution of (2.8)

$$\big\|V[w]\big\|_{W^{1,\infty}(0,2\pi)} \leqslant C\big(\|w\|_X + \|D\|_{L^1(0,2\pi)}\big).$$

Using (2.6), (2.10) and (2.11) we hence obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w\|_X^2 - \left(\sigma + \frac{\beta}{2}\right)\|w\|_X^2 \\
&\quad \leqslant \iint vw\Omega\big[V_x(t)\big]w \leqslant \big\|V_x(t)\big\|_\infty \|vw\|_2\|w\|_2 \\
&\quad \leqslant C\big(\|w\|_X + \|D\|_1\big)\|vw\|_2\|w^I\|_2 e^{(\beta/2)t} \\
&\quad \leqslant C\|w^I\|_2 e^{(\beta/2)t}\big(\|w\|_X^2 + \|w\|_X\|D\|_1\big) \\
&\quad \leqslant C\|w^I\|_2 e^{(\beta/2)t}\big(\|w\|_X^2 + \|D\|_1^2\big).
\end{aligned}$$

Thus

$$\frac{d}{dt}\|w\|_X^2 \leqslant a(t)\|w\|_X^2 + b(t),$$

where

$$a(t) = C\|w^I\|_2 e^{(\beta/2)t} + \beta + 2\sigma,$$
$$b(t) = C\|w^I\|_2 e^{(\beta/2)t}\|D\|_1^2.$$

Finally, applying Gronwall's inequality yields

$$\left\|w(t)\right\|_X^2 \leqslant \|w^I\|_X^2 e^{\int_0^t a(s)\,ds} + \int_0^t b(s) e^{\int_s^t a(\tau)\,d\tau}\,ds, \quad t \geqslant 0. \tag{2.12}$$

Hence $t_{\max} = \infty$ holds.

Step 2. Since (2.12) only involves $\|w^I\|_X$ this result carries over to w^I only in X by the following density argument.

For $w^I \in X$ let (w_n^I) be a sequence in $D(\overline{A})$ such that $w_n^I \to w^I$ in X. Using (2.12) we have for every w_n^I an a priori estimate for the corresponding classical solution:

$$\left\|w_n(t)\right\|_X \leqslant h(t), \quad \forall t \geqslant 0,\ n \in \mathbb{N},$$

with $h \in C[0, \infty)$ independent of n.

Let $w \in C([0, t_{\max}(w^I)), X)$ be the unique mild solution for w^I, which exists according to the first part of this theorem.

Next we assume $t_{\max}(w^I) < \infty$. Thus $\lim_{t \nearrow t_{\max}(w^I)} \|w(t)\|_X = \infty$. For the continuous, monotonously increasing function $g(t) := \max\{\|w(\tau)\|_X,\ 0 \leqslant \tau \leqslant t\}$ we also have $\lim_{t \nearrow t_{\max}(w^I)} g(t) = \infty$.

Choose $N \in \mathbb{N}$ with $N \geqslant 2\max\{h(t),\ t \in [0, t_{\max}(w^I)]\}$. Then there exists a $t_N < t_{\max}(w^I)$ such that

$$\begin{aligned} &g(t_N) = N, \\ &g(t) \leqslant N, \quad t \leqslant t_N, \\ &g(t) \geqslant N, \quad t_N \leqslant t < t_{\max}(w^I). \end{aligned} \tag{2.13}$$

We denote by L_N the Lipschitz constant of the operator B on

$$B_N := \left\{u \in X,\ \|u\|_X < N\right\}.$$

Let $\hat{B}$ be a (globally) Lipschitz extension of B outside of B_N. Thus, applying Theorem 6.1.2 in [10] on $[0, t_N]$ we obtain a Lipschitz dependence of the solutions on their initial values,

$$\|w - w_n\|_{C([0,t_N],X)} \leqslant C(L_N)\left\|w^I - w_n^I\right\|_X.$$

Thus, $w_n \to w$ in $C([0, t_N], X)$, and $\|w(t)\|_X \leqslant h(t) \leqslant N/2$ for $0 \leqslant t \leqslant t_N$ follows. This contradicts the assumption (2.13). □

Appendix A. Proof of Lemma 2.2

To prove the assertion we shall construct for each $f \in D(\overline{P}) \subset L^2((0,2\pi)\times\mathbb{R})$ a sequence $\{f_n\} \subset D(P)$ such that $f_n \to f$ in the graph norm $\|f\|_P = \|f\|_{L^2} + \|vf\|_{L^2} + \|Pf\|_{L^2} + \|vPf\|_{L^2}$.

To shorten the proof we shall consider here only the case

$$P = \mu + \nu v\partial_x + \beta v\partial_v + \sigma\partial_v^2 + 2\gamma\partial_v\partial_x + \alpha\partial_x^2$$

(cf. the definition of the operator A in (2.2)), but exactly the same strategy extends to the general case.

First we define the mollifying delta sequence

$$\varphi_n(x,v) := n^2\varphi(nx,nv), \quad n \in \mathbb{N},\ x,v \in \mathbb{R},$$

with the properties

$$\varphi \in C_0^\infty(\mathbb{R}^2), \qquad \varphi(x,v) \geqslant 0, \qquad \iint \varphi(x,v)\,dx\,dv = 1,$$
$$\operatorname{supp}\varphi \subset \left\{|x|^2 + |v|^2 \leqslant 1\right\}.$$

The velocity-cutoff function

$$\psi_n(v) := \psi\left(\frac{v}{n}\right), \quad n \in \mathbb{N},\ v \in \mathbb{R},$$

is assumed to have the properties

$$\psi \in C_0^\infty(\mathbb{R}), \qquad 0 \leqslant \psi(v) \leqslant 1, \qquad \left|\psi^{(j)}(v)\right| \leqslant C_j \quad \forall v \in \mathbb{R},\ j = 1,2,$$
$$\operatorname{supp}\psi \subset [-1,1], \qquad \psi|_{[-1/2,1/2]} \equiv 1.$$

We now define the approximating sequence

$$\tilde{f}_n(x,v) := (\tilde{f} * \varphi_n)(x,v)\cdot\psi_n(v), \quad n \in \mathbb{N},$$

where $*$ denotes the convolution in x and v. Remember that $\tilde{f}$ denotes the (in x) 2π-periodic extension of the function $f \in X$ to $\mathbb{R}^2$. By construction we have $\tilde{f}_n \in C^\infty(\mathbb{R}^2)$ and $\tilde{f}_n$ is 2π-periodic in x with compact support in v. Now, let R denote the restriction operator of (in x) 2π-periodic functions to $(0,2\pi)\times\mathbb{R}$. Then, $f_n := R\tilde{f}_n \in D(P)$. According to the 4 terms of the graph norm we split the proof into 4 steps:

Step 1. Since $\varphi_n \to \delta$ in $\mathcal{D}'(\mathbb{R}^2)$ and $\psi_n(v) \to 1$ pointwise, we have $\tilde{f}_n \to \tilde{f}$ in $L^2_{\text{loc}}(\mathbb{R}_x) \times L^2(\mathbb{R}_v)$ and

$$f_n \to f \quad \text{in } L^2\big((0,2\pi)\times\mathbb{R}\big).$$

Step 2. For the second term of the graph norm we write

$$v\tilde{f}_n = (v\tilde{f} * \varphi_n)\psi_n + (\tilde{f} * v\varphi_n)\psi_n.$$

The restriction of the first summand converges to vf in $L^2((0,2\pi)\times\mathbb{R})$ and the second term converges to 0 since $v\varphi_n \to 0$ in $\mathcal{D}'(\mathbb{R}^2)$. Hence we have

$$f_n \to f \quad \text{in } X.$$

Step 3. To prove that $Pf_n \to Pf$ in $L^2((0,2\pi)\times\mathbb{R})$ we write

$$\begin{aligned} P\tilde{f}_n &= \mu(\tilde{f}*\varphi_n)\psi_n + \nu(v\tilde{f}_x*\varphi_n)\psi_n + \beta(v\tilde{f}_v*\varphi_n)\psi_n \\ &\quad + \sigma(\tilde{f}_{vv}*\varphi_n)\psi_n + 2\gamma(\tilde{f}_{xv}*\varphi_n)\psi_n + \alpha(\tilde{f}_{xx}*\varphi_n)\psi_n \\ &\quad + r_n^1(x,v) \\ &= (P\tilde{f}*\varphi_n)\psi_n + r_n^1(x,v). \end{aligned}$$

As we shall show, the restriction of all six terms of the remainder

$$\begin{aligned} r_n^1 &= \nu(\tilde{f}*v\partial_x\varphi_n)\psi_n + \beta(\tilde{f}*\varphi_n)(v\partial_v\psi_n) \\ &\quad + \beta\big(\tilde{f}*\partial_v(v\varphi_n)\big)\psi_n + 2\sigma\left(\tilde{f}*\left(\frac{1}{n}\partial_v\varphi_n\right)\right)(n\partial_v\psi_n) \\ &\quad + \sigma(\tilde{f}*\varphi_n)\partial_v^2\psi_n + 2\gamma\left(\tilde{f}*\left(\frac{1}{n}\partial_x\varphi_n\right)\right)(n\partial_v\psi_n) \end{aligned}$$

converge to 0 in $L^2((0,2\pi)\times\mathbb{R})$.

In the first term $v\partial_x\varphi_n \to 0$ in $\mathcal{D}'(\mathbb{R}^2)$. Hence we have

$$R(\tilde{f}*v\partial_x\varphi_n) \to 0 \quad \text{in } L^2\big((0,2\pi)\times\mathbb{R}\big),$$

and the same argument holds for the third term.

For the second term we have

$$v\partial_v\psi_n = \frac{v}{n}\psi'\left(\frac{v}{n}\right),$$

which is in $L^\infty(\mathbb{R})$, uniformly for $n\in\mathbb{N}$ and with support in $[-n,-n/2]\cup[n/2,n]$. Hence, the second term converges to 0 in $L^2((0,2\pi)\times\mathbb{R})$.

In the fourth term $(1/n)\partial_v\varphi_n \to 0$ in $\mathcal{D}'(\mathbb{R}^2)$, and hence

$$R\left(\tilde{f}*\left(\frac{1}{n}\partial_v\varphi_n\right)\right) \to 0 \quad \text{in } L^2\big((0,2\pi)\times\mathbb{R}\big).$$

Furthermore, $n\partial_v\psi_n = \psi'(v/n)$ with $|\psi'| \leqslant C_1$. By the same argument also the sixth term converges to 0 in $L^2((0,2\pi)\times\mathbb{R})$.

Finally, the fifth term converges to 0 since

$$\partial_v^2\psi_n = \frac{1}{n^2}\psi''\left(\frac{v}{n}\right) \quad \text{with } |\psi''| \leqslant C_2.$$

Step 4. To prove that $vPf_n \to vPf$ in $L^2((0,2\pi)\times\mathbb{R})$ we write

$$
\begin{aligned}
vP\tilde{f}_n &= \mu(v\tilde{f} * \varphi_n)\psi_n + \nu(v^2\tilde{f}_x * \varphi_n)\psi_n + \beta(v^2\tilde{f}_v * \varphi_n)\psi_n \\
&\quad + \sigma(v\tilde{f}_{vv} * \varphi_n)\psi_n + 2\gamma(v\tilde{f}_{xv} * \varphi_n)\psi_n + \alpha(v\tilde{f}_{xx} * \varphi_n)\psi_n \\
&\quad + r_n^2(x,v) \\
&= \big((vP\tilde{f}) * \varphi_n\big)\psi_n + r_n^2(x,v),
\end{aligned}
$$

with the remainder

$$
\begin{aligned}
r_n^2 &= \mu(\tilde{f} * v\varphi_n)\psi_n + 2\nu(v\tilde{f} * v\partial_x\varphi_n)\psi_n + \nu(\tilde{f} * v^2\partial_x\varphi_n)\psi_n \\
&\quad + \beta(v\tilde{f} * \varphi_n + \tilde{f} * v\varphi_n)v\partial_v\psi_n + 2\beta\big(v\tilde{f} * \partial_v(v\varphi_n)\big)\psi_n \\
&\quad + \beta(\tilde{f} * v^2\partial_v\varphi_n)\psi_n + \sigma\big(\tilde{f} * \partial_{vv}(v\varphi_n)\big)\psi_n \\
&\quad + 2\sigma\left(v\tilde{f} * \frac{\partial_v\varphi_n}{n} + \tilde{f} * \frac{v\partial_v\varphi_n}{n}\right)\psi'\left(\frac{v}{n}\right) \\
&\quad + \sigma(\tilde{f} * \varphi_n)v\partial_v^2\psi_n + 2\gamma\big(\tilde{f} * \partial_{xv}(v\varphi_n)\big)\psi_n \\
&\quad + 2\gamma\left(v\tilde{f} * \frac{\partial_x\varphi_n}{n} + \tilde{f} * \frac{v\partial_x\varphi_n}{n}\right)\psi'\left(\frac{v}{n}\right) + \alpha(\tilde{f} * v\partial_{xx}\varphi_n)\psi_n.
\end{aligned}
$$

For proving that the restriction of all terms of r_n^2 converge to 0 in $L^2((0,2\pi)\times\mathbb{R})$ we recall that both $f, vf \in L^2((0,2\pi)\times\mathbb{R})$. Since the strategy of the proof is the same as in step 3 we shall only give the key points:

The distributions $v\varphi_n$, $v\partial_x\varphi_n$, $v^2\partial_x\varphi_n$, $\partial_v(v\varphi_n)$, $v^2\partial_v\varphi_n$, $\partial_{vv}(v\varphi_n)$, $\partial_v\varphi_n/n$, $v\partial_v\varphi_n/n$, $\partial_{xv}(v\varphi_n)$, $\partial_x\varphi_n/n$, $v\partial_x\varphi_n/n$, and $v\partial_{xx}\varphi_n$ all converge to 0 in $\mathcal{D}'(\mathbb{R}^2)$. Further, $v\partial_v^2\psi_n \to 0$ in $L^\infty(\mathbb{R})$ and the term $v\partial_v\psi_n$ was already discussed in step 3. □

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References

[1] A. Arnold, Self-consistent relaxation-time models in quantum mechanics, Comm. Partial Differential Equations 21 (1995) 473–506.

[2] A. Arnold, J.L. Lopez, P.A. Markowich, J. Soler, An analysis of quantum Fokker–Planck models: A Wigner function approach, Rev. Mat. Iberoamericana (2001), submitted.

[3] A. Arnold, C. Ringhofer, An operator splitting method for the Wigner–Poisson problem, SIAM J. Numer. Anal. 33 (1996) 1622–1643.

[4] A.O. Caldeira, A.J. Leggett, Path integral approach to quantum Brownian motion, Phys. A 121 (1983) 587–616.

[5] J.A. Carrillo, J. Soler, J.L. Vázquez, Asymptotic behaviour and selfsimilarity for the three dimensional Vlasov–Poisson–Fokker–Planck system, J. Funct. Anal. 141 (1996) 99–132.

[6] L. Diósi, On high-temperature Markovian equation for quantum Brownian motion, Europhys. Lett. 22 (1993) 1–3.
[7] H.L. Gruvin, T.R. Govindan, J.P. Kreskovsky, M.A. Stroscio, Transport via the Liouville equation and moments of quantum distribution functions, Solid State Electron. 36 (1993) 1697–1709.
[8] P.A. Markowich, C. Ringhofer, An analysis of the quantum Liouville equation, Z. Angew. Math. Mech. 69 (1989) 121–127.
[9] P.A. Markowich, C. Ringhofer, C. Schmeiser, Semiconductor Equations, Springer, New York, 1990.
[10] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, Berlin, 1983.
[11] M. Reed, B. Simon, Methods of Modern Mathematical Physics I, Functional Analysis, Academic Press, London, 1980.
[12] H. Risken, The Fokker–Planck Equation, Springer, Berlin, 1984.
[13] M.A. Stroscio, Moment-equation representation of the dissipative quantum Liouville equation, Superlattices Microstruct. 2 (1986) 83–87.
[14] E. Wigner, On the quantum correction for thermodynamic equilibrium, Phys. Rev. 40 (1932) 749–759.