Note

Two Transformations of Series That Commute with Compositional Inversion

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It is shown that the compositional inverse of either of two transformations of a given series can be determined from the compositional inverse of the series. Specifically, if \( t \cdot f(t) \) and \( t \cdot g(t) \) are compositional inverses, then so are \( t \cdot f_k(t) \) and \( t \cdot g^*_k(t) \), where \( f_k(t) \) is the \( k \)th Euler transformation of \( f(t) \) and \( g^*_k(t) = g(t)/(1 - kt \cdot g(t)) \).

1. INTRODUCTION

Compositional (or functional) inversion of series is defined in two forms in [3, pp. 149, 177]. Given two series \( f(t) = 1 + a_1t + a_2t^2 + \cdots \) and \( g(t) = 1 + b_1t + b_2t^2 + \cdots \), \( t \cdot f(t) \) and \( t \cdot g(t) \) are compositional inverses if they satisfy

\[
t \cdot f(t) \cdot g(t \cdot f(t)) = t \cdot g(t) \cdot f(t \cdot g(t)) = t.
\]

(1)

It is sometimes convenient to rewrite (1) as

\[
f(t \cdot g(t)) = 1/g(t) \quad \text{and} \quad g(t \cdot f(t)) = 1/f(t).
\]

(2)

The classical method for determining the compositional inverse of a series is to compute the first few terms, guess at the general term, and then check the guess against Eqs. (1) or (2). Finding the compositional inverse of a series would frequently be made easier if it were possible to derive it from the compositional inverse of a related series. However, the only transformation of series which commutes with the compositional inverse heretofore known is the obvious one of replacing \( t \) by \( kt \) for some constant \( k \).

In this paper I present two nontrivial transformations of series, the well-known Euler transformation and what I call the star transformation, and I show how to determine the compositional inverse of each of these transformations using the compositional inverse of the original series.
The $k$th Euler transformation of a series, as determined by Touchard [5], cf. [3, p. 156], is

$$f_k(t) = \frac{1}{1 + kt} f\left(\frac{t}{1 + kt}\right) = \sum_{n=0}^{\infty} (f - k)^n \frac{t^n}{n!}, \quad f^n \equiv f_n. \quad (3)$$

The $k$th star transformation of series is

$$f_k^*(t) = \left[\left[f(t)\right]^{-1} - kt\right]^{-1} = \frac{f(t)}{1 - kt \cdot f(t)} = \sum_{n=0}^{\infty} k^n t^n [f(t)]^{n+1}. \quad (4)$$

Both of these transformations are closed under composition in the sense that the $i$th transformation of the $j$th transformation of a series equals its $(i + j)$th transformation. That is, $[f_i][j](t) = f_{i+j}(t)$ and $[g_i^*][j^*](t) = g_{i+j}^*(t)$ for any series $f(t)$ and $g(t)$.

Furthermore, the two transformations commute with each other in the sense that the $i$th Euler transformation of the $j$th star transformation of a series equals the $j$th star transformation of the $i$th Euler transformation of a series. That is, $[f_i]^*[j^*](t) = [f^*_i][j^*](t)$ for any series $f(t)$.

It will be convenient to call two series $f(t)$ and $g(t)$ an inverse pair of series provided that $t \cdot f(t)$ and $t \cdot g(t)$ are functional inverses which satisfy Eqs. (1).

It will be shown in the next section that if $f(t)$ and $g(t)$ are an inverse pair of series, then so are $f_k(t)$ and $g_k^*(t)$, where these are, respectively, the Euler and star transformations of $f(t)$ and $g(t)$. By symmetry, it follows that $f_k^*(t)$ and $g_k(t)$ are also an inverse pair of series.

The paper concludes with a number of examples, both combinatorial and algebraic, for concreteness.

2. COMPOSITIONAL INVERSES OF THE TRANSFORMATIONS

**Theorem.** If $f(t)$ and $g(t)$ are an inverse pair of series satisfying Eqs. (1), then $f_k(t)$ and $g_k^*(t)$, as defined by Eqs. (3) and (4), respectively, are also an inverse pair of series.

**Proof.**

(1) $f_k(t \cdot g_k^*(t)) = f_k\left(\frac{t \cdot g(t)}{1 - kt \cdot g(t)}\right)$

$$= \frac{1}{1 + k \left(\frac{t \cdot g(t)}{1 - kt \cdot g(t)}\right)}$$

$$\cdot f\left(\frac{t \cdot g(t)}{1 - kt \cdot g(t)}\right) \left[1 + k \left(\frac{t \cdot g(t)}{1 - kt \cdot g(t)}\right)\right]$$

$$= (1 - kt \cdot g(t)) \cdot f(t \cdot g(t)) = \frac{1 - kt \cdot g(t)}{g(t)} = \frac{1}{g_k^*(t)}.$$
\[ (2) \quad g_k^*(t \cdot f_k(t)) = g_k^* \left( \frac{t}{1 + kt} \cdot f \left( \frac{t}{1 + kt} \right) \right) = g_k^*(u \cdot f(u)), \]

\[ u = \frac{t}{1 + kt}, \]

\[ = \frac{g(u \cdot f(u))}{1 - ku \cdot f(u) \cdot g(u \cdot f(u))} = \frac{1/f(u)}{1 - ku \cdot f(u)/f(u)} \]

\[ = \frac{1}{f(u)} \cdot \frac{1}{1 - ku} = \frac{1 + kt}{f(t/(1 + kt))} = \frac{1}{f_k(t)}. \]

Therefore \( f_k(t) \) and \( g_k^*(t) \) satisfy Eqs. (2) and, hence, are an inverse pair of series.

**Corollary.** If \( f(t) \) and \( g(t) \) are an inverse pair of series and \( k \) is any integer, the following diagram is commutative where \( \rightarrow^E \) represents the Euler transformation, \( \leftrightarrow^* \) represents the star transformation, and \( \leftrightarrow^C \) is used between inverse pairs of series.

\[ f(t) \xrightarrow{E} f_k^*(t) \]

\[ g(t) \xrightarrow{\ast} g_k^*(t) \]

\[ * \]

\[ \ast \]

\[ \ast \]

\[ \ast \]

\[ \ast \]

3. **Examples**

A class of combinatorial examples of the application of the Euler and star transformations can be found in the Catalan domain. Consider the series

\[ c(t) = c_0 + c_1 t + c_2 t^2 + \cdots = \frac{1 - (1 - 4t)^{1/2}}{2t}, \]

the generating function of the well-known Catalan [1-5] numbers \( c_n = \binom{2n}{n}/(n + 1) \). Applying the star transformation to \( c(t) \) yields \( c_k^*(t) = c(t)/(1 - t \cdot c(t)) \) which equals \( c^2(t) \) because
\[ \frac{c(t)}{1 - t \cdot c(t)} = c^2(t) \iff c(t) = c^2(t)(1 - t \cdot c(t)) \iff 1 = c(t) - t \cdot c^2(t), \]

a well-known identity for the Catalan numbers.

Because

\[ c^2(t) = \frac{c(t) - 1}{t} = \frac{1 - 2t - (1 - 4t)^{1/2}}{2t^2}, \]

applying the Euler transformation to \( c^2(t) \) yields

\[ \frac{1 - t - (1 - (1 - t)^2 - 4t^2)^{1/2}}{2t^2} = m(t), \]

the generating function of Motzkin numbers discussed in [1, 2, 4]. Applying the Euler transformation to the Motzkin numbers then yields

\[ \frac{1 - (1 - 4t^2)^{1/2}}{2t^2} = c(t^2). \]

Returning to \( c(t) \) and applying the Euler transformation yields

\[ c_1(t) = \frac{1 - (1 - 4t/(1 + t))^{1/2}}{2t} = \frac{(1 + t) - ((1 - t)^2 - 4t^2)^{1/2}}{2t(1 + t)} = \gamma(t), \]

the generating function of gamma numbers discussed in [2, 4], where it is noted that the gamma numbers and the Motzkin numbers satisfy the identity \( \gamma_n + \gamma_{n+1} = m_n \). Applying the star transformation to \( \gamma(t) \) also yields the Motzkin numbers.

It follows, because \( c(t) \) and \( 1 - t \) are an inverse pair of series, that the following diagram is commutative:
Here are a number of algebraic examples of inverse pairs of series obtained through the application of the Euler and star transformations.

(1) \( f(t) = 1 \) and \( g(t) = 1 \) are (trivially) an inverse pair of series. Hence, \( f_\varepsilon(t) = 1/(1 + kt) \leftrightarrow^C g_\varepsilon(t) = 1/(1 - kt) \).

(2) \( f(t) = t^n \) and \( g(t) = t^{-n/(n+1)} \), for \( n \neq -1 \), are an inverse pair of series. Hence, \( f_\varepsilon(t) = t^n/(1 + kt)^{n+1} \leftrightarrow^C g_\varepsilon(t) = 1/(t^n/(n+1) - kt) \) and \( f_\varepsilon(t) = t^n/(1 - kt^{n+1}) \leftrightarrow^C g_\varepsilon(t) = 1/[t^n(1 + kt)]^{1/(n+1)} \).

(3) \( f(t) = 1/(1 - t^n)^{1/2} \) and \( g(t) = 1/(1 + t^n)^{1/2} \) are an inverse pair of series. Hence, \( f_\varepsilon(t) = 1/((1 + kt)^2 - t^n)^{1/2} \leftrightarrow^C g_\varepsilon(t) = 1/((1 + t^n)^{1/2} - kt) \).

(4) \( f(t) = 1/(1 + t^n) \) and \( g(t) = 2/(1 + (1 - 4t^n)^{1/2}) \) are an inverse pair of series. Hence, \( f_\varepsilon(t) = (1 + kt)/(1 + kt)^2 + t^n \leftrightarrow^C g_\varepsilon(t) = 2/((1 - 2kt + (1 - 4t^n)^{1/2}) \) and \( f_\varepsilon(t) = 1/(1 - kt + t^2) \leftrightarrow^C g_\varepsilon(t) = 2/(1 + kt + ((1 + kt)^2 - 4t^n)^{1/2}) \).

(5) \( f(t) = 1 - t \) and \( g(t) = 2/(1 + (1 - 4t^n)^{1/2}) \) are an inverse pair of series. Hence, \( f_\varepsilon(t) = (1 + kt - t)/(1 + kt)^2 \leftrightarrow^C g_\varepsilon(t) = 2/((1 - 2kt + (1 - 4t^n)^{1/2}) \) and \( f_\varepsilon(t) = (1 - t)/(1 - kt + kt^2) \leftrightarrow^C g_\varepsilon(t) = 2/(1 + kt + ((1 + kt)^2 - 4t^n)^{1/2}) \).

(6) \( f(t) = 2 - t \) and \( g(t) = 1/(1 + (1 - t)^{1/2}) \) are an inverse pair of series. Hence, \( f_\varepsilon(t) = (2 + 2kt - t)/(1 + kt)^2 \leftrightarrow^C g_\varepsilon(t) = 1/(1 - kt + (1 - t)^{1/2}) \) and \( f_\varepsilon(t) = (2 + t)/(1 - 2kt - kt^2) \leftrightarrow^C g_\varepsilon(t) = 1/(1 + kt + ((1 + kt)^2 - t(1 + kt))^{1/2}) \).

More examples may be obtained from these by applying the two transformations one after the other.

REFERENCES