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Multivariate tight affine frames with a small number of generators

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Abstract

We give a simple and explicit construction of compactly supported affine tight frames with small number of generators, associated to multivariate box splines (with respect to the dilation matrix $2I$). Moreover, the same technique applied to the case of bivariate box splines on the four-directions mesh with dilation matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ gives tight frames with at most five generators.

Keywords: Tight affine frames; Compactly supported wavelets; Multivariate box spline

1. Introduction

In the recent literature, a great deal of attention was devoted to the construction of compactly supported tight frames both in the univariate and the multivariate cases, see for example [4–8,10–16].

Compactly supported tight frames are a good replacement of compactly supported orthonormal wavelets when the system generated by integer translations of the corresponding scaling function φ is not orthogonal or, more generally, when φ is simply refinable and $\{\varphi(\cdot - k),\ k \in \mathbb{Z}\}$ may be not stable. Furthermore, in the multivariate case, even if φ is stable, and leaving aside tensor products, the construction of the corresponding (orthogonal, compactly supported) wavelets is a

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difficult task in general and so far no universal explicit algorithm is available. On the other hand, the recent papers [13,14] contain a general theory of affine systems and in particular a general unitary extension principle, which makes the construction of tight affine frames somewhat easier than the solution of the matrix extension problem which arises in the orthogonal wavelets construction. The results in our paper are based on the theory given by Ron and Shen [13,15].

A technique for the constructions of frame wavelets based on the unitary extension principle was applied to univariate box splines, to some multivariate compactly supported box splines [15] and to convolutions of self-similar compact sets in any dimension [10]. However, the actual application of this technique in these papers gives a number of mother wavelets (the frame generators) which grows at least proportionally with the regularity. In contrast with this, the papers [5,12] contain a general result (in the univariate case), where the number of mother wavelets is always two (three in the symmetric case).

Using the unitary extension principle and a new technique based on a Kronecker product approach, Chui and He [6] showed how to construct tight affine frames for a general class of multivariate compactly supported box splines. However, this construction yields a large number of mother wavelets, namely at least $2^K - 1$ generators (where K is the number of distinct vectors in the matrix Ξ defining the box spline).

This paper addresses the problem of constructing multivariate compactly supported tight affine frames with a small number of generators, for box splines belonging to the same class as in Ref. [6]. We apply the method of Ron and Shen [15], which consists in replacing the original box spline φ_Ξ with a new scaling function φ of larger support. Then, exactly as in Ref. [15], we define the first $2^N - 1$ wavelet filters to be the N-dimensional Haar wavelet filters. The novelty of our result lies in the technique of constructing the remaining filters, which allows us to reduce the number of frame generators. As a result, we are able to construct explicitly tight affine frames with at most $2^N - 1 + K$ generators where, as above, K is the number of distinct vectors in the matrix Ξ. To give an example, any bivariate box spline on the three-directions mesh gives rise to a tight frame with at most six generators (only four for the Courant element).

We prove our result in the case where the dilation matrix is $2I$ (where I is the identity) even though the construction could be carried out also in the more general cases considered in the paper [16]. As an example, in Section 4 we will deal with the case of box splines on the four-directions mesh with dilation matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ (see [15,9]).

2. Notation

We say that a function $\varphi \in L^2(\mathbb{R}^N)$ is a scaling function if the following conditions hold: (i) φ satisfies a refinement equation of the type

$$\hat{\varphi}(2\cdot) = m_0 \hat{\varphi}, \tag{1}$$

where m_0 is a 2π-periodic bounded function called the (symbol of the) mask of the refinement Eq. (1); (ii) $\hat{\varphi}$ is continuous at 0 and $\hat{\varphi}(0) = 1$.

In this paper, m_0 will always be a trigonometric polynomial and φ will always be compactly supported.

Let us denote V_0 the closed linear span of the translates of φ by means of the vectors in $\mathbb{Z}^N$. For every integer j let us denote by V_j the 2^j-dilate of V_0. Clearly the refinement Eq. (1) implies that the sequence V_j is increasing with j. If $\hat{\varphi}$ is continuous at 0 and $\hat{\varphi}(0) = 1$, then it is possible to prove that $\cup_j V_j$ is dense in $L^2(\mathbb{R}^N)$ and $\cap_j V_j = \{0\}$ [1]. As in this paper we will be dealing with integrable box splines, these conditions are automatically satisfied. However, it is worth pointing out that, unlike the case of multiresolution analyses, the scaling functions considered in this paper in general do not generate by translation Riesz bases of V_0. In fact they do not even need generating a frame.

Now let $\psi_1, \ldots, \psi_L$ be elements of V_1. Then, there exist 2π-periodic functions m_ℓ, for $\ell = 1, \ldots, L$, such that

$$\hat{\psi}_\ell(2\cdot) = m_\ell \hat{\varphi},$$

holds [2, Theorem 2.14]. We say that the functions ψ_ℓ are the generators of a tight affine frame if for every $f \in L^2(\mathbb{R}^N)$ one has

$$\sum_{j,k,\ell} |\langle f, \psi_{j,k,\ell} \rangle|^2 = \|f\|_2^2.$$

Here we made, as usual,

$$\psi_{j,k,\ell} = 2^{j/2} \psi_\ell(2^j \cdot - k)$$

with $j \in \mathbb{Z}, k \in \mathbb{Z}^N$, $\ell = 1, \ldots, L$.

Set $E = \{0,1\}^N$ and denote by $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)$ any element of E.

Ron and Shen [15] have a general technique (based on the unitary extension principle) for the construction of tight affine frames. We summarize this technique, in the case where the dilation is $2I$, in the following theorem.

Theorem 1 (Ron and Shen [15, Section 4]). *Let φ_1 and φ_2 be refinable distributions with bounded masks τ_1 and τ_2 and assume that $\varphi_1 * \varphi_2$ is a scaling function. Assume that T_1 and T_2 are collections of bounded periodic functions such that, for $\varepsilon \in E$,*

$$\tau_1(\omega)\overline{\tau_1(\omega + \pi\varepsilon)} + \sum_{\tau \in T_1} \tau(\omega)\overline{\tau(\omega + \pi\varepsilon)} = \delta_{0\varepsilon} \tag{2}$$

and

$$|\tau_2(\omega)|^2 + \sum_{\tau \in T_2} |\tau(\omega)|^2 = 1. \tag{3}$$

Define

$$T = T_1 \cup \tau_1 T_2(2\cdot). \tag{4}$$

Then the functions $\hat{\psi}(2\cdot) = \tau\hat{\varphi}$, *where* $\tau \in T$ *and* $\hat{\varphi} = \hat{\varphi}_1\hat{\varphi}_2(2\cdot)$, *are the generators of a tight affine frame.*

Remark. In the original formulation of this result it is required a mild decay condition on $\hat{\varphi}$. However, it follows from the result [8, Theorem 2] that the conclusion of Theorem 1 is true also without any decay assumption. In any case, the scaling function we will be dealing with is a box spline, which satisfies even stronger decay conditions.

3. Tight affine frames

Let

$$\Xi = \left[\xi_1, \quad \cdots \quad \xi_N, \quad \underbrace{\xi_{N+1}, \ldots, \xi_{N+1}}_{k_1 \text{ times}}, \quad \cdots \quad \underbrace{\xi_{N+s}, \ldots, \xi_{N+s}}_{k_s \text{ times}} \right]$$

be a full rank matrix with integer entries such that the vectors $\xi_1, \xi_2, \ldots, \xi_N$ are a basis for $\mathbb{Z}^N$ (of course one or more of the vectors ξ_{N+j} $j = 1, \ldots, s$, is allowed to be one of the first N vectors). The box spline φ_Ξ associated with Ξ is the $L^2(\mathbb{R}^N)$ function whose Fourier transform is

$$\hat{\varphi}_\Xi(\omega) = \prod_{j=1}^{N} \left(\frac{1 - \exp(-i\langle \xi_j, \omega \rangle)}{i\langle \xi_j, \omega \rangle} \right) \prod_{j=1}^{s} \left(\frac{1 - \exp(-i\langle \xi_{N+j}, \omega \rangle)}{i\langle \xi_{N+j}, \omega \rangle} \right)^{k_j}. \tag{5}$$

Note that we do not assume that Ξ is unimodular, so that, in general, φ_Ξ is not the scaling function of a multiresolution analysis (see [3]).

Let, throughout this paper,

$$H(t) = \frac{1 + e^{-it}}{2} \tag{6}$$

denote the one-dimensional Haar low-pass filter.

Upon performing an unimodular change of variables we may suppose, without loss of generality, that the first N columns in Ξ are the fundamental vectors of the axes, i.e.

$$\xi_1 = (1, 0, \ldots, 0)^t, \;\; \xi_2 = (0, 1, \ldots, 0)^t, \ldots, \;\; \xi_N = (0, 0, \ldots, 1)^t.$$

Therefore (5) takes the form

$$\hat{\varphi}_\Xi(\omega) = \prod_{j=1}^{N} \left(\frac{1 - e^{-i\omega_j}}{i\omega_j} \right) \prod_{j=1}^{s} \left(\frac{1 - \exp(-i\langle \xi_{N+j}, \omega \rangle)}{i\langle \xi_{N+j}, \omega \rangle} \right)^{k_j}.$$

Let m_Ξ denote the refinement mask of φ_Ξ. Then we have that

$$m_\Xi(\omega) = \frac{\hat{\varphi}_\Xi(2\omega)}{\hat{\varphi}_\Xi(\omega)} = \tau_1(\omega)\tau_2(\omega),$$

where we made

$$\tau_1(\omega) = \prod_{j=1}^{N} H(\omega_j)$$

and

$$\tau_2(\omega) = \prod_{j=1}^{s} H(\langle \xi_{N+j}, \omega \rangle)^{k_j}.$$

Note that $\varphi_\Xi = \varphi_1 * \varphi_2$, where φ_1 is the N-dimensional Haar scaling function and $\hat{\varphi}_2(\omega) = \prod_{j=1}^{\infty} \tau_2(2^{-j}\omega)$. According to Theorem 1, we define a new compactly supported scaling function φ via the formula

$$\varphi = \frac{1}{2^N} \varphi_2\left(\frac{\cdot}{2}\right) * \varphi_1. \tag{7}$$

Clearly, the refinement mask of φ is

$$m_0(\omega) = \tau_1(\omega)\tau_2(2\omega).$$

Theorem 2. *Let Ξ be a full rank matrix with integer entries such that the vectors $\xi_1, \xi_2, \ldots, \xi_N$ are a basis for $\mathbb{Z}^N$ and let φ_Ξ be the corresponding box spline. Then, with notation as above, there exist $(2^N - 1 + s)$ trigonometric polynomials m_ℓ such that the functions ψ_ℓ with*

$$\hat{\psi}_\ell(2\omega) = m_\ell(\omega)\hat{\varphi}(\omega) = m_\ell(\omega)\tau_2(\omega)\hat{\varphi}_\Xi(\omega), \quad \ell = 1, \ldots, 2^N - 1 + s \tag{8}$$

generate a compactly supported tight affine frame. The explicit expression of the polynomials m_ℓ is given in Eqs. (9) *and* (11).

Proof. We start by constructing explicitly the polynomial m_ℓ and then we will apply Theorem 1 to show that the compactly supported functions ψ_ℓ, given by Eq. (8) are actually the generators of a tight affine frame.

To every integer $\ell = 1, \ldots, 2^N - 1$ we associate the element $\varepsilon \in E \backslash \{0\}$ such that $\ell = \sum_{j=1}^{N} 2^{j-1}\varepsilon_j$. Then, as in [15], we define m_ℓ as

$$m_\ell(\omega) = \prod_{j=1}^{N} H(\omega_j + \pi\varepsilon_j), \tag{9}$$

i.e., the orthonormal N-dimensional Haar wavelet masks.

As for the remaining masks we proceed as follows. For every $h = 1, \ldots, s$, the trigonometric polynomial

$$1 - |H(\langle \xi_{N+h}, \omega \rangle)|^{2k_h}$$

has real coefficients, is positive and even in its argument. Hence, by the Riesz Lemma, there exists a (not unique) trigonometric polynomial Q_h such that

$$|Q_h(\langle \xi_{N+h}, \omega \rangle)|^2 = 1 - |H(\langle \xi_{N+h}, \omega \rangle)|^{2k_h}, \quad h = 1, \ldots, s. \tag{10}$$

For every $h = 1, \ldots, s$, we define

$$m_{2^N-1+h}(\omega) = \tau_1(\omega) Q_h(\langle \xi_{N+h}, 2\omega \rangle) \prod_{j=h+1}^{s} H(\langle \xi_{N+j}, 2\omega \rangle)^{k_j}, \tag{11}$$

where, if $h = s$, the last product in (11) must be interpreted as 1.

We define T_1 to be the set of the $2^N - 1$ polynomials in (9). Obviously Eq. (2) is satisfied. Next we set

$$T_2 = \left\{ Q_h(\langle \xi_{N+h}, \omega \rangle) \prod_{j=h+1}^{s} H(\langle \xi_{N+j}, \omega \rangle)^{k_j},\ h = 1, \ldots, s \right\}.$$

Note that the set of the polynomials in (11) is exactly of the form $\tau_1 T_2(2\cdot)$.

Next we show that condition (3) in Theorem 1 is satisfied. First we have that

$$\begin{aligned} &|\tau_2(\omega)|^2 + |Q_1(\langle \xi_{N+1}, \omega \rangle)|^2 \prod_{j=2}^{s} |H(\langle \xi_{N+j}, \omega \rangle)|^{2k_j} \\ &\quad = \prod_{j=2}^{s} |H(\langle \xi_{N+j}, \omega \rangle)|^{2k_j} \{ |H(\langle \xi_{N+1}, \omega \rangle)|^{2k_1} + |Q_1(\langle \xi_{N+1}, \omega \rangle)|^2 \} \\ &\quad = \prod_{j=2}^{s} |H(\langle \xi_{N+j}, \omega \rangle)|^{2k_j}. \end{aligned} \tag{12}$$

Arguing as before, and taking into account the last equation in (12), we get,

$$\begin{aligned} &|\tau_2(\omega)|^2 + \sum_{h=1}^{2} |Q_h(\langle \xi_{N+h}, \omega \rangle)|^2 \prod_{j=h+1}^{s} |H(\langle \xi_{N+j}, \omega \rangle)|^{2k_j} \\ &\quad = \prod_{j=3}^{s} |H(\langle \xi_{N+j}, \omega \rangle)|^{2k_j}. \end{aligned}$$

Carrying on this process, we finally arrive at the equation

$$|\tau_2(\omega)|^2 + \sum_{h=1}^{s} |Q_h(\langle \xi_{N+h}, \omega \rangle)|^2 \prod_{j=h+1}^{s} |H(\langle \xi_{N+j}, \omega \rangle)|^{2k_j} = 1.$$

Since the set of the set of all the polynomials m_ℓ in (9) and (11) is of the form (4) the result follows by Theorem 1. □

We observe that the number of wavelets obtained by the construction in [15, Section 4] is, with notation as above, $2^N - 1 + k_1 + \cdots + k_s$, while Theorem 2 gives only $2^N - 1 + s$ wavelets. (Note that $s \leqslant K$, where K is the number of distinct vectors of Ξ). However, most of these wavelets will lack symmetry properties.

Remark. According to [5] a tight affine frame with scaling function φ and generators ψ_ℓ is said to be a minimum energy frame if, for every $f \in L^2$,

$$\sum_k |\langle f, \varphi_{1,k} \rangle|^2 = \sum_k |\langle f, \varphi_{0,k} \rangle|^2 + \sum_{\ell,k} |\langle f, \psi_{0,k,\ell} \rangle|^2.$$

By [14, Corollary 6.7], every tight frame constructed using the unitary extension principle is minimum energy. Hence the frames constructed in Theorem 2 are minimum energy with respect to the scaling function φ in (7). However they are not minimum energy, in general, with respect to the scaling function φ_Ξ (see Example 2 below).

Example 1. Suppose φ_Ξ is the bivariate box spline on the three-directions mesh with Fourier transform given by

$$\hat{\varphi}_\Xi(\omega) = \left(\frac{1 - e^{-i\omega_1}}{i\omega_1}\right)^a \left(\frac{1 - e^{-i\omega_2}}{i\omega_2}\right)^b \left(\frac{1 - e^{-i(\omega_1+\omega_2)}}{i(\omega_1 + \omega_2)}\right)^c,$$

where a, b, $c \geqslant 1$ are integers.

In this case $m_\Xi(\omega) = \tau_1(\omega)\tau_2(\omega) = H(\omega_1)H(\omega_2)\tau_2(\omega)$, where

$$\tau_2(\omega) = \left(\frac{1 + e^{-i\omega_1}}{2}\right)^{a-1} \left(\frac{1 + e^{-i\omega_2}}{2}\right)^{b-1} \left(\frac{1 + e^{-i(\omega_1+\omega_2)}}{2}\right)^c.$$

The construction in the proof of Theorem 2 gives the following six trigonometric polynomials m_ℓ

$$m_1(\omega) = H(\omega_1 + \pi)H(\omega_2),$$

$$m_2(\omega) = H(\omega_1)H(\omega_2 + \pi),$$

$$m_3(\omega) = H(\omega_1 + \pi)H(\omega_2 + \pi)$$

and

$$m_4(\omega) = H(\omega_1)H(\omega_2)Q_1(2\omega_1)H(2\omega_2)^{b-1}H(2\omega_1 + 2\omega_2)^c,$$

$$m_5(\omega) = H(\omega_1)H(\omega_2)Q_2(2\omega_2)H(2\omega_1 + 2\omega_2)^c,$$

$$m_6(\omega) = H(\omega_1)H(\omega_2)Q_3(2\omega_1 + 2\omega_2),$$

where H is as in (6) and $Q_h, h = 1, \ldots, 3$, is the square root defined in (10) with $\xi_3 = (1,0)^t$, $\xi_4 = (0,1)^t$, $\xi_5 = (1,-1)^t$ and $k_1 = a-1$, $k_2 = b-1$, $k_3 = c$. Clearly, if $a = 1$ (resp. $b = 1$) then $m_4 = 0$ (resp. $m_5 = 0$). In particular, in the case of the Courant element ($a = b = c = 1$) only four wavelets are needed, while the method of Chui and He [6] gives seven wavelets.

The actual wavelets filters, with respect to φ_Ξ, are obtained by multiplying the m_ℓ by τ_2.

The following figures illustrate the case where $a = b = 1$ and $c = 2$.

The function φ_Ξ is symmetric with respect to the lines $y = x$ and $y = 3 - x$. Note that this spline is only C^0 across diagonals, but is otherwise C^1 (Fig. 1).

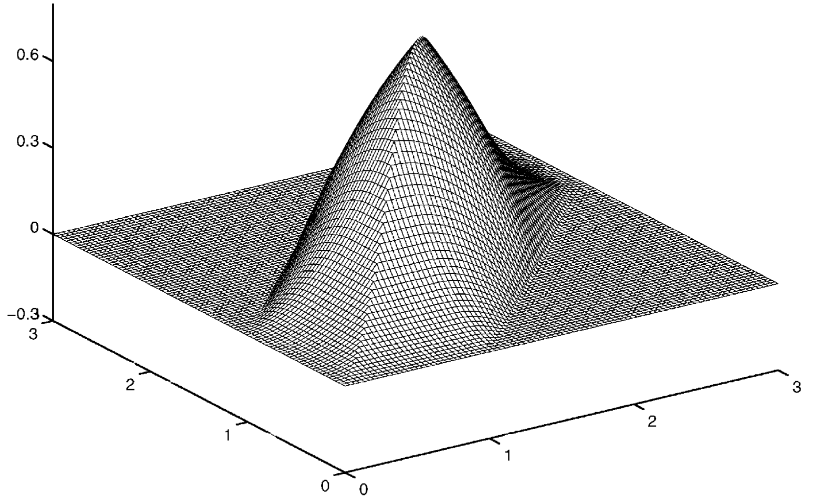

Fig. 1. The scaling function φ_Ξ with $\Xi = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$.

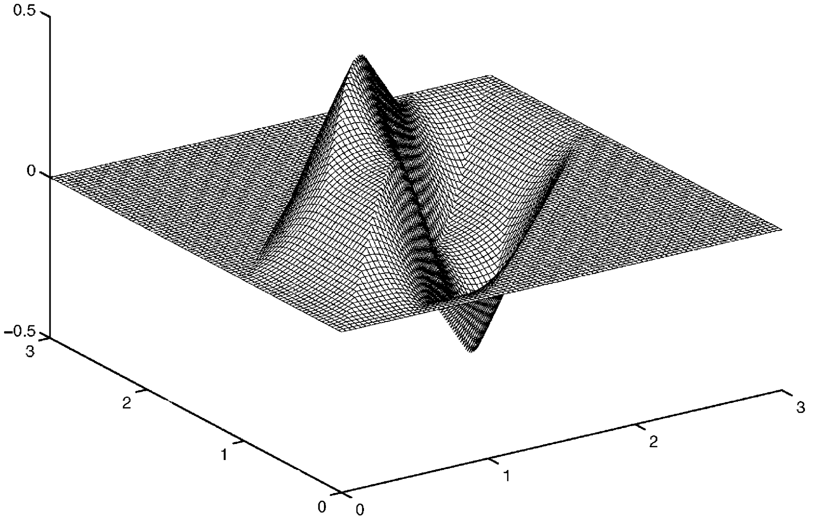

Fig. 2. The wavelet ψ_1.

The wavelet ψ_1 is antisymmetric with respect to the line $y = x$ and to the point $(3/2, 3/2)$ (Fig. 2). The figure of the wavelet ψ_2 is obtained by exchanging x with y.

The wavelet ψ_3 has the two axes of symmetry $y = x$ and $y = 3 - x$ (Fig. 3). Moreover, the scaling function and the first three wavelets have the same hexagonal

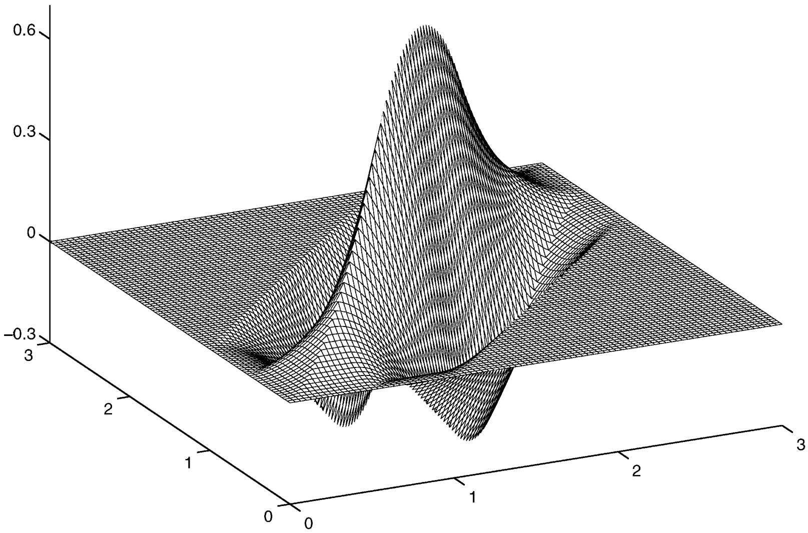

Fig. 3. The wavelet ψ_3.

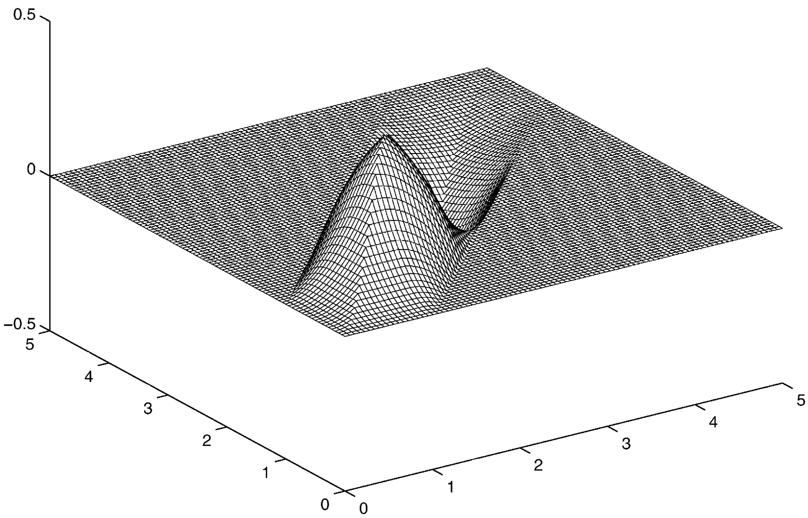

Fig. 4. The wavelet ψ_4.

support $[1, 1, 2]$ where we set

$$[a_1, a_2, a_3] = \left\{ \sum_{j=1}^{3} t_j \xi_j, \ 0 \leqslant t_j \leqslant a_j \right\}$$

and $\xi_1 = (1, 0)^t$, $\xi_2 = (0, 1)^t$, $\xi_3 = (1, 1)^t$.

Finally, ψ_4 has support $[1,1,3]$ and is only symmetric with respect to $y=x$ (Fig. 4).

Analogous computations can be carried out for box splines on the four-directions mesh. In this case the number of wavelets is at most seven and for the Zwart–Powell element only five are needed. In the next section we discuss in more detail these splines with respect to the dilation matrix $\binom{1\ \ 1}{1\ -1}$.

Example 2. The univariate box spline has Fourier transform

$$\hat{\varphi}_\Xi(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^k$$

and the refinement mask is

$$m_\Xi(\omega) = \tau_1(\omega)\tau_2(\omega) = H(\omega)H(\omega)^{k-1}.$$

We have that the function φ is

$$\hat{\varphi}(\omega) = \tau_2(\omega)\hat{\varphi}_\Xi(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^{k-1}\hat{\varphi}_\Xi(\omega).$$

For $k>1$ we have two wavelets

$$\hat{\psi}_\ell(2\omega) = m_\ell(\omega)\tau_2(\omega)\hat{\varphi}_\Xi(\omega), \quad \ell = 1,2,$$

where

$$m_1(\omega) = H(\omega+\pi)$$

and

$$m_2(\omega) = H(\omega)\sqrt{1-|H(2\omega)|^{2(k-1)}}.$$

The frame generated by these wavelets is minimum energy with respect to the scaling function φ, but, in view of [5, Lemma 1], it is not minimum energy with respect to φ_Ξ. Namely, the vectors $(m_\Xi, m_1\tau_2, m_2\tau_2)$ and $(m_\Xi(\cdot+\pi),\ m_1(\cdot+\pi)\tau_2(\cdot+\pi),\ m_2(\cdot+\pi)\tau_2(\cdot+\pi))$ are not orthogonal.

4. Tight frames with dilation matrix $\binom{1\ \ 1}{1\ -1}$

In this section we show how the methods used to prove Theorem 2 can be extended to the case where the dilation matrix is not diagonal. For sake of simplicity we confine ourselves to the case of bivariate box splines on the four-directions mesh with dilation matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{13}$$

Namely, it is known (see [15]) that the spline φ_Ξ, with Fourier transform

$$\hat{\varphi}_\Xi(\omega) = \left(\frac{1 - e^{-i\omega_1}}{i\omega_1}\right)^{k_1} \left(\frac{1 - e^{-i\omega_2}}{i\omega_2}\right)^{k_2} \left(\frac{1 - e^{-i(\omega_1+\omega_2)}}{i(\omega_1 + \omega_2)}\right)^{k_1} \left(\frac{1 - e^{-i(\omega_1-\omega_2)}}{i(\omega_1 - \omega_2)}\right)^{k_2},$$

is refinable with respect to M. Here k_1 and k_1 are integers greater or equal to one. A straightforward computation, see [15], shows that

$$\hat{\varphi}_\Xi(M\cdot) = m_\Xi \hat{\varphi}_\Xi$$

where

$$m_\Xi(\omega) = \left(\frac{1 + e^{-i\omega_1}}{2}\right)^{k_1} \left(\frac{1 + e^{-i\omega_2}}{2}\right)^{k_2}.$$

We show how to modify the arguments of the proof of Theorem 2 to construct tight frames with scaling function φ_Ξ. We write

$$m_\Xi(\omega) = \tau_1(\omega)\tau_2(\omega),$$

where $\tau_1(\omega) = H(\omega_1)H(\omega_2)$ and

$$\tau_2(\omega) = \left(\frac{1 + e^{-i\omega_1}}{2}\right)^{k_1-1} \left(\frac{1 + e^{-i\omega_2}}{2}\right)^{k_2-1}.$$

Let

$$m_0(\omega) = \tau_1(\omega)\tau_2(M\omega)$$

and define the refinable function φ via the formula

$$\varphi = \frac{1}{\sqrt{2}}\varphi_2(M^{-1}\cdot) * \varphi_1.$$

Theorem 1, stated in the particular case of the diagonal dilation matrix $2I$, actually holds for general dilation matrices. In the case of the dilation matrix (13), Eq. (2) involves the representatives of the quotient group $2\pi(M^{-1}\mathbb{Z}^2/\mathbb{Z}^2)$. We choose the representatives $(0, 0)$ and (π, π). In this case, condition (2) in Theorem 1 becomes

$$\tau_1(\omega)\overline{\tau_1(\omega + v)} + \sum_{\tau \in T_1} \tau(\omega)\overline{\tau(\omega + v)} = \delta_{0v}, \tag{14}$$

where $v \in \{(0, 0),\ (\pi, \pi)\}$. We define T_1 to be the set of the polynomials

$$m_1(\omega) = H(\omega_1 + \pi)H(\omega_2),$$

$$m_2(\omega) = H(\omega_1)H(\omega_2 + \pi),$$

$$m_3(\omega) = H(\omega_1 + \pi)H(\omega_2 + \pi).$$

Obviously (14) is satisfied.

Now, let Q_j, $j = 1, 2$, denote a square root (by the Riesz Lemma) of the polynomial $1 - |H(\omega_j)|^{2k_j-2}$.

We define T_2 to be the set

$$\{Q_1(\omega_1)\, H(\omega_2)^{k_2-1},\ Q_2(\omega_2)\}.$$

Since $M((\omega_1,\omega_2)^t) = (\omega_1+\omega_2, \omega_1-\omega_2)^t$, the last two masks are

$$m_4(\omega) = H(\omega_1)H(\omega_2)Q_1(\omega_1+\omega_2)H(\omega_1-\omega_2)^{k_2-1},$$

$$m_5(\omega) = H(\omega_1)H(\omega_2)Q_2(\omega_1-\omega_2).$$

Therefore, the functions

$$\hat{\psi}_\ell(M\omega) = m_\ell(\omega)\hat{\varphi}(\omega) = m_\ell(\omega)\tau_2(\omega)\hat{\varphi}_\Xi(\omega),$$

generate a tight affine frame.

Remark. It is worthwhile pointing out that the above construction yields tight affine frames with exactly five generators for every $k_1 > 1$ and $k_2 > 1$. Clearly, if $k_1 = 1$ (resp. $k_2 = 1$), then $\psi_4 = 0$ (resp. $\psi_5 = 0$). It is easy to compute the support of the wavelets. Let

$$\xi_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad \xi_2 = \begin{pmatrix}0\\1\end{pmatrix}, \quad \xi_3 = \begin{pmatrix}1\\1\end{pmatrix}, \quad \xi_4 = \begin{pmatrix}1\\-1\end{pmatrix}.$$

Then the function φ has the octagonal support $[2k_1-1, 2k_2-1, k_1, k_2]$, where we set

$$[a_1,a_2,a_3,a_4] = \left\{\sum_{j=1}^{4} t_j\xi_j, \ 0 \leqslant t_j \leqslant a_j\right\}.$$

The first three wavelets have the same support as φ_Ξ i.e. $[k_1,k_2,k_1,k_2]$. The supports of ψ_4 and ψ_5 are, respectively $[2k_1-1, 2k_2-1, k_1, k_2]$ and $[k_1, 2k_2-1, k_1, k_2]$.

This result should be compared with [15] where two different constructions are given. In the first one $(k_1+1)(k_2+1)-1$ generators, with the same support as φ_Ξ, are constructed. In the second one the number of generators is k_1+k_2, of which k_2 have the same support as φ_Ξ, while k_1 have the octagonal support $[2k_1, k_2, k_1, k_2]$.

In conclusion the advantage of our construction is evident for large values of k_1+k_2, but in general there is a tradeoff between the number of generators and the size and symmetry of the support.

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