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Toric degenerations of Gelfand–Cetlin systems and potential functions

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Abstract

We define a toric degeneration of an integrable system on a projective manifold, and prove the existence of a toric degeneration of the Gelfand–Cetlin system on the flag manifold of type A. As an application, we calculate the potential function for a Lagrangian torus fiber of the Gelfand–Cetlin system.

Keywords: Toric degeneration; Gelfand–Cetlin system; Lagrangian intersection Floer theory

1. Introduction

It is well known that a polarized toric variety $(X, \mathcal{L})$ is related to a convex polytope $\Delta_{\mathcal{L}}$, the moment polytope, in two different ways:

- $\Delta_{\mathcal{L}}$ is the image of the moment map for the standard torus action on X, and
- the space $H^0(X, \mathcal{L})$ of holomorphic sections of $\mathcal{L}$ has a basis consisting of Laurent monomials, or equivalently, the weight decomposition of $H^0(X, \mathcal{L})$ with respect to the torus action is multiplicity-free, and each monomial corresponds to an integral point of $\Delta_{\mathcal{L}}$.

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Similar relations are known also for flag manifolds. Let

$$\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n)$$

be a non-increasing sequence of real numbers and consider the orbit

$$\mathcal{O}_\lambda = \mathrm{Ad}_{U(n)} \cdot \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

of the Hermitian matrix $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ under the adjoint action of the unitary group $U(n)$. This orbit has a natural Kähler structure, where the complex structure comes from an identification with the flag manifold $F = GL(n, \mathbb{C})/P$ of type A, and the Kähler form ω_λ comes from the Kostant–Kirillov symplectic form on the coadjoint orbit in the dual space $\mathfrak{u}(n)^*$ of the Lie algebra $\mathfrak{u}(n)$, identified with $\mathcal{O}_\lambda$ by the Killing form.

When all λ_i are integral, there is a $U(n)$-equivariant ample line bundle $\mathcal{L}_\lambda$ on $\mathcal{O}_\lambda$ whose first Chern class $c_1(\mathcal{L}_\lambda)$ is represented by ω_λ. The Borel–Weil theory states that $H^0(F, \mathcal{L}_\lambda)$ is an irreducible representation of $U(n)$ of highest weight λ. In this setting, a convex polytope Δ_λ, called the *Gelfand–Cetlin polytope*, appears in two different ways:

- Δ_λ is the image of the moment map of a completely integrable system on F called the *Gelfand–Cetlin system* [15].
- $H^0(F, \mathcal{L}_\lambda)$ admits a multiplicity-free decomposition into one-dimensional subspaces with respect to the action of a chain

  $$U(1) \subset U(2) \subset \cdots \subset U(n-1) \subset U(n)$$

  of subgroups. Each of these subspaces is parametrized by a sequence

  $$\big(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n-1)}, \lambda^{(n)}\big)$$

  of highest weights, which is in one-to-one correspondence with an integral point of Δ_λ. By choosing a non-zero element of each subspace, one obtains the *Gelfand–Cetlin basis* of $H^0(F, \mathcal{L}_\lambda)$ [10].

Despite the similarities between the toric moment map and the Gelfand–Cetlin system, there are marked differences: The Hamiltonian torus action of the Gelfand–Cetlin system does not preserve the complex structure on the flag manifold unlike the case of a toric variety. Although the fibers over the interior of the moment polytope are Lagrangian tori in both cases, the fibers over the boundary of Δ_λ are not necessarily isotropic tori in contrast to the case of $\Delta_{\mathcal{L}}$. In Example 3.8, we will see that the moment polytope for the full flag manifold in dimension three has a vertex where the fiber is a Lagrangian three-sphere.

It is known [13,4,19] that there is a flat family $f : (\mathfrak{X}, \mathfrak{L}) \to \mathbb{C}$ of polarized varieties such that

- $(X_t = f^{-1}(t), \mathcal{L}_t = \mathfrak{L}|_{X_t})$ is isomorphic to $(F, \mathcal{L}_\lambda)$ as a polarized manifold for any $t \neq 0$, and
- $(X_0, \mathcal{L}_0)$ is the polarized toric variety associated with the Gelfand–Cetlin polytope Δ_λ.

In this paper, we study the relation between the Gelfand–Cetlin system on $(F, \mathcal{L}_\lambda)$ and the moment map of $(X_0, \mathcal{L}_0)$. To state our main result, we make the following definition:

Definition 1.1. Let (X, ω) be a projective manifold X with a Kähler form ω and $\Phi : X \to \mathbb{R}^N$ be a completely integrable system on it. A *toric degeneration* of Φ consists of a flat family $f : \mathfrak{X} \to B$ of algebraic varieties over a complex manifold B, a Kähler form $\widetilde{\omega}$ on $\mathfrak{X}$, a piecewise smooth path $\gamma : [0,1] \to B$, a continuous map $\widetilde{\Phi} : \mathfrak{X}|_{\gamma([0,1])} \to \mathbb{R}^N$ on the total space $\mathfrak{X}|_{\gamma([0,1])} = f^{-1}(\gamma([0,1]))$ of the family restricted to the path, and a flow ϕ_t on $\mathfrak{X}|_{\gamma([0,1])}$ which covers the path γ and is defined away from the union $\bigcup_{t\in[0,1]} \operatorname{Sing}(X_t)$ of the singular loci of the fibers $X_t = f^{-1}(\gamma(t))$ such that

- for each $t \in [0,1]$, $\Phi_t = \widetilde{\Phi}|_{X_t}$ is a completely integrable system on the Kähler variety $(X_t, \omega_t = \widetilde{\omega}|_{X_t})$, whose image $\Phi_t(X_t)$ is a convex polytope Δ independent of t,
- (X_1, ω_1) is isomorphic to (X, ω) as a Kähler manifold,
- Φ_1 coincides with Φ under the above isomorphism $X_1 \cong X$,
- (X_0, ω_0) is a toric variety with a torus-invariant Kähler form,
- $\Phi_0 : X_0 \to \mathbb{R}^N$ is the moment map for the torus action on X_0 (hence Δ is a moment polytope of X_0), and
- if we set $\Delta^\circ = \Delta \backslash \Phi_0(\operatorname{Sing}(X_0))$ and $X_t^\circ = \Phi_t^{-1}(\Delta^\circ)$, then the flow ϕ_t sends $X_{t'}^\circ$ to another fiber $X_{t'-t}^\circ$ preserving the symplectic structures and the completely integrable systems:

$$\begin{tikzcd}[column sep=small] (X_{t'}^\circ, \omega_{t'}) \arrow[rr, "\phi_t"] \arrow[dr, "\Phi_{t'}"'] & & (X_{t'-t}^\circ, \omega_{t'-t}) \arrow[dl, "\Phi_{t'-t}"] \\ & \mathbb{R}^N & \end{tikzcd}$$

Note that the existence of a toric degeneration of a projective manifold with a structure of an integrable system does not imply the existence of a toric degeneration of that integrable system. For example, $\mathbb{P}^1 \times \mathbb{P}^1$ admits a flat degeneration into the Hirzebruch surface $\mathbb{F}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$, although the corresponding toric integrable structures cannot be related by a degeneration since their moment polytopes are distinct.

Now the main theorem in this paper is the following:

Theorem 1.2. *For any non-increasing sequence $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n)$ of real numbers, the Gelfand–Cetlin system on $(\mathcal{O}_\lambda, \omega_\lambda)$ admits a toric degeneration.*

Essential ingredients of the proof are the *degeneration in stages* of the flag manifold, introduced by Kogan and Miller [19] to relate the Gelfand–Cetlin basis of $H^0(F, \mathcal{L}_\lambda)$ with the monomial basis of $H^0(X_0, \mathcal{L}_0)$ in a geometric way, and the *gradient-Hamiltonian flow*, introduced by W.-D. Ruan [21] to construct Lagrangian torus fibrations on Calabi–Yau manifolds.

As an application of Theorem 1.2, we compute the *potential function* of the Gelfand–Cetlin system in Theorem 10.1 by reducing to the case of toric Fano manifolds, first studied by Cho and Oh [6] and further elaborated by Fukaya, Oh, Ohta and Ono [9]. The potential function is a Floer theoretic invariant of a Lagrangian submanifold introduced by Fukaya, Oh, Ohta and Ono [8], which encodes the information of holomorphic disks with Lagrangian boundary condition. It will be used in Theorem 12.1 to show the existence of a non-displaceable Lagrangian torus in the flag manifold just as in the toric case [9, Theorem 1.5].

In the case of a toric Fano manifold, the potential function gives the *Landau–Ginzburg potential* after the substitution of e^{-1} into the indeterminant element T of the Novikov ring and a suitable change of variables. The Landau–Ginzburg potential appears in Givental's integral representation of the *J-function*, which generates the *quantum D-module* encoding the information of Gromov–Witten invariants. As a corollary to this integral representation, one obtains an isomorphism between the quantum cohomology ring and the Jacobi ring of the Landau–Ginzburg potential. Such properties continue to hold in the case of a full flag manifold, where the J-function gives a solution to the quantum completely integrable system called the *quantum Toda lattice*, although it fails for more general flag manifolds.

The organization of this paper is as follows: In Section 2, we fix notation and recall basic facts on flag manifolds which are used through this paper. In Section 3, we recall the construction of the Gelfand–Cetlin system. In Section 5, we introduce toric degenerations of flag manifolds in stages following [19]. A toric degeneration of the Gelfand–Cetlin system is constructed in Section 6 and Section 7. In Section 6 we construct a map $\widetilde{\Phi}$ in Definition 1.1 using the degeneration in stages, and prove in Section 7 that the gradient-Hamiltonian flow sends the flag manifold to the toric variety X_0 preserving the structure of completely integrable systems. In Section 8, we construct another, not in-stages, toric degeneration of the Gelfand–Cetlin system so that X_t is biregular to X for any $t \neq 0$. This will be used in Section 9 to compare the moduli spaces of holomorphic disks in the flag manifold and the Gelfand–Cetlin toric variety. In Section 10, we recall the definition of the potential function, and compute it for a Lagrangian torus fiber in the Gelfand–Cetlin system. In Section 11, we study the case of the full flag manifold $F^{(3)}$ and the Grassmannian $Gr(2,4)$ in some detail. In Section 12, we prove the existence of a non-displaceable Lagrangian torus in the flag manifold along the lines of [9]. In Section 13, we recall Givental's integral representation of the J-function for the full flag manifold, and discuss its relation with the potential function.

2. Partial flag manifolds

Fix a sequence $0 = n_0 < n_1 < \cdots < n_r < n_{r+1} = n$ of integers, and set $k_i = n_i - n_{i-1}$ for $i = 1, \dots, r+1$. The partial flag manifold $F = F(n_1, \dots, n_r, n)$ is a complex manifold parameterizing nested subspaces

$$0 \subset V_1 \subset \cdots \subset V_r \subset \mathbb{C}^n, \qquad \dim V_i = n_i.$$

Let $F^{(n)}$ denote the full flag manifold $F(1, 2, \dots, n)$ for short. The dimension of $F(n_1, \dots, n_r, n)$ is given by

$$N = N(n_1, \dots, n_r, n) := \dim_{\mathbb{C}} F(n_1, \dots, n_r, n) = \sum_{i=1}^{r} (n_i - n_{i-1})(n - n_i).$$

Let $P = P(n_1, \dots, n_r, n) \subset GL(n, \mathbb{C})$ be the isotropic subgroup of the standard flag $V_i = \langle e_1, \dots, e_{n_i} \rangle$, where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{C}^n$. Then the intersection of P and $U(n)$ is $U(k_1) \times \cdots \times U(k_{r+1})$, and F is written as

$$F = GL(n, \mathbb{C})/P = U(n)/\bigl(U(k_1) \times \cdots \times U(k_{r+1})\bigr).$$

In particular, the full flag manifold is given by $F^{(n)} = GL(n,\mathbb{C})/B = U(n)/T$, where $B \subset GL(n,\mathbb{C})$ is a Borel subgroup consisting of upper triangular invertible matrices, and T is a maximal torus in $U(n)$ consisting of diagonal matrices.

In this paper we will use two descriptions of flag manifolds, (co)adjoint orbits and Plücker embeddings. First we recall the (co)adjoint orbit description. Using a $U(n)$-invariant inner product $\langle,\rangle$ on the Lie algebra $\mathfrak{u}(n)$ of $U(n)$, we identify the dual $\mathfrak{u}(n)^*$ of $\mathfrak{u}(n)$ with the space $\sqrt{-1}\mathfrak{u}(n)$ of Hermitian matrices. We fix $\lambda = \operatorname{diag}(\lambda_1,\dots,\lambda_n) \in \sqrt{-1}\mathfrak{u}(n)$ with

$$\underbrace{\lambda_1 = \cdots = \lambda_{n_1}}_{k_1} > \underbrace{\lambda_{n_1+1} = \cdots = \lambda_{n_2}}_{k_2} > \cdots > \underbrace{\lambda_{n_r+1} = \cdots = \lambda_n}_{k_{r+1}}. \tag{1}$$

Then F is identified with the adjoint orbit $\mathcal{O}_\lambda \subset \sqrt{-1}\mathfrak{u}(n)$ of λ by

$$F = U(n)/\big(U(k_1)\times\cdots\times U(k_{r+1})\big) \xrightarrow{\sim} \mathcal{O}_\lambda, \qquad [g] \longmapsto g\lambda g^*.$$

Note that $\mathcal{O}_\lambda$ consists of Hermitian matrices with fixed eigenvalues $\lambda_1,\dots,\lambda_n$. $\mathcal{O}_\lambda$ has a standard symplectic form ω_λ called the Kostant–Kirillov form. Recall that tangent vectors of $\mathcal{O}_\lambda$ at x can be written as $\operatorname{ad}_\xi(x) = [x,\xi]$ for $\xi \in \mathfrak{u}(n)$. Then ω_λ is defined by

$$\omega_\lambda\big(\operatorname{ad}_\xi(x), \operatorname{ad}_\eta(x)\big) = \frac{1}{2\pi}\big\langle x, [\xi,\eta]\big\rangle.$$

Note that ω_λ is the unique $U(n)$-invariant Kähler form in its cohomology class $[\omega_\lambda]$.

Next we recall the Plücker embedding of F. For each $k = 1,\dots,n-1$, we set $\mathbb{P}_k := \mathbb{P}(\bigwedge^k \mathbb{C}^n) = \mathbb{P}^{\binom{n}{k}-1}$. Then the Plücker embedding is given by

$$\iota : F \hookrightarrow \prod_{i=1}^r \mathbb{P}_{n_i}, \quad \big(0 \subset V_1 \subset \cdots \subset V_r \subset \mathbb{C}^n\big) \mapsto \big(\textstyle\bigwedge^{n_1} V_1, \dots, \bigwedge^{n_r} V_r\big).$$

Note that we have a natural projection

$$\begin{array}{cccc} \pi = \pi_{n_1,\dots,n_r}: & \prod_{k=1}^{n-1}\mathbb{P}_k & \longrightarrow & \prod_{i=1}^r \mathbb{P}_{n_i} \\ & \cup & & \cup \\ & F^{(n)} & \longrightarrow & F(n_1,\dots,n_r,n). \end{array}$$

For an $n\times n$ matrix $z = (z_{ij})$ and $I = \{i_1 < \cdots < i_k\} \subset \{1,\dots,n\}$, we set

$$z_I = \begin{pmatrix} z_{i_1 1} & z_{i_1 2} & \cdots & z_{i_1 k} \\ z_{i_2 1} & z_{i_2 2} & \cdots & z_{i_2 k} \\ \vdots & \vdots & & \vdots \\ z_{i_k 1} & z_{i_k 2} & \cdots & z_{i_k k} \end{pmatrix}.$$

Then the Plücker coordinates are given by

$$p_I(z) := \det z_I$$

for I with $|I| = n_1, \dots, n_l$. In other words, F can be obtained as a "multiple Proj" of $\mathbb{C}[p_I;\ |I| = n_1, \dots, n_r]$:

$$F(n_1, \dots, n_r, n) = \text{multiple Proj}\, \mathbb{C}\big[p_I;\ |I| = n_1, \dots, n_r\big] \subset \prod_{i=1}^{r} \mathbb{P}_{n_i},$$

which means that F is a subvariety in $\prod_{i=1}^r \mathbb{P}_{n_i} = \prod_{i=1}^r \operatorname{Proj}\mathbb{C}[Z_I;\ |I| = n_i]$ corresponding to $\mathbb{C}[p_I;\ |I| = n_1, \dots, n_r]$. In this setting, ω_λ coincides with the restriction $\iota^*\widetilde{\omega}_\lambda$ of a Kähler form

$$\widetilde{\omega}_\lambda = \sum_{i=1}^{r} (\lambda_{n_i} - \lambda_{n_{i+1}})\omega_{\mathrm{FS},n_i} \tag{2}$$

on $\prod_{i=1}^r \mathbb{P}_{n_i}$, where $\omega_{\mathrm{FS},k}$ is the Fubini–Study form on $\mathbb{P}_k$.

Example 2.1. The full flag manifold $F^{(3)}$ for $n = 3$ is three-dimensional and embedded into $\mathbb{P}_1 \times \mathbb{P}_2 = \mathbb{P}^2 \times \mathbb{P}^2$ as a hypersurface by

$$\iota = \big([p_1 : p_2 : p_3], [p_{12} : p_{13} : p_{23}]\big) : F^{(3)} \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2.$$

The defining equation (i.e. the Plücker relation) is given by

$$Z_1 Z_{23} - Z_2 Z_{13} + Z_3 Z_{12} = 0,$$

where $[Z_1 : Z_2 : Z_3]$, $[Z_{12} : Z_{13} : Z_{23}]$ are homogeneous coordinates on $\mathbb{P}_1$ and $\mathbb{P}_2$ respectively.

Example 2.2. $F^{(4)}$ is of dimension six and embedded into $\prod_k \mathbb{P}_k = \mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^3$. The Plücker relations are given by ten quadrics, and hence $F^{(4)}$ is not a complete intersection. The projection $\pi_2 : \prod_k \mathbb{P}_k \to \mathbb{P}_2 = \mathbb{P}^5$ maps $F^{(4)}$ to the Grassmannian $F(2,4) = Gr(2,4)$ of two-planes in a four-space, which is a hypersurface in $\mathbb{P}_2$ defined by

$$Z_{12}Z_{34} - Z_{13}Z_{24} + Z_{14}Z_{23} = 0.$$

We consider the case where $\lambda_i \in \mathbb{Z}$ for $i = 1, \dots, n$. Then λ can be regarded as a character of T by

$$T \longrightarrow \mathbb{C}^*, \qquad \operatorname{diag}(t_1, \dots, t_n) \longmapsto t_1^{\lambda_1} \cdots t_n^{\lambda_n},$$

and hence gives an action of T on $\mathbb{C}$. Using this T-action, we define a line bundle on $F^{(n)}$ by

$$\big(U(n) \times \mathbb{C}\big)/T \longrightarrow F^{(n)} = U(n)/T.$$

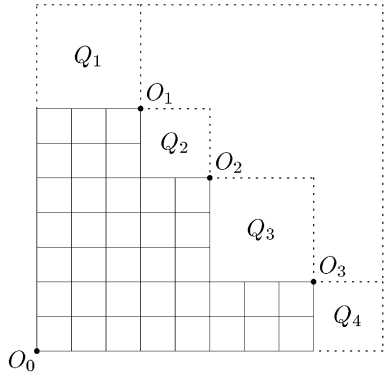

Fig. 1. The ladder diagram for $F(3, 5, 8, 10)$.

It is easy to see that this descends to a line bundle $\mathcal{L}_\lambda$ on $F(n_1, \dots, n_r, n)$ under the condition (1). Note that $\mathcal{L}_\lambda$ is also written as

$$\mathcal{L}_\lambda = \iota^* \mathcal{O}_{\mathbb{P}_{n_1}}(\lambda_{n_1} - \lambda_{n_2}) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}_{n_r}}(\lambda_{n_r} - \lambda_n),$$

where $\boxtimes$ is the outer tensor product. Hence ω_λ represents the first Chern class $c_1(\mathcal{L}_\lambda)$ of $\mathcal{L}_\lambda$.

We recall the description of the anti-canonical bundle $\mathcal{K}_F^{-1}$ on F in terms of characters λ. Note that the holomorphic tangent space of $F^{(n)}$ at the standard flag can be identified with $\bigoplus_{i<j} \mathbb{C} E_{ij}$, where E_{ij} is the matrix whose (i, j)-entry is 1 and the other entries are zero. Hence the anti-canonical bundle $\mathcal{K}_{F^{(n)}}^{-1}$ of the full flag manifold corresponds to the sum of "positive roots" $(0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0)$:

$$\mathcal{K}_{F^{(n)}}^{-1} = L_{2\rho}, \quad 2\rho = (n-1, n-3, \dots, -n+3, -n+1).$$

Similarly, the anti-canonical bundle of the partial flag manifold $F(n_1, \dots, n_r, n)$ is given by

$$\lambda = (\underbrace{n - n_1, \dots,}_{k_1} \underbrace{n - n_1 - n_2, \dots,}_{k_2} \dots, \underbrace{n - n_{r-1} - n_r, \dots,}_{k_r} \underbrace{-n_r, \dots, -n_r}_{k_{r+1}}), \tag{3}$$

the sum of roots corresponding to E_{ij} above the diagonal squares Q_k of the ladder diagram, which will be introduced in the following. For example, the anti-canonical bundle $\mathcal{K}_{Gr(r,n)}^{-1}$ of the Grassmannian $Gr(r, n)$ corresponds to

$$\lambda = (\underbrace{n - r, \dots, n - r}_{r}, \underbrace{-r, \dots, -r}_{n-r}).$$

Now we introduce the standard *ladder diagram*, which is used in [4] to describe the toric degeneration of F. (See Fig. 1.)

Definition 2.3. We consider an $n \times n$ square Q and place squares Q_l of size $k_l \times k_l$ $(l = 1, \dots, r+1)$ on the diagonal. The ladder diagram is the set of boxes below the diagonal squares. Let O_0 denote the lower left corner of the ladder diagram. For $l = 1, \dots, r$, the lower right corner of Q_l is denoted by O_l.

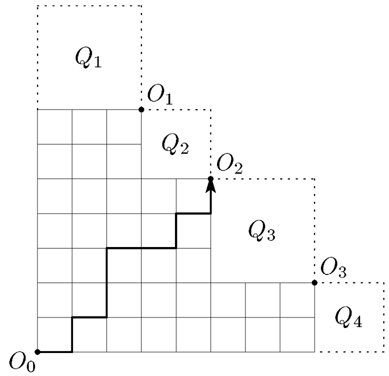

Fig. 2. A positive path.

Note that the number of boxes in the ladder diagram is equal to $N(n_1, \ldots, n_r, n) = \dim_{\mathbb{C}} F(n_1, \ldots, n_r, n)$. Any matrix in $P(n_1, \ldots, n_r, n)$ has 0's in its entries which correspond to boxes in the ladder diagram, and $U(k_l)$ is placed in the diagonal square Q_l.

Definition 2.4. A *positive path* is a path on the ladder diagram, starting at the lower left corner O_0 and moving either upward or to the right along edges, until one of O_k is reached.[1]

For each positive path ending at O_k, we can associate a homogeneous coordinate on $\mathbb{P}_{n_k} = \mathbb{P}^{\binom{n}{n_k}-1}$. Note that the number of positive paths reaching O_k is $\binom{n}{n_k}$. If the path is horizontal in the $i_1, \ldots, i_{n_k}$-th steps, then the corresponding coordinate is $Z_{i_1,\ldots,i_{n_k}}$. For example, the positive path in Fig. 2 corresponds to $Z_{1,3,6,7,9}$ on $\mathbb{P}_5$.

3. The Gelfand–Cetlin system

In this section we recall the construction of the Gelfand–Cetlin system. First we consider the case of full flag manifold $F^{(n)} = F(1, 2, \ldots, n)$. Note that this corresponds to the case where all λ_i are distinct:

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n.$$

For $x \in \mathcal{O}_\lambda$ and $k = 1, \ldots, n-1$, let $x^{(k)}$ denote the upper-left $k \times k$ submatrix of x. Since $x^{(k)}$ is also a Hermitian matrix, it has real eigenvalues $\lambda_1^{(k)}(x) \geqslant \lambda_2^{(k)}(x) \geqslant \cdots \geqslant \lambda_k^{(k)}(x)$. By taking the eigenvalues for all $k = 1, \ldots, n-1$, we obtain a set of functions

$$\Phi_\lambda : \mathcal{O}_\lambda \longrightarrow \mathbb{R}^{n(n-1)/2}, \quad x \longmapsto \left(\lambda_i^{(k)}(x)\right)_{\substack{k=1,\ldots,n-1,\\ i=1,\ldots,k}}. \tag{4}$$

Recall that $\dim_{\mathbb{C}} F^{(n)} = n(n-1)/2$.

[1] The choice of the orientation of positive paths is different from the one in [4].

Theorem 3.1. *(See Guillemin and Sternberg [15].)* $\{\lambda_i^{(k)}\}_{k,i}$ *is a completely integrable system on* $(\mathcal{O}_\lambda, \omega_\lambda)$.

$\{\lambda_i^{(k)}\}_{k,i}$ is called the *Gelfand–Cetlin system* on $(F^{(n)}, \omega_\lambda)$.

Remark 3.2. For $k = 1, \dots, n-1$, we regard $U(k)$ as a subgroup of $U(n)$ by

$$U(k) \cong \left(\begin{array}{c|c} U(k) & 0 \\ \hline 0 & 1_{n-k} \end{array}\right) \subset U(n). \tag{5}$$

Then the map $x \mapsto x^{(k)}$ gives a moment map of the $U(k)$-action on $(\mathcal{O}_\lambda, \omega_\lambda)$.

For later use, we present a proof of this theorem. We first recall some basic facts on moment maps. Let G be a compact Lie group acting on a symplectic manifold (M, ω) with a moment map $\mu : M \to \mathfrak{g}^*$. Note that $\mathfrak{g}^*$ has a Poisson structure induced from the Kostant–Kirillov form. For functions f_1, f_2 on $\mathfrak{g}^*$, their Poisson bracket $\{f_1, f_2\}_{\mathfrak{g}^*}$ at $x \in \mathfrak{g}^*$ is defined to be the Poisson bracket at x of the restrictions $f_i|_{\mathcal{O}_x}$ to the coadjoint orbit $\mathcal{O}_x$ of x.

Lemma 3.3. *For* $f_1, f_2 \in C^\infty(\mathfrak{g}^*)$, *it follows that*

$$\{\mu^* f_1, \mu^* f_2\}_M = \mu^*\{f_1, f_2\}_{\mathfrak{g}^*},$$

where $\{,\}_M$ *is the Poisson bracket on* M. *In particular, if* f_1 (*or* f_2) *is* $\mathrm{Ad}(G)^*$*-invariant, then we have*

$$\{\mu^* f_1, \mu^* f_2\}_M = 0.$$

See [14] for a proof. We also recall the following Noether type theorem, which will be used in Section 7.

Lemma 3.4. *If* $f \in C^\infty(M)$ *is* G*-invariant, then* μ *is constant along the Hamiltonian flow of* f.

Now we go back to our situation and prove Theorem 3.1. Since $\lambda_i^{(k)}$ is a pull-back of a $U(k)$-invariant function on $\sqrt{-1}\mathfrak{u}(k)$ by the moment map $x \mapsto x^{(k)}$, Lemma 3.3 implies that

$$\{\lambda_i^{(k)}, \lambda_j^{(l)}\} = 0$$

on $(\mathcal{O}_\lambda, \omega_\lambda)$.

Next we see the image $\Phi_\lambda(\mathcal{O}_\lambda)$ of $\mathcal{O}_\lambda$. We first consider the eigenvalues of x and $x^{(n-1)}$. The mini-max principle implies that

$$\lambda_1 \geqslant \lambda_1^{(n-1)} \geqslant \lambda_2 \geqslant \lambda_2^{(n-1)} \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_{n-1} \geqslant \lambda_{n-1}^{(n-1)} \geqslant \lambda_n.$$

Hence $(\lambda_i^{(k)}(x))$ satisfies

$$
\begin{array}{ccccccccccccccccc}
\lambda_1 & & & & \lambda_2 & & & & \lambda_3 & & \cdots & & \lambda_{n-1} & & & & \lambda_n \\
& \geqslant & & \leqslant & & \geqslant & & \leqslant & & & & & & \geqslant & & \leqslant & \\
& & \lambda_1^{(n-1)} & & & & \lambda_2^{(n-1)} & & & & & & & & \lambda_{n-1}^{(n-1)} & & \\
& & & \geqslant & & \leqslant & & & & & & & & \leqslant & & & \\
& & & & \lambda_1^{(n-2)} & & & & & & & & \lambda_{n-2}^{(n-2)} & & & & \\
& & & & & \geqslant & & & & & & \leqslant & & & & & \\
& & & & & & \ddots & & & & \iddots & & & & & & \\
& & & & & & & \geqslant & & \leqslant & & & & & & & \\
& & & & & & & & \lambda_1^{(1)} & & & & & & & &
\end{array}
\tag{6}
$$

An array of real numbers satisfying (6) is called a *Gelfand–Cetlin pattern* for λ. The *Gelfand–Cetlin polytope* Δ_λ is a polytope consisting of Gelfand–Cetlin patterns for λ. The above argument means the image $\Phi_\lambda(\mathcal{O}_\lambda)$ is contained in Δ_λ.

Lemma 3.5. *Let $a_1, \ldots, a_{k+1}, b_1, \ldots, b_k$ be real numbers satisfying*

$$a_1 \geqslant b_1 \geqslant a_2 \geqslant \cdots \geqslant a_k \geqslant b_k \geqslant a_{k+1}.$$

Then there exist $x_1, \ldots, x_k \in \mathbb{C}$ and $x_{k+1} \in \mathbb{R}$ such that

$$
\begin{pmatrix}
b_1 & & 0 & \bar{x}_1 \\
& \ddots & & \vdots \\
0 & & b_k & \bar{x}_k \\
x_1 & \cdots & x_k & x_{k+1}
\end{pmatrix}
$$

has eigenvalues $a_1, \ldots, a_{k+1}$.

We omit the proof. Using this lemma successively, we can prove that $\Phi_\lambda(\mathcal{O}_\lambda) = \Delta_\lambda$. The fact that $\dim \Delta_\lambda = n(n-1)/2$ implies the functional independence of $\lambda_i^{(k)}$'s.

Remark 3.6. Recall that a moment map of the T-action on $\mathcal{O}_\lambda$ is given by

$$
x = (x_{ij}) \longmapsto \begin{pmatrix} x_{11} & & 0 \\ & \ddots & \\ 0 & & x_{nn} \end{pmatrix}.
$$

Since

$$x_{kk} = \operatorname{tr} x^{(k)} - \operatorname{tr} x^{(k-1)} = \sum_i \lambda_i^{(k)} - \sum_i \lambda_i^{(k-1)},$$

the T-action is contained in the Gelfand–Cetlin system.

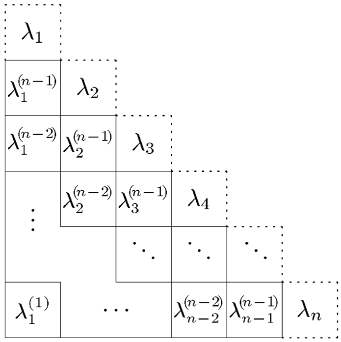

Fig. 3. The ladder diagram as a container of Gelfand–Cetlin patterns.

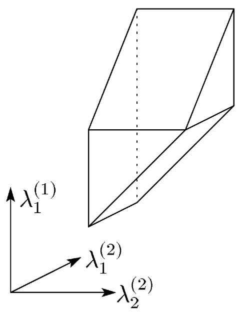

Fig. 4. The Gelfand–Cetlin polytope for $F^{(3)}$.

Remark 3.7. We can think of the ladder diagram as a container of a Gelfand–Cetlin pattern leaning to the right, here $\lambda_1, \dots, \lambda_n$ are placed in the diagonal squares $Q_1, \dots, Q_n$ respectively (see Fig. 3). A vertex of the Gelfand–Cetlin polytope is given by a Gelfand–Cetlin pattern each of whose entry is connected to some λ_i by a chain of equalities. By putting arrows on edges of the ladder diagram where the adjacent entries are distinct, we obtain a tree of positive paths, which is called a *meander* in [4].

Example 3.8. In the case of $F^{(3)}$, the Gelfand–Cetlin system consists of three functions

$$\Phi_\lambda = \left(\lambda_1^{(2)}, \lambda_2^{(2)}, \lambda_1^{(1)}\right) : F^{(3)} \longrightarrow \mathbb{R}^3,$$

and the Gelfand–Cetlin polytope is illustrated in Fig. 4. Topology of the fibers are quite similar to the toric case: for almost every point in Δ_λ, if it is contained in an i-dimensional face, its fiber is an i-dimensional torus. A difference appears at the vertex where four edges are intersecting. The fiber of this point is a three-dimensional sphere S^3. This can be seen as follows. The vertex is given by the equations

$$\begin{array}{ccccc} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$=$} & \\ \lambda_1^{(2)} & & & & \lambda_2^{(2)} \\ & \rotatebox{-45}{$=$} & & \rotatebox{45}{$=$} & \\ & & \lambda_1^{(1)} & & \end{array} .$$

Hence a point $x \in \mathcal{O}_\lambda$ in the fiber of this point must satisfy

$$x^{(2)} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

or equivalently, x must have the form

$$x = \begin{pmatrix} \lambda_2 & & z_1 \\ & \lambda_2 & z_2 \\ \bar{z}_1 & \bar{z}_2 & \nu \end{pmatrix}.$$

The condition that x has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ is equivalent to

$$\nu = \lambda_1 - \lambda_2 + \lambda_3, \qquad |z_1|^2 + |z_2|^2 = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3),$$

which means that the fiber is isomorphic to S^3.

Remark 3.9. The torus action given by the Gelfand–Cetlin system does not preserve the complex structure on F. In fact any torus acting holomorphically on F must be contained in a maximal torus of $PGL(n, \mathbb{C})$ which is the holomorphic automorphism group of the flag manifold. Thus the inverse image of a face of the Gelfand–Cetlin polytope is not necessarily a complex subvariety in F. In the case of $F^{(3)}$, for two faces of dimension two in the back side of Δ_λ in Fig. 4, their inverse images are complex subvarieties. On the other hand, it is not true for other four faces of dimension two.

We move on to the case of a partial flag manifold $F(n_1, \ldots, n_r, n) \cong \mathcal{O}_\lambda$ where λ satisfies (1). (See Fig. 5.) We can consider the functions (4) also in this case. Under the condition (1), (6) implies that

$$\begin{aligned} \lambda_1^{(n-1)} &= \cdots = \lambda_{n_1-1}^{(n-1)} = \lambda_{n_1}, \\ \lambda_{n_1+1}^{(n-1)} &= \cdots = \lambda_{n_2-1}^{(n-1)} = \lambda_{n_2}, \\ &\vdots \end{aligned}$$

which mean that $\lambda_i^{(k)}$ contained in Q_l is a constant function λ_{n_l}. In other words, non-constant $\lambda_i^{(k)}$ exactly corresponds to a box in the ladder diagram. In particular, we have the right number of Poisson commuting functions

$$\Phi_\lambda : \bigl(F(n_1, \ldots, n_r, n), \omega_\lambda\bigr) \longrightarrow \mathbb{R}^{N(n_1,\ldots,n_r,n)}, \quad x \longmapsto \bigl(\lambda_j^{(i)}(x)\bigr).$$

We call this the Gelfand–Cetlin system on $F(n_1, \ldots, n_r, n)$.

Example 3.10. We consider the case of $Gr(2,4)$, where the condition $\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4$ is satisfied. The Gelfand–Cetlin system consists of four functions

$$\Phi_\lambda = \bigl(\lambda_2^{(3)}, \lambda_1^{(2)}, \lambda_2^{(2)}, \lambda_1^{(1)}\bigr) : Gr(2,4) \longrightarrow \mathbb{R}^4.$$

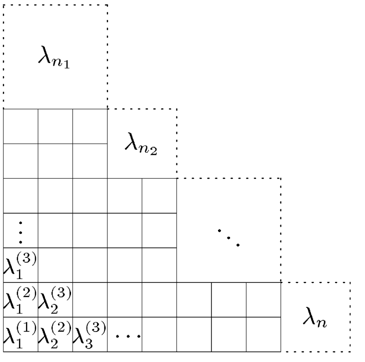

Fig. 5. Gelfand–Cetlin patterns in a partial flag case.

Under the projection $\mathbb{R}^4 \to \mathbb{R}$, $(\lambda_2^{(3)}, \lambda_1^{(2)}, \lambda_2^{(2)}, \lambda_1^{(1)}) \to \lambda_2^{(3)}$, Δ_λ is fibered by Gelfand–Cetlin polytopes for $F^{(3)}$, and the fiber shrinks to a two-dimensional triangle on the boundaries $\lambda_2^{(3)} = \lambda_1, \lambda_3$. We see a fiber of a boundary point given by $\lambda_2^{(3)} = \lambda_1^{(2)} = \lambda_2^{(2)} = \lambda_1^{(1)} = s$. From the argument in the $F^{(3)}$-case, each matrix x in this fiber satisfies

$$x^{(3)} = \begin{pmatrix} s & & z_1 \\ & s & z_2 \\ \bar{z}_1 & \bar{z}_2 & \alpha \end{pmatrix}$$

with $\alpha = \lambda_1 + \lambda_3 - s$ and $|z_1|^2 + |z_2|^2 = (\lambda_1 - s)(s - \lambda_3)$. We assume that $\lambda_1 > s > \lambda_3$, and take $g \in U(3)$ such that $g^* x^{(3)} g = \operatorname{diag}(\lambda_1, s, \lambda_3)$. Note that such g is unique up to scalar. Then it is easy to check that x has the form

$$\begin{pmatrix} g & \\ & 1 \end{pmatrix}^* x \begin{pmatrix} g & \\ & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & & & 0 \\ & s & & z_3 \\ & & \lambda_3 & 0 \\ 0 & \bar{z}_3 & 0 & \alpha \end{pmatrix}$$

with $|z_3|^2 = (\lambda_1 - s)(s - \lambda_3)$. This means that the fiber is a Lagrangian $S^3 \times S^1$. When s goes to a boundary, say λ_1, we have $x = \operatorname{diag}(\lambda_1, \lambda_1, \lambda_3, \lambda_3)$, which means that the fiber shrinks to a point.

In the rest of this section, we see some properties of the Gelfand–Cetlin polytope. Readers who are interested only in toric degeneration of the Gelfand–Cetlin system can skip to the next section. First we consider the case where ω_λ represents the anti-canonical class, or equivalently, λ is given by (3).

Definition 3.11. An N-dimensional integral polytope $\Delta \subset \mathbb{R}^N$ is said to be *reflexive* if the following two conditions fold:

(i) all codimension one faces of Δ are supported by an affine hyperplane of the form $\{u \in \mathbb{R}^N \mid \langle u, v \rangle = -1\}$ for some $v \in \mathbb{Z}^N$, where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^N$.
(ii) Δ contains only one integral point 0 in its interior.

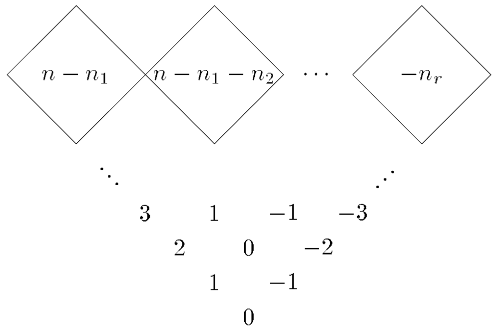

Fig. 6. An integral point in the Gelfand–Cetlin polytope.

It is proved in [2] that Δ is reflexive if and only if the toric variety obtained from Δ by the Delzant construction is Fano.

Lemma 3.12. *Under the condition* (3)*, the Gelfand–Cetlin polytope* Δ_λ *is reflexive after a translation.*

See also [4], where the same is proved in the dual side.

Proof. For each k and i, we set $\lambda_i^{(k)} = k - 2i + 1$. Then it is easy to check that the collection of these $\lambda_i^{(k)}$ gives a Gelfand–Cetlin pattern. The definition implies that

$$\lambda_i^{(k+1)} = \lambda_i^{(k)} + 1 > \lambda_i^{(k)} > \lambda_i^{(k)} - 1 = \lambda_{i+1}^{(k+1)}$$

for each k and i, which means that this $(\lambda_i^{(k)})$ is the unique integral point in the interior of Δ_λ. (See Fig. 6.) $\square$

Next we compute the volume of the Gelfand–Cetlin polytope Δ_λ, though it is not used in the following.

Proposition 3.13. *Under the condition* (1)*, the volume of the Gelfand–Cetlin polytope* Δ_λ *is given by*

$$\mathrm{Vol}(\Delta_\lambda) = \frac{\prod_{i<j} (\lambda_{n_i} - \lambda_{n_j})^{k_i k_j}}{\prod_{k=1}^{n-1} k!}.$$

Proof. When all λ_i are integers, the Borel–Weil theory states that the space $H^0(F, \mathcal{L}_\lambda)$ of holomorphic sections of $\mathcal{L}_\lambda$ is an irreducible representation of $U(n)$ of highest weight λ. Gelfand and Cetlin [10] constructed a basis of $H^0(F, \mathcal{L}_\lambda)$ called the Gelfand–Cetlin basis, which is indexed by integral points of Δ_λ. In particular, the dimension of $H^0(F, \mathcal{L}_\lambda)$ is equal to the number

$\#(\Delta_\lambda \cap \mathbb{Z}^N)$ of integral points in Δ_λ. On the other hand, the dimension of $H^0(F, \mathcal{L}_\lambda)$ is given by the Weyl dimension formula

$$\dim H^0(F, \mathcal{L}_\lambda) = \frac{\prod_{i<j}(\lambda_i - \lambda_j + j - i)}{\prod_{k=1}^{n-1} k!}.$$

The proposition follows from

$$\frac{1}{m^N}\#\bigl(\Delta_{m\lambda} \cap \mathbb{Z}^N\bigr) = \mathrm{Vol}(\Delta_\lambda) + O\left(\frac{1}{m}\right)$$

and

$$\begin{aligned}\frac{1}{m^N}\dim H^0(F, \mathcal{L}_{m\lambda}) &= \frac{1}{m^N}\frac{\prod_{i<j}(m\lambda_i - m\lambda_j + j - i)}{\prod_{k=1}^{n-1} k!} \\ &= \frac{\prod_{i<j}(\lambda_{n_i} - \lambda_{n_j})^{k_i k_j}}{\prod_{k=1}^{n-1} k!} + O\left(\frac{1}{m}\right)\end{aligned}$$

for sufficiently large $m \in \mathbb{Z}_{>0}$. □

For example, in the full flag case, the volume of Δ_λ is given by the difference product of $\lambda_1, \dots, \lambda_n$:

$$\mathrm{Vol}(\Delta_\lambda) = \frac{\prod_{i<j}(\lambda_i - \lambda_j)}{\prod_{k=1}^{n-1} k!}.$$

We close this section with computation of the volume of the dual polytope $(\Delta_\lambda)^*$ of Δ_λ for full flag manifolds and Grassmannians, which will be used in Section 12. First we consider the case of full flag manifolds. Let $e_j^{(k)}$ denote the unit vector corresponding to the coordinate $\lambda_j^{(k)}$. Then $(\Delta_\lambda)^*$ is a convex hull of $\pm e_j^{(n-1)}$, $e_j^{(k+1)} - e_j^{(k)}$, and $e_j^{(k)} - e_{j+1}^{(k+1)}$.

Lemma 3.14. *In the full flag case, the volume of $(\Delta_\lambda)^*$ is equal to $2^N/N!$, where $N = n(n-1)/2 = \dim F^{(n)}$.*

To see this, we observe that Δ^* can be constructed successively as follows. We start with $\pm e_1^{(n-1)}$, which give an interval $[-1, 1]$. Adding $\pm e_2^{(n-1)}$ we obtain a square which is a union of two triangles of height 1 over the base $[-1, 1]$, glued along the interval. By adding $\pm e_3^{(n-1)}$ further, we get an octahedron, two pyramids of height 1 over the square, glued along the common base square, and so on. The construction is almost the same for rows below $\lambda_j^{(n-1)}$: adding $e_j^{(k+1)} - e_j^{(k)}$ and $e_j^{(k)} - e_{j+1}^{(k+1)}$ we also obtain two cones of height 1 over the polytope constructed at this stage, glued along their bases. Therefore the volume of $(\Delta_\lambda)^*$ is given by

$$2 \cdot \frac{2}{2} \cdot \frac{2}{3} \cdots \frac{2}{N} = \frac{2^N}{N!}.$$

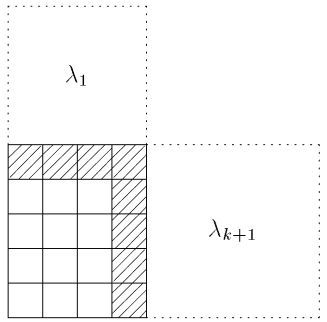

Fig. 7. A Gelfand–Cetlin pattern for a Grassmannian.

In the case of Grassmannian $Gr(k,n)$, we start with the shaded boxes in Fig. 7. Since

$$\lambda_1 \geqslant u_1 \geqslant u_2 \geqslant \cdots \geqslant u_{n-1} \geqslant \lambda_{k+1}$$

are the only nontrivial relations, this part gives a simplex with vertices

$$\begin{pmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \\ 0 \\ 1 \end{pmatrix}.$$

Its volume is given by

$$\frac{1}{(n-1)!} \det \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 0 & & & \\ 0 & -1 & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & 0 & -1 & 1 & 0 \\ 0 & & & 0 & -1 & 1 \end{pmatrix} = \frac{n}{(n-1)!}.$$

Since contributions of the remaining boxes are the same as in the full flag case, we obtain the following.

Lemma 3.15. *The volume of* $(\Delta_\lambda)^*$ *for the Grassmannian* $Gr(k,n)$ *is given by*

$$\mathrm{Vol}(\Delta_\lambda)^* = \frac{n 2^{N-(n-1)}}{N!},$$

where $N = k(n-k) = \dim Gr(k,n)$.

4. Refinement of Gelfand–Cetlin polytopes and small resolutions

In this section, we show the following result about the geometry of Gelfand–Cetlin polytopes:

Proposition 4.1. *Every refinement of the fan Σ of a Gelfand–Cetlin toric variety into simplicial cones, without adding new rays, gives a resolution of the Gelfand–Cetlin toric variety. In particular, this gives a small resolution.*

Remark 4.2. This statement was shown in [1] for a particular refinement of the fan. See also the discussion in the last part of Section 5 for the full flag case. Below we give a different proof, which gives the result for any refinement.

To prove the proposition, it suffices to show the following lemma.

Lemma 4.3. *Let σ be any N-dimensional simplicial cone given by N-rays of a maximal-dimensional cone in the fan Σ. Then the monoid $\mathbb{Z}^N \cap \sigma$ is generated by the integral generators of the rays of σ.*

Proof. Since each ray in Σ corresponds to a toric divisor, which is given by Gelfand–Cetlin patterns such that just one of the inequalities is chosen to be an equality, the choice of rays can be expressed as a set of equalities in a Gelfand–Cetlin pattern. Each equality has the form

$$\lambda_i - \lambda_i^{(n-1)} = 0 \quad \text{or} \quad \lambda_i^{(n-1)} - \lambda_{i+1} = 0$$

if the equality appears in the top row of the Gelfand–Cetlin pattern, and

$$\lambda_i^{(k-1)} - \lambda_{i+1}^{(k)} = 0 \quad \text{or} \quad \lambda_i^{(k)} - \lambda_i^{(k-1)} = 0$$

otherwise. Hence the generators of one-dimensional cones in Σ have the form $(0, \dots, 0, \pm 1, 0, \dots, 0)$ or $(0, \dots, 0, \pm 1, 0, \dots, 0, \mp 1, 0, \dots, 0)$. Recall that each cone in Σ of maximal dimension corresponds to a vertex of the Gelfand–Cetlin polytope, and the vertex is given by a Gelfand–Cetlin pattern such that every entry is connected to some λ_i in the top row by a chain of equalities. Note also that every singular locus is a toric stratum such that the corresponding Gelfand–Cetlin patterns contain a loop of equalities such as

$$\begin{array}{ccccc} & & \lambda_{i+1}^{(k+1)} & & \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$=$} & \\ \lambda_i^{(k)} & & & & \lambda_{i+1}^{(k)} \\ & \rotatebox{-45}{$=$} & & \rotatebox{45}{$=$} & \\ & & \lambda_i^{(k-1)} & & \end{array},$$

and such a loop makes the corresponding cone to be non-simplicial. Hence choosing N-rays which give a simplicial cone is equivalent to removing some of the equalities in the Gelfand–Cetlin pattern so that the resulting set of equalities does not form any loop. Note that the resulting chain of equalities may contain a part which does not occur in Gelfand–Cetlin patterns such as

$$\begin{array}{ccccc} & & \lambda_{i+1}^{(k+1)} & & \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$=$} & \\ \lambda_i^{(k)} & & & & \lambda_{i+1}^{(k)} \\ & \rotatebox{-45}{$=$} & & \rotatebox{60}{$\gneq$} & \\ & & \lambda_i^{(k-1)} & & \end{array}.$$

By arraying the generators of the rays of σ considered as column vectors, we obtain an $N \times N$ matrix A. Then Lemma 4.3 follows from the following:

Claim 1. *The matrix A has determinant* ± 1.

Proof. Since the chain of equalities above is a tree when it is regarded as a graph in such a way that its edges are given by the equalities, we can take a univalent end. The row in A corresponding to this univalent end has the form $(0, \dots, 0, \pm 1, 0, \dots, 0)$, and hence the calculation of the determinant can be reduced to that for an $(N-1) \times (N-1)$ matrix. We obtain the above claim by repeating this process. $\square$

5. Degeneration of flag manifolds in stages

It is known that $F(n_1, \dots, n_r, n)$ degenerates into the *Gelfand–Cetlin toric variety*, a toric variety which corresponds to the Gelfand–Cetlin polytope [13,19,4]. In this section, we recall the construction of toric degenerations in [19], with minor changes.

The toric degeneration is given by deforming the Plücker embedding. For that purpose, we introduce a weight w_{ij} of each variable z_{ij} given by

$$w_{ij} = \begin{cases} 3^{i-j-1}, & i > j, \\ 0, & i \leqslant j. \end{cases}$$

Namely, the matrix of weights w_{ij} is given by

$$w = (w_{ij}) = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ 3 & 1 & 0 & & \\ \vdots & \ddots & \ddots & \ddots & \\ 3^{n-2} & \dots & 3 & 1 & 0 \end{pmatrix}.$$

For each $I = \{i_1 < \dots < i_k\} \subset \{1, \dots, n\}$, we set

$$q_I(z,t) := t^{-\operatorname{tr} w_I} p_I\big(t^{w_{ij}} z_{ij}\big) = t^{-\operatorname{tr} w_I} \det\big(t^{w_{ij}} z_{ij}\big)_I.$$

Since the diagonal term

$$d_I(z) = z_{i_1 1} z_{i_2 2} \dots z_{i_k k}$$

of $p_I(z_{ij}) = \det z_I$ is the unique term of the lowest weight, $q_I(z,t)$ is a polynomial in z_{ij} and t. From the construction, $q_I(z,1) = p_I(z)$ is a Plücker coordinate for $t = 1$, and $q_I(z,0) = d_I(z)$ is a *monomial* for $t = 0$. We define a one-parameter family of projective varieties by

$$\mathfrak{X} = \mathfrak{X}(n_1, \dots, n_l, n) = \text{multiple} \operatorname{Proj} \mathbb{C}\big[t, q_I;\ |I| = n_1, \dots, n_r\big] \subset \prod_{i=1}^{r} \mathbb{P}_{n_i} \times \mathbb{C}.$$

In the full flag case, we simply write $\mathfrak{X}(1,\dots,n)=\mathfrak{X}^{(n)}$. Note that $\mathfrak{X}(n_1,\dots,n_r,n)$ is obtained from $\mathfrak{X}^{(n)}$ by the natural projection

$$\begin{array}{ccc} \pi_{n_1,\dots,n_l}: \mathfrak{X}^{(n)} & \longrightarrow & \mathfrak{X}(n_1,\dots,n_r,n) \\ \downarrow & & \downarrow \\ \mathbb{C} & = & \mathbb{C}. \end{array}$$

Theorem 5.1. *$f:\mathfrak{X}\to\mathbb{C}$ is a flat family of projective varieties such that $X_1:=f^{-1}(1)$ is the flag manifold F, and the central fiber $X_0:=f^{-1}(0)$ is the Gelfand–Cetlin toric variety.*

The existence of toric degenerations is first proved by Gonciulea and Lakshmibai [13], and the fact that X_0 is isomorphic to the Gelfand–Cetlin toric variety is proved by Kogan and Miller [19] in the full flag case. The results are generalized to partial flag cases by Batyrev et al. [3,4]. Note that X_0 is singular except for the trivial case, i.e. the case of projective spaces. It is proved in [4] that the singular locus of X_0 consists of codimension three conifold strata.

Remark 5.2. For every $t\neq 0$, $X_t=f^{-1}(t)$ is isomorphic to the flag manifold F as a complex manifold. On the other hand, the restriction $\widetilde{\omega}_\lambda|_{X_t}$ of the Kähler form defined in (2) coincides with the Kostant–Kirillov form only when $|t|=1$. Note also that the natural $U(k)$-action on $\prod\mathbb{P}_{n_i}=\prod\mathbb{P}(\bigwedge^{n_i}\mathbb{C}^n)$ $(k\geqslant 2)$ induced from the inclusion (5) does not preserve X_t for $|t|\neq 1$ in general.

Example 5.3. The degenerating family for the full flag manifold $F^{(3)}$ of dimension three is given by

$$\mathfrak{X}=\bigl\{\bigl([Z_1:Z_2:Z_3],[Z_{12}:Z_{13}:Z_{23}],t\bigr)\bigm| Z_1Z_{23}-Z_2Z_{13}+tZ_3Z_{12}=0\bigr\},$$

with the central fiber

$$X_0=\bigl\{\bigl([Z_1:Z_2:Z_3],[Z_{12}:Z_{13}:Z_{23}]\bigr)\in\mathbb{P}^2\times\mathbb{P}^2\bigm| Z_1Z_{23}-Z_2Z_{13}=0\bigr\}.$$

Note that X_0 has a singularity at $([0:0:1],[1:0:0])$, which corresponds to the vertex of the Gelfand–Cetlin polytope emanating four edges.

Example 5.4. Recall that $Gr(2,4)$ is embedded into $\mathbb{P}^5$ with its defining equation $Z_{12}Z_{34}-Z_{13}Z_{24}+Z_{14}Z_{23}=0$. The equation for the toric degeneration of $Gr(2,4)$ is given by

$$tZ_{12}Z_{34}-Z_{13}Z_{24}+Z_{14}Z_{23}=0.$$

Then X_0 has conifold singularities along $\{Z_{13}=Z_{24}=Z_{14}=Z_{23}=0\}=\mathbb{P}^1$.

Now we see the Gelfand–Cetlin toric variety X_0 more closely. Let $I=\{i_1<\dots<i_k\}$ and $J=\{j_1<\dots<j_l\}$ be subsets of $\{1,\dots,n\}$ with $k\leqslant l$, and γ_I, γ_J the corresponding positive paths. We define their meet and join by

$$\begin{aligned} I\wedge J&=\bigl\{\min(i_1,j_1),\dots,\min(i_k,j_k),j_{k+1},\dots,j_l\bigr\},\\ I\vee J&=\bigl\{\max(i_1,j_1),\dots,\max(i_k,j_k)\bigr\}. \end{aligned}$$

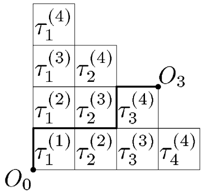

Fig. 8. Positive paths and monomials on X_0.

Then $I \wedge J$ (resp. $I \vee J$) corresponds to a positive path moving along the lower (resp. upper) route of the union $\gamma_I \cup \gamma_J$. The defining equations for the Gelfand–Cetlin toric variety $X_0 \subset \prod_i \mathbb{P}_{n_i}$ are given by the following binomial relations

$$Z_I Z_J - Z_{I\wedge J} Z_{I\vee J} = 0 \tag{7}$$

(see [13] or [19]). Next we see the monomial embedding of X_0 into $\prod \mathbb{P}_{n_i}$. Let T_k be a torus corresponding to the k-th row $(\lambda_i^{(k)})_i$ from the bottom of Gelfand–Cetlin patterns. In the full flag case, T_k is a k-dimensional torus T^k. We take natural coordinates $(\tau_i^{(k)})_i$ on $T_k^{\mathbb{C}} = T_k \otimes \mathbb{C}$, and consider the following matrix

$$\tau = \begin{pmatrix} \tau_1^{(n-1)} \cdots \tau_1^{(2)} \tau_1^{(1)} & & & & \\ \tau_1^{(n-1)} \cdots \tau_1^{(2)} & \tau_2^{(n-1)} \cdots \tau_2^{(2)} & & & \\ \vdots & \vdots & \ddots & & \\ \tau_1^{(n-1)} \tau_1^{(n-2)} & \tau_2^{(n-1)} \tau_2^{(n-2)} & \cdots \ \tau_{n-2}^{(n-1)} \tau_{n-2}^{(n-2)} & & \\ \tau_1^{(n-1)} & \tau_2^{(n-1)} & \cdots \ \tau_{n-2}^{(n-1)} & \tau_{n-1}^{(n-1)} & \\ 1 & 1 & \cdots \ 1 & 1 & 1 \end{pmatrix},$$

where we assume that $\tau_i^{(k)} = 1$ if the corresponding $\lambda_i^{(k)}$ is contained in a diagonal square Q_l. Then the embedding of X_0 into $\prod_i \mathbb{P}_{n_i}$ is given by the monomials

$$Z_I = d_I(\tau), \quad |I| = n_1, \ldots, n_r. \tag{8}$$

These monomials can be described using the ladder diagram in the following way. We put $\tau_i^{(k)}$ on the ladder diagram in the same way as for $\lambda_i^{(k)}$. Then each monomial is written as $d_I(\tau) = \prod \tau_i^{(k)}$, where the product is taken over $\tau_i^{(k)}$'s placed above the positive path γ_I corresponding to I. For example, the path in Fig. 8 corresponds to $\tau_1^{(4)}\tau_1^{(3)}\tau_1^{(2)}\tau_2^{(4)}\tau_2^{(3)}$. It is easy from this expression to see that these monomials satisfy the binomial relations (7).

We extend $f : \mathfrak{X} \to \mathbb{C}$ to an $(n-1)$-parameter family to define the *degeneration of flag manifolds in stages* which is also introduced in [19], on which we will construct a toric degeneration of the Gelfand–Cetlin system. Let $\boldsymbol{t} = (t_2, \ldots, t_n)$ be parameters and set

$$\widetilde{w}_{k,ij} = \begin{cases} 0, & i < k, \\ w_{kj} - w_{k-1,j}, & i \geqslant k \end{cases}$$

for $k = 2, \dots, n$. Then the weight of z_{ij} is extended to a multi-weight

$$\boldsymbol{t}^{\widetilde{w}_{ij}} := t_2^{\widetilde{w}_{2,ij}} t_3^{\widetilde{w}_{3,ij}} \cdots t_n^{\widetilde{w}_{n,ij}}.$$

Then the matrix of multi-weights is given by

$$\left(\boldsymbol{t}^{\widetilde{w}_{ij}}\right)_{ij} = \begin{pmatrix} 1 & & & & & \\ t_2 & 1 & & & & \\ t_2 t_3^2 & t_3 & 1 & & & \\ t_2 t_3^2 t_4^6 & t_3 t_4^2 & t_4 & 1 & & \\ \vdots & \vdots & & \ddots & \ddots & \\ t_2 t_3^2 t_4^6 \dots t_n^{2\cdot 3^{n-3}} & t_3 t_4^2 \dots t_n^{2\cdot 3^{n-4}} & \dots & & t_n & 1 \end{pmatrix}.$$

Note that t_k does not appear above the k-th row. We set

$$\tilde{q}_I(z_{ij}, t_2, \dots, t_n) = d_I \left(\boldsymbol{t}^{\widetilde{w}_{ij}}\right)^{-1} p_I \left(\boldsymbol{t}^{\widetilde{w}_{ij}} z_{ij}\right) \in \mathbb{C}[z_{ij}, t_k].$$

Since $\boldsymbol{t}^{\widetilde{w}_{ij}} = t^{w_{ij}}$ for $\boldsymbol{t} = (t, \dots, t)$, we have $\tilde{q}_I(z_{ij}, t, \dots, t) = q_J(z_{ij}, t)$. By taking multiple Proj of $\mathbb{C}[t_i, \tilde{q}_I]$, we obtain an $(n-1)$-parameter family $\tilde{f} : \widetilde{\mathfrak{X}} \to \mathbb{C}^{n-1}$ of projective varieties. From the construction, $X_{(1,\dots,1)} := \tilde{f}^{-1}(1, \dots, 1)$ is isomorphic to the flag manifold F, and $X_{(0,\dots,0)} = \tilde{f}^{-1}(0, \dots, 0)$ is the Gelfand–Cetlin toric variety X_0. An important point is that the $U(k-1)$-action on $\prod_i \mathbb{P}_{n_i}$ preserves each fiber $X_{(1,\dots,1,t_k,\dots,t_n)}$ for $t_2 = \dots = t_{k-1} = 1$, and the action of $T_{n-1} \times \dots \times T_k$ preserves $X_{(t_2,\dots,t_k,0,\dots,0)}$ for $t_{k+1} = \dots = t_n = 0$ (see [19], or the discussion below). Hence we consider a sequence of degenerations given by varying the parameters as follows:

$$\boldsymbol{t} = (1, \dots, 1) \rightsquigarrow (1, \dots, 1, 0) \rightsquigarrow (1, \dots, 1, 0, 0) \rightsquigarrow \dots \rightsquigarrow (0, \dots, 0).$$

Let

$$\begin{array}{rccc} f_k : & \mathfrak{X}_k = \widetilde{\mathfrak{X}}|_{\substack{t_2 = \dots = t_{k-1} = 1 \\ t_{k+1} = \dots = t_n = 0}} & \longrightarrow & \mathbb{C} \\ & \cup & & \Psi \\ & X_{(1,\dots,1,t_k,0,\dots,0)} & \longrightarrow & t_k \end{array} \tag{9}$$

denote the $(n-k+1)$-th stage of the degeneration, i.e. a one-parameter sub-family given by fixing $t_2 = \dots = t_{k-1} = 1$ and $t_{k+1} = \dots = t_n = 0$. We write $X_{k,t_k} := f_k^{-1}(t_k) = X_{(1,\dots,1,t_k,0,\dots,0)}$ for short. Note that each $X_{k,t}$ has actions of $T_{n-1} \times \dots \times T_k$ and $U(k-1)$. In particular, $X_{k,1} = X_{k+1,0}$ admits actions of $T_{n-1} \times \dots \times T_k$ and $U(k)$.

Remark 5.5. The final stage $f_2 : \mathfrak{X}_2 \to \mathbb{C}$ of the toric degeneration is a trivial family. For example, the two-parameter family for $F^{(3)}$ is given by

$$t_2 Z_1 Z_{23} - t_2 Z_2 Z_{13} + t_3^2 Z_3 Z_{12} = 0.$$

This fact is related to the fact that $F^{(2)} = \mathbb{P}^1$ is a toric variety and the family in this case is trivial (see the discussion below).

Now we consider the full flag case, and see $X_{n,0} = X_{n-1,1} = X_{(1,\dots,1,0)}$ in more detail, where the actions of T_{n-1} and $U(n-1)$ can be seen explicitly as follows:

Lemma 5.6. *$X_{n-1,1}$ has a resolution $h_{n-1,1} : Y_{n-1,1} \to X_{n-1,1}$ such that $Y_{n-1,1}$ has a structure of $(\mathbb{P}^1)^{n-1}$-bundle over a smaller flag manifold $F^{(n-1)}$. The $U(n-1)$-action on $X_{n-1,1}$ is induced from the standard one on $F^{(n-1)}$, and the T_{n-1}-action comes from the natural torus action on the $(\mathbb{P}^1)^{n-1}$-fibers.*

Proof. We consider the multi-parameter family $\widetilde{\mathfrak{X}}^{(n-1)} \to \mathbb{C}^{n-2}$ for $F^{(n-1)}$. Recall that each fiber is a subvariety in $\mathbb{P}^{(n-1)} := \prod_k \mathbb{P}(\bigwedge^k \mathbb{C}^{n-1})$. Let $\mathcal{E}_0 = \mathcal{E}_{n-1} = \mathcal{O}_{\mathbb{P}^{(n-1)}}$ be trivial bundles and $\mathcal{E}_k = \mathrm{pr}_k^* \mathcal{O}_{\mathbb{P}(\bigwedge^k \mathbb{C}^{n-1})}(1)$ for each k, where $\mathrm{pr}_k : \mathbb{P}^{(n-1)} \to \mathbb{P}(\bigwedge^k \mathbb{C}^{n-1})$ is the natural projection. We define a $(\mathbb{P}^1)^{n-1}$-bundle on $\mathbb{P}^{(n-1)}$ by

$$E := \mathbb{P}(\mathcal{E}_0 \oplus \mathcal{E}_1) \times_{\mathbb{P}^{(n-1)}} \mathbb{P}(\mathcal{E}_1 \oplus \mathcal{E}_2) \times_{\mathbb{P}^{(n-1)}} \cdots \times_{\mathbb{P}^{(n-1)}} \mathbb{P}(\mathcal{E}_{n-2} \oplus \mathcal{E}_{n-1}).$$

Restricting this to each fiber $X^{(n-1)}_{(t_2,\dots,t_{n-1})}$, we obtain a family

$$\begin{array}{ccc} \mathfrak{Y} & \longrightarrow & \widetilde{\mathfrak{X}}^{(n-1)} \\ \cup & & \cup \\ Y_{(t_2,\dots,t_{n-1})} & \longrightarrow & X^{(n-1)}_{(t_2,\dots,t_{n-1})} \end{array}$$

of $(\mathbb{P}^1)^{n-1}$-bundles. We claim that there exists a surjective birational morphism

$$\begin{array}{ccc} \mathfrak{Y} & \longrightarrow & \widetilde{\mathfrak{X}}|_{t_n=0} \\ \cup & & \cup \\ Y_{(t_2,\dots,t_{n-1})} & \longrightarrow & X_{(t_2,\dots,t_{n-1},0)}. \end{array}$$

To see this, we observe that

$$\tilde{q}_I(z, t_2, \dots, t_{n-1}, 0) = \begin{cases} \tilde{q}_I\big(z^{(n-1)}, t_2, \dots, t_{n-1}\big), & i_k < n, \\ z_{nk}\tilde{q}_{i_1,\dots,i_{k-1}}\big(z^{(n-1)}, t_2, \dots, t_{n-1}\big), & i_k = n \end{cases} \tag{10}$$

for $I = \{i_1 < \cdots < i_k\} \subset \{1, \dots, n\}$, where $\tilde{q}_I$ in the right-hand side is regarded as a function of $(n-1) \times (n-1)$ matrices $z^{(n-1)}$. Let Z'_I, $I \subset \{1, \dots, n-1\}$ be the homogeneous coordinates of $\prod_k \mathbb{P}(\bigwedge^k \mathbb{C}^{n-1})$, where we assume that $Z'_\emptyset = Z'_{1,\dots,n-1} = 1$, and $[u_i : v_i]$ the fiber coordinates of $\mathbb{P}(\mathcal{E}_{i-1} \oplus \mathcal{E}_i)$ with $u_i \in \mathcal{E}_{i-1}$ and $v_i \in \mathcal{E}_i$. Then (10) implies that

$$Z_I = \begin{cases} u_k Z'_I, & \text{if } n \notin I, \\ v_k Z'_{i_1,\dots,i_{k-1}}, & \text{if } I = \{i_1, \dots, i_{k-1}, i_k = n\} \end{cases} \tag{11}$$

on $\mathbb{P}_k$ gives a surjective birational morphism

$$Y_{(t_2,\dots,t_{n-1})} \subset E \longrightarrow X_{(t_2,\dots,t_{n-1},0)} \subset \prod \mathbb{P}_k$$

for each $(t_2, \dots, t_{n-1})$ (we may think that $z_{nk} = v_k/u_k$). In particular, we have a resolution

$$h_{n-1,1} : Y_{n-1,1} := Y_{(1,\dots,1)} \longrightarrow X_{n-2,1} = X_{(1,\dots,1,0)}$$

of $X_{n-1,1}$. We can see from (11) that the $U(n-1)$-action on $X_{n-1,1}$ is induced from the standard one on $F^{(n-1)} = X^{(n-1)}_{(1,\dots,1)}$. On the other side, (8) implies that the T_{n-1}-action on $\prod \mathbb{P}_k$ is given by

$$Z_I \longmapsto \begin{cases} \tau_1^{(n-1)} \dots \tau_k^{(n-1)} Z_I, & \text{if } n \notin I, \\ \tau_1^{(n-1)} \dots \tau_{k-1}^{(n-1)} Z_I, & \text{if } I = \{i_1, \dots, i_{k-1}, i_k = n\} \end{cases}$$

on $\mathbb{P}_k$, or equivalently,

$$[Z_I : Z_{I'n}]_{I,I' \subset \{1,\dots,n-1\}} \longmapsto \left[\tau_k^{(n-1)} Z_I : Z_{I'n}\right]_{I,I' \subset \{1,\dots,n-1\}}.$$

This together with (11) mean that the action of T_{n-1} is induced from a natural action on the fiber $(\mathbb{P}^1)^{n-1}$. $\square$

By repeating this process, we have an iterated fibration

$$Y_{m,1} \xrightarrow{(\mathbb{P}^1)^{n-1}} \cdots \xrightarrow{(\mathbb{P}^1)^{m}} F^{(m)}$$

over $F^{(m)}$ and a resolution $h_{m,1} : Y_{m,1} \longrightarrow X_{m,1}$ for each $m = n-1, \dots, 2$ such that the $U(m)$-action on $X_{m,1}$ is induced from the standard one on $F^{(m)}$, and the torus action comes from a natural action on the fibers. Taking the Plücker coordinates $([Z'_I]_{|I|=k})_k$ on $F^{(m)}$ and fiber coordinates $([u_i^n : v_i^n])_{i=1,\dots,n-1} \in (\mathbb{P}^1)^{n-1}, \dots, ([u_i^{m+1} : v_i^{m+1}])_{i=1,\dots,m} \in (\mathbb{P}^1)^m$, $h_{m,1}$ is given by

$$Z_I = \chi_{l,I''}(u,v) \cdot Z'_{I'} \tag{12}$$

for $I = \{i_1 < \dots < i_k < i_{k+1} < \dots < i_l\} \subset \{1,\dots,n\}$ with $I' = \{i_1,\dots,i_k\} \subset \{1,\dots,m\}$ and $I'' = \{i_{k+1},\dots,i_l\} \subset \{m+1,\dots,n\}$, where $\chi_{l,I''}(u,v)$ is a monomial in u_i^k, v_j^l defined inductively by (11). $\chi_{l,I''}(u,v)$ is explicitly given by

$$\chi_{l,I''}(u,v) = u_l^n \cdots u_l^{i_l+1} v_l^{i_l} u_{l-1}^{i_l-1} \cdots u_{l-1}^{i_{l-1}+1} v_{l-1}^{i_{l-1}} u_{l-2}^{i_{l-1}-1} \cdots v_{k+1}^{i_{k+1}} u_k^{i_{k+1}-1} \cdots u_k^{m+1}.$$

Note that (12) also gives a birational surjective morphism $h_{m,t} : Y_{m,t} \to X_{m,t}$ for each t.

In particular, we obtain a resolution $Y_0 \to X_0$ of the Gelfand–Cetlin toric variety such that Y_0 has a structure of iterated fibration over $\mathbb{P}^1$. This is a small resolution of X_0 constructed in [4].

6. Toric degeneration of Gelfand–Cetlin systems

In this and the next sections we construct a toric degeneration of the Gelfand–Cetlin system using the degeneration in stages discussed in the previous section.

For each $m = 1, \dots, n-1$, we consider the natural $U(m)$-action on $\prod_{j=1}^r \mathbb{P}_{n_j}$ which is an extension of the action on $F(n_1, \dots, n_r, n)$. Let

$$\mu^{(m)} : \prod_{j=1}^r \mathbb{P}_{n_j} \longrightarrow \sqrt{-1}\mathfrak{u}(m)$$

denote the moment map of the $U(m)$-action, and

$$\tilde{\lambda}_i^{(m)} : \prod_{j=1}^r \mathbb{P}_{n_j} \longrightarrow \mathbb{R}, \quad \tilde{\lambda}_1^{(m)} \geqslant \cdots \geqslant \tilde{\lambda}_m^{(m)}$$

be functions which associate eigenvalues of $\mu^{(m)}(Z)$ for each $Z \in \prod_{j=1}^r \mathbb{P}_{n_j}$. From the construction, the collection of $\tilde{\lambda}_j^{(m)}$'s restricted to X_1 gives the Gelfand–Cetlin system on $F(n_1, \ldots, n_r, n)$. Hereafter we do not assume that indices of Z_I are increasing, and use the convention

$$Z_{\sigma(i_1),\ldots,\sigma(i_k)} = (\operatorname{sgn}\sigma) Z_{i_1,\ldots,i_k}$$

for $\sigma \in \mathfrak{S}_k$. Then the moment map $\mu^{(n)}$ of the $U(n)$-action is given by

$$\mu^{(n)}(Z) = \sum_{k=1}^r \frac{\lambda_{n_k} - \lambda_{n_{k+1}}}{\sum_{|I|=n_k} |Z_I|^2} \left(\sum_{|I'|=n_k-1} Z_{iI'} \overline{Z}_{jI'} \right)_{i,j=1,\ldots,n} + \lambda_n \cdot 1_n,$$

and $\mu^{(m)}$ is its $m \times m$ upper left block

$$\mu^{(m)}(Z) = \sum_{k=1}^r \frac{\lambda_{n_k} - \lambda_{n_{k+1}}}{\sum_{|I|=n_k} |Z_I|^2} \left(\sum_{|I'|=n_k-1} Z_{iI'} \overline{Z}_{jI'} \right)_{i,j=1,\ldots,m} + \lambda_n \cdot 1_m. \tag{13}$$

We also extend to $\prod_{j=1}^r \mathbb{P}_{n_j}$ the torus action on the Gelfand–Cetlin toric variety X_0, and consider the moment map

$$\tilde{\nu}_i^{(m)} : \prod_{j=1}^r \mathbb{P}_{n_j} \longrightarrow \mathbb{R}$$

of the action of $\tau_i^{(m)}$. From (8), the torus action is given by

$$Z_I \longmapsto d_I(\tau) \cdot Z_I.$$

In particular, the action of T_m is given by

$$Z_I \longmapsto \tau_1^{(m)} \ldots \tau_k^{(m)} Z_I$$

for $I = \{i_1 < \cdots < i_k < i_{k+1} < \cdots < i_l\}$ with $\{i_1 < \cdots < i_k\} \subset \{1, \ldots, m\}$ and $\{i_{k+1} < \cdots < i_l\} \subset \{m+1, \ldots, n\}$, where we assume that $\tau_i^{(m)} = 1$ if it is contained in a diagonal square Q_j. Hence we have

$$\tilde{\nu}_j^{(m)} = \sum_{k=1}^r \frac{\lambda_{n_k} - \lambda_{n_{k+1}}}{\sum_{|I|=n_k} |Z_I|^2} \sum_{\substack{|I|=n_k, \\ i_j \leqslant m}} |Z_I|^2 + \lambda_n. \tag{14}$$

Note that $\tilde{\lambda}_1^{(1)} = \tilde{\nu}_1^{(1)}$. For each t, we define

$$\Phi_{m,t} = \big(\tilde{\nu}_i^{(n-1)}, \dots, \tilde{\nu}_j^{(m)}, \tilde{\lambda}_k^{(m-1)}, \dots, \tilde{\lambda}_1^{(1)}\big)\big|_{X_{m,t}} : X_{m,t} \longrightarrow \mathbb{R}^{N(n_1,\dots,n_r,n)}.$$

Then $\Phi_{n,1}$ coincides with the Gelfand–Cetlin system on $X_{n,1} = F$, while $\Phi_{2,0}$ is the moment map of the torus action on the Gelfand–Cetlin toric variety X_0.

Theorem 6.1. *Let $f_m : \mathfrak{X}_m \to \mathbb{C}$ be the $(n-m+1)$-th stage of the toric degeneration defined in* (9) *($m = n, \dots, 2$). Then, for each t, $\Phi_{m,t} : X_{m,t} \to \mathbb{R}^N$ is a completely integrable system on $(X_{m,t}, \widetilde{\omega}_\lambda|_{X_{m,t}})$. Moreover, for $t = 0$,*

$$\tilde{\lambda}_k^{(m-1)} = \tilde{\nu}_k^{(m-1)}$$

on $X_{m,0} = X_{m-1,1}$. Namely, $\Phi_{m,0}$ coincides with the initial map $\Phi_{m-1,1}$ in the next stage.

Proof. Functional independence for $\tilde{\lambda}_i^{(k)}$'s and $\tilde{\nu}_j^{(l)}$'s follows from the fact that, for all t, the image $\Phi_{m,t}(X_{m,t})$ coincides with the Gelfand–Cetlin polytope Δ_λ, whose dimension is equal to $\frac{1}{2}\dim_{\mathbb{R}} X_{m,t}$. This will be proved in the next section (Corollary 7.3). We prove here the Poisson commutativity of the functions. Since the symplectic structure and the moment maps $\mu^{(k)}$, $\tilde{\nu}_j^{(l)}$ in the full flag case descend to those in the partial flag case under the condition (1), it suffices to prove in the full flag case.

Since the $U(m-1)$-action preserves $X_{m,t}$, the restriction $\mu^{(m-1)}|_{X_{m,t}}$ gives a moment map of the $U(m-1)$-action on $X_{m,t}$. From Lemma 3.3, we have

$$\big\{\tilde{\lambda}_i^{(k)}, \tilde{\lambda}_j^{(l)}\big\} = 0, \quad k, l \leqslant m-1$$

on $X_{m,t}$. Similarly, the restrictions of $\tilde{\nu}_i^{(k)}$ to $X_{m,t}$ $(k \geqslant m)$ give a moment map of the $T_{n-1} \times \cdots \times T_m$-action on $X_{m,t}$, and hence we obtain

$$\big\{\tilde{\nu}_i^{(k)}, \tilde{\nu}_j^{(l)}\big\} = 0$$

for $k, l \geqslant m$ on $X_{m,t}$.

To see that $\tilde{\lambda}_i^{(k)}$ commutes with $\tilde{\nu}_j^{(l)}$ $(k < m, l \geqslant m)$, we pull them back to $Y_{m,t}$. Hereafter we normalize homogeneous coordinates so that $\sum_{|I|=k} |Z_I|^2 = \sum_{|I|=k} |Z'_I|^2 = |u_i^k|^2 + |v_i^k|^2 = 1$, and assume that $\lambda_n = 0$ for simplicity. From (13) and (12) we have

$$\begin{aligned} h_m^* \mu^{(m)} &= \sum_{l=1}^{n-1} (\lambda_l - \lambda_{l+1}) \left(\sum_{k=1}^{l} \sum_{\substack{I' \subset \{1,\dots,m\}, \\ |I'| = k-1, \\ I'' \subset \{m+1,\dots,n\}, \\ |I''| = l-k}} \chi_{l,I''} Z'_{iI'} \overline{\chi_{l,I''} Z'_{jI'}} \right)_{i,j=1,\dots,m} \\ &= \sum_{l=1}^{n-1} (\lambda_l - \lambda_{l+1}) \left\{ \sum_{k=1}^{l} \left(\sum_{\substack{I'' \subset \{m+1,\dots,n\}, \\ |I''| = l-k}} |\chi_{l,I''}|^2 \right) \left(\sum_{\substack{I' \subset \{1,\dots,m\}, \\ |I'| = k-1}} Z'_{iI'} \overline{Z}'_{jI'} \right)_{i,j=1,\dots,m} \right\}. \end{aligned}$$

Since T_l acts only on the fibers $([u_i^j : v_i^j])_i$, $h_m^*\mu^{(m)}$ (and hence $h_m^*\mu^{(k)}$, $k < m$) is T_l-invariant. In particular, we have

$$\{\tilde{\lambda}_i^{(k)}, \tilde{\nu}_j^{(l)}\} = \xi_{\tilde{\nu}_j^{(l)}} \tilde{\lambda}_i^{(k)} = 0, \quad k \leqslant m-1,\ l \geqslant m$$

on $X_{m,t}$, where $\xi_{\tilde{\nu}_j^{(l)}}$ is the Hamiltonian vector field of $\tilde{\nu}_j^{(l)}$.

Finally we check that $\tilde{\lambda}_j^{(m)} = \tilde{\nu}_j^{(m)}$ on $X_{m,1} = X_{m+1,0}$, which is also seen on the resolution $Y_{m,1}$. Since $h_m^*\mu^{(m)}$ has a form of a moment map of the standard $U(m)$-action on $F^{(m)}$, the eigenvalues $h_{m,1}^*\tilde{\lambda}_j^{(m)}$ are $U(m)$-invariant. In particular, $h_{m,1}^*\tilde{\lambda}_j^{(m)}$ is determined by its values on the fiber over the standard flag in $F^{(m)}$. Recall that the standard flag is given by

$$Z'_I = \begin{cases} 1, & \text{if } I = \{1, \dots, k\}, \\ 0, & \text{otherwise} \end{cases}$$

on $\mathbb{P}_k$. Then we have

$$h_m^*\mu^{(m)} = \sum_{l=1}^{n-1} (\lambda_l - \lambda_{l+1}) \left\{ \sum_{k=1}^{l} \left(\sum_{\substack{I'' \subset \{m+1,\dots,n\}, \\ |I''| = l-k}} |\chi_{l,I''}|^2 \right) \begin{pmatrix} 1_k & \\ & 0_{m-k} \end{pmatrix} \right\}$$

on the fiber of this point. Thus its eigenvalues are given by

$$h_m^*\tilde{\lambda}_j^{(m)} = \sum_{l=1}^{n-1} (\lambda_l - \lambda_{l+1}) \left\{ \sum_{k=j}^{l} \left(\sum_{\substack{I'' \subset \{m+1,\dots,n\}, \\ |I''| = l-k}} |\chi_{l,I''}|^2 \right) \right\}. \tag{15}$$

On the other hand, from (14) we have

$$\begin{aligned} h_m^*\tilde{\nu}_j^{(m)} &= \sum_{l=1}^{n-1} (\lambda_l - \lambda_{l+1}) \left(\sum_{k=j}^{l} \sum_{\substack{I' \subset \{1,\dots,m\}, \\ |I'| = k, \\ I'' \subset \{m+1,\dots,n\}, \\ |I''| = l-k}} \left| \chi_{l,I''} Z'_{I'} \right|^2 \right) \\ &= \sum_{l=1}^{n-1} (\lambda_l - \lambda_{l+1}) \left\{ \sum_{k=j}^{l} \left(\sum_{\substack{I'' \subset \{m+1,\dots,n\}, \\ |I''| = l-k}} |\chi_{l,I''}|^2 \right) \left(\sum_{\substack{I' \subset \{1,\dots,m\}, \\ |I'| = k,}} \left| Z'_{I'} \right|^2 \right) \right\}. \end{aligned}$$

Since $\sum_{|I'|=k} |Z'_{I'}|^2 = 1$, it follows that

$$h_m^*\tilde{\nu}_j^{(m)} = \sum_{l=1}^{n-1} (\lambda_l - \lambda_{l+1}) \left(\sum_{k=j}^{l} \sum_{\substack{I'' \subset \{m+1,\dots,n\}, \\ |I''| = l-k}} |\chi_{l,I''}|^2 \right),$$

which coincides with (15). □

7. Gradient-Hamiltonian flows

For a family of hypersurfaces in a Kähler manifold, W.-D. Ruan [21] constructed a flow, called the *gradient-Hamiltonian flow*, which sends a member of the family to another. We apply this to the toric degenerations of flag manifolds in this section and finish the construction of a toric degeneration of the Gelfand–Cetlin system from the previous section.

Let $(\mathfrak{X}, \widetilde{\omega})$ be a Kähler variety. Assume that we have a family $X_t = \{f = t\}$ ($t \in \mathbb{C}$) of complex hypersurfaces defined by a meromorphic function f on $\mathfrak{X}$. For example, if X_t is given by $X_t = \{s_0 - t s_\infty = 0\}$ for some holomorphic sections $s_0, s_\infty \in H^0(\mathfrak{X}, \mathfrak{L})$ of a line bundle $\mathfrak{L}$ on $\mathfrak{X}$, then we choose $f = s_0/s_\infty$. Let $\nabla(\Re f)$ be the gradient vector field of the real part of f, and $\xi_{\Im f}$ the Hamiltonian vector field of the imaginary part of f, which are defined on the smooth locus of $\mathfrak{X}$. Then the Cauchy–Riemann equation implies that

$$\nabla(\Re f) = -\xi_{\Im f}.$$

We define the *gradient-Hamiltonian vector field* of f by

$$V = -\frac{\nabla(\Re f)}{|\nabla(\Re f)|^2} = \frac{\xi_{\Im f}}{|\xi_{\Im f}|^2}.$$

The flow of V is called the gradient-Hamiltonian flow. From the definition, we have

$$V(\Re f) = -\frac{1}{|\nabla(\Re f)|^2}\langle \nabla(\Re f), \nabla(\Re f)\rangle = -1$$

and

$$V(\Im f) = \frac{1}{|\xi_{\Im f}|^2}\{\Im f, \Im f\} = 0.$$

Therefore the gradient-Hamiltonian flow sends (an open dense subset of) X_1 to another member X_{1-t} of the family. Note that V does not preserve the symplectic structure $\widetilde{\omega}$, because V is normalized and hence not a Hamiltonian vector field. However we can check that the restrictions $\omega_t := \widetilde{\omega}|_{X_t}$ to X_t are preserved. In other words, the gradient-Hamiltonian flow gives a map

$$\phi_t = \exp(tV) : (X_1, \omega_1) \longrightarrow (X_{1-t}, \omega_{1-t})$$

between (open dense subsets of) symplectic varieties.

Remark 7.1. (See Ruan [22].) If we write f locally as u/v for some holomorphic functions u, v, then V can be written as

$$V = \frac{-2\Re(\bar{v}(\nabla v - t\nabla u))}{|du - t\,dv|^2}. \tag{16}$$

In particular, V is smooth on the smooth part of X_t. Note that (16) makes sense even on the locus where f is not defined, if it is smooth.

Proposition 7.2. *Assume that $(\mathfrak{X}, \widetilde{\omega})$ has a Hamiltonian action of a compact Lie group G, which preserves each X_t. Let $\mu : \mathfrak{X} \to \mathfrak{g}^*$ be the moment map. Then, for $h \in C^\infty(\mathfrak{g}^*)$, $\mu^* h$ is invariant under the gradient-Hamiltonian flow of f.*

Proof. G-invariance of f and Lemma 3.4 implies that

$$\xi_{\Im f}\big(\mu^* h\big) = 0,$$

which proves the proposition. □

We apply this to each stage $f_m : \mathfrak{X}_m \to \mathbb{C}$ of the toric degeneration. We take a Kähler form on $\mathfrak{X}_m$, which is invariant under the actions of $U(m-1)$ and $T_{n-1} \times \cdots \times T_m$, and the restriction to each $X_{m,t}$ coincides with $\widetilde{\omega}_\lambda|_{X_{m,t}}$. Then we obtain the gradient-Hamiltonian vector field V_m of f_m and its flow

$$\phi_{m,t} = \exp(t V_m) : \big(X^\circ_{m,1}, \widetilde{\omega}_\lambda|_{X^\circ_{m,1}}\big) \longrightarrow \big(X^\circ_{m,1-t}, \widetilde{\omega}_\lambda|_{X^\circ_{m,1-t}}\big),$$

where $X^\circ_{m,t}$ is an open dense subset of $X_{m,t}$. Proposition 7.2 implies that the gradient-Hamiltonian flow $\phi_{m,t}$ preserves the values of $\tilde{\nu}_i^{(n-1)}, \dots, \tilde{\nu}_j^{(m)}$ and $\tilde{\lambda}_k^{(m-1)}, \dots, \tilde{\lambda}_1^{(1)}$. In particular, the image $\Phi_{m,t}(X^\circ_{m,t})$ is a dense subset of the Gelfand–Cetlin polytope Δ_λ for each t, and hence $\Phi_{m,t}(X_{m,t}) = \overline{\Delta^\circ_\lambda} = \Delta_\lambda$. The above fact also implies that $X^\circ_{m,t}$ can be taken to be $\Phi^{-1}_{m,t}(\Delta^\circ_\lambda)$, where $\Delta^\circ_\lambda = \Delta_\lambda \setminus \mathrm{Sing}(X_0)$. Hence we have

Corollary 7.3. *The gradient-Hamiltonian flow $\phi_{m,t}$ gives a deformation of $X^\circ_{m,t}$ preserving the structure of completely integrable systems. In particular, the image $\Phi_{m,t}(X_{m,t})$ is the Gelfand–Cetlin polytope Δ_λ for $t \geqslant 0$:*

$$\begin{tikzcd} X^\circ_{m,1} \arrow[rr, "\phi_{m,1-t}"] \arrow[dr, "\Phi_{m,1}"'] & & X^\circ_{m,t} \arrow[dl, "\Phi_{m,t}"] \\ & \Delta_\lambda & \end{tikzcd}$$

Combining this with Theorem 6.1, we obtain a toric degeneration of the Gelfand–Cetlin system.

Remark 7.4. By changing the phase of the rational function f_m, we obtain a flow sending $X_{m,t}$ to $X_{m,0}$ for each t not necessarily real. Note that, if $|t_0| = 1$, then $(X_{n,t_0}, \widetilde{\omega}_\lambda|_{X_{n,t_0}})$ is isomorphic to (F, ω_λ), and the restriction Φ_{n,t_0} coincides with the Gelfand–Cetlin system. Applying the same argument, we have $\Phi_{m,t}(X_{m,t}) = \Delta_\lambda$ for each $t \in \mathbb{C}$.

Remark 7.5. The toric degeneration of the Gelfand–Cetlin system gives an isomorphism between geometric quantizations for the flag manifold and Gelfand–Cetlin toric variety. To see this, we recall the method of geometric quantization via Lagrangian torus fibrations. Let (M, ω) be a symplectic manifold, $\mathcal{L}$ a complex line bundle on M with a unitary connection whose first Chern form coincides with ω (such a line bundle is called a *prequantum bundle*). We further assume that M admits a Lagrangian torus fibration $\Phi : M \to B$. A fiber $L(u) := \Phi^{-1}(u)$ of Φ is said to be *Bohr–Sommerfeld* if the restriction $\mathcal{L}|_{L(u)}$ has trivial holonomies. The *real quantization* is defined to be the space of covariantly constant sections of $\mathcal{L}$ restricted to Bohr–Sommerfeld fibers.

Assume that $\lambda_{n_i} - \lambda_{n_{i+1}} \in \mathbb{Z}$ for all i, and consider a family of line bundles $\mathfrak{L}_\lambda \to \mathfrak{X}_m$ given by

$$\mathcal{O}_{\mathbb{P}_1}(\lambda_1 - \lambda_2) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}_{n-1}}(\lambda_{n-1} - \lambda_n),$$

on $\prod_i \mathbb{P}_{n_i}$. Then the restriction $\mathcal{L}_{m,t} = \mathfrak{L}_\lambda|_{X_{m,t}}$ gives a prequantum line bundle on each $X_{m,t}$. Using the completely integrable systems $\Phi_{m,t}$, we obtain a real quantization for each $X_{m,t}$. It is proved in [15] that Bohr–Sommerfeld fibers for the Gelfand–Cetlin system exist exactly on integral points of the Gelfand–Cetlin polytope. It is easy to check that the vector field V_m lifts to $\mathfrak{L}_\lambda$ preserving the unitary connection on each $\mathcal{L}_{m,t}$. In particular, the gradient-Hamiltonian flow preserves the Bohr–Sommerfeld condition, and hence it gives an isomorphism between real quantizations on the flag manifold and the Gelfand–Cetlin toric variety.

Example 7.6. We see the gradient-Hamiltonian flow for $F^{(3)}$ in $\mathbb{P}^2 \times \mathbb{P}^2$ instead of in the total space $\mathfrak{X}$ of the deformation. Recall that the degenerating family is given by

$$X_t = \bigl\{\bigl([Z_1 : Z_2 : Z_3], [Z_{12} : Z_{13} : Z_{23}]\bigr) \bigm| Z_1 Z_{23} - Z_2 Z_{13} + t Z_3 Z_{12} = 0\bigr\}.$$

Hence the rational function in this case is

$$f = \frac{Z_2 Z_{13} - Z_1 Z_{23}}{Z_3 Z_{12}}.$$

Theorem 6.1 says that the restriction

$$\Phi_t := \bigl(\tilde{\lambda}_1^{(2)}, \tilde{\lambda}_2^{(2)}, \tilde{\lambda}_1^{(1)}\bigr)\big|_{X_t} : (X_t, \widetilde{\omega}_\lambda|_{X_t}) \longrightarrow \mathbb{R}^3$$

to X_t is a completely integrable system for each t, and Φ_0 coincides with the moment map of the torus action on the Gelfand–Cetlin toric variety. The gradient-Hamiltonian vector field V vanishes on $X_1 \cap X_0$, which is the inverse image of two faces in the back side of Δ_λ in Fig. 4. Hence $X_1 \cap X_0$ is fixed under the gradient-Hamiltonian flow. We see the behavior of the S^3-fiber of the Gelfand–Cetlin system. Recall that this S^3 is given by $\lambda_1^{(2)} = \lambda_2^{(2)} = \lambda_1^{(1)} = \lambda_2$. It is easy to check that the image in $\mathbb{P}^2 \times \mathbb{P}^2$ of the S^3-fiber is

$$\bigl\{\bigl([z_1 : z_2 : \lambda_1 - \lambda_2], [\lambda_2 - \lambda_3 : \bar{z}_2 : -\bar{z}_1]\bigr) \bigm| |z_1|^2 + |z_2|^2 = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)\bigr\}.$$

From Corollary 7.3, the image of S^3 under the flow is given by $\Phi_t^{-1}(\lambda_2, \lambda_2, \lambda_2)$, or equivalently,

$$\phi_{1-t}\bigl(S^3\bigr) = \left\{\mu^{(2)}(Z) = \begin{pmatrix} \lambda_2 & \\ & \lambda_2 \end{pmatrix}\right\} \cap X_t.$$

Let $(y_1, y_2, y_{23}, y_{13})$ be local coordinates on $\mathbb{P}^2 \times \mathbb{P}^2$ given by

$$y_i = \frac{Z_i}{\sqrt{\sum_j |Z_j|^2}}, \qquad y_{ij} = \frac{Z_{ij}}{\sqrt{\sum_{k,l} |Z_{kl}|^2}}.$$

Then we have

$$\begin{aligned}
\mu^{(2)} &= \frac{\lambda_1 - \lambda_2}{\sum |Z_i|^2} \begin{pmatrix} |Z_1|^2 & \overline{Z}_1 Z_2 \\ Z_1 \overline{Z}_2 & |Z_2|^2 \end{pmatrix} \\
&\quad + \frac{\lambda_2 - \lambda_3}{\sum |Z_{ij}|^2} \begin{pmatrix} |Z_{12}|^2 + |Z_{13}|^2 & \overline{Z}_{13} Z_{23} \\ Z_{13} \overline{Z}_{23} & |Z_{12}|^2 + |Z_{23}|^2 \end{pmatrix} + \begin{pmatrix} \lambda_3 & \\ & \lambda_3 \end{pmatrix} \\
&= \frac{\lambda_1 - \lambda_2}{\sum |Z_i|^2} \begin{pmatrix} |Z_1|^2 & \overline{Z}_1 Z_2 \\ Z_1 \overline{Z}_2 & |Z_2|^2 \end{pmatrix} + \frac{\lambda_2 - \lambda_3}{\sum |Z_{ij}|^2} \begin{pmatrix} -|Z_{23}|^2 & \overline{Z}_{13} Z_{23} \\ Z_{13} \overline{Z}_{23} & -|Z_{13}|^2 \end{pmatrix} + \begin{pmatrix} \lambda_2 & \\ & \lambda_2 \end{pmatrix} \\
&= (\lambda_1 - \lambda_2) \begin{pmatrix} |y_1|^2 & \bar{y}_1 y_2 \\ y_1 \bar{y}_2 & |y_2|^2 \end{pmatrix} + (\lambda_2 - \lambda_3) \begin{pmatrix} -|y_{23}|^2 & \bar{y}_{13} y_{23} \\ y_{13} \bar{y}_{23} & -|y_{13}|^2 \end{pmatrix} + \begin{pmatrix} \lambda_2 & \\ & \lambda_2 \end{pmatrix}.
\end{aligned}$$

It is easy from this to see that $\phi_{1-t}(S^3)$ is given by

$$\phi_{1-t}\big(S^3\big) = \big\{\sqrt{t}\, y = (\sqrt{t}\, y_1, \sqrt{t}\, y_2, \sqrt{t}\, y_{23}, \sqrt{t}\, y_{13}) \bigm| y \in S^3\big\},$$

and this means that the S^3-fiber shrinks to the singular point of X_0 under the flow. In particular, the gradient-Hamiltonian flow extends to $X_t \to X_0$.

8. A not-in-stages degeneration of the Gelfand–Cetlin system

First we summarize what we have obtained and then we will mention what we will need for the application of the toric degeneration to the Floer theory of flag manifolds, which is the content of the latter half of the paper.

So far we considered degeneration of flag manifolds in stages and we saw that the gradient-Hamiltonian flow connects the two integrable system structures: Gelfand–Cetlin and toric. The degeneration is parametrized by $\mathbb{C}^{n-1}$. The point $(0, \dots, 0)$ corresponds to the Gelfand–Cetlin toric variety. The point $(1, \dots, 1)$ corresponds to the flag manifold embedded in the product of projective spaces by the Plücker embedding. This has a natural action of $U(n)$ (which can be extended to an action on the ambient multiple projective space), and from this we construct the Gelfand–Cetlin system.

The gradient-Hamiltonian flow maps the fiber over $(1, \dots, 1)$ to the fiber over $(1, \dots, 1, 0)$, then to the fiber over $(1, \dots, 1, 0, 0)$ and so on, along a piecewise linear path on the base. From the point of view of finding structures of integrable systems, the problem is that when we move from $(1, \dots, 1)$, we cannot see the natural $U(n)$ action any more. More precisely, since we use only $U(n-1)$ action to construct Gelfand–Cetlin systems, true problem emerges after the first degeneration. For example, there is only natural $U(n-2)$-action on the fiber over $(1, \dots, t, 0)$, $0 \leqslant t < 1$. So we cannot apply the construction of the Gelfand–Cetlin integrable system on these fibers. The only structures of integrable systems we have on the fibers over $(1, \dots, t, 0)$, $0 < t < 1$ are the ones which are the push-forward of the Gelfand–Cetlin system on the fiber over $(1, \dots, 1)$ by the gradient-Hamiltonian flow.

On $(1, \dots, 1, 0)$, there is another natural integrable structure, induced by the combination of the $U(n-2)$ action and the new action of the torus T^{n-1} (Theorem 6.1). The good point is that this natural structure of an integrable system on the fiber over $(1, \dots, 1, 0)$ coincides with the structure of the integrable system induced by the push-forward of the Gelfand–Cetlin system by the gradient-Hamiltonian flow (Corollary 7.3). The same procedure can be applied along the path from $(1, \dots, 1, 0)$ to $(1, \dots, 1, 0, 0)$ and so on, giving the degeneration of the Gelfand–Cetlin integrable system on the flag manifold to the Lagrangian fibration of the Gelfand–Cetlin toric variety.

What we want to do from now is to compute some Floer theoretical quantity of the flag manifolds. Namely, we want to compute the *potential function* of the Lagrangian torus fibers of the Gelfand–Cetlin system. Such a computation was done by Cho and Oh [6] for toric manifolds. Since the Gelfand–Cetlin system degenerates to the toric integrable system, we want to use their result in our case, too. This works for $F^{(3)}$, which degenerates directly (not in stages) to the toric integrable system, but in general, we cannot directly apply Cho–Oh's computation.

The first problem is we have to care about the singularity of the toric variety ([6] deals with smooth toric manifolds), but this point is not very problematic for the calculation of the potential functions, for which we need to consider only those disks with Maslov index two. We study this point in the next section.

The other problem, which is also related to the first, is that since the gradient-Hamiltonian flow does not preserve the complex structure, the moduli space of holomorphic disks with Lagrangian boundary condition may change along the flow (note that in the degeneration in stages, the variety near the Gelfand–Cetlin toric variety is in general singular, and it is not isomorphic to flag manifold. To reach to the flag manifold by chasing back the gradient-Hamiltonian flow, we have to go the long way along the piecewise linear path, which will change the complex structure widely).

This problem will be resolved as follows, assuming we have a fiber preserving flow on the family over the segment $(t, \dots, t)$, $0 \leqslant t \leqslant 1$, which preserves the symplectic forms and the Lagrangian fibration structures. Note that in this case the fiber over $(0, \dots, 0)$ is the Gelfand–Cetlin toric variety, and the others are flag manifolds. Let ϵ be a positive small number. The fiber over $(\epsilon, \dots, \epsilon)$ has structure of an integrable system, which is the push-forward of (so identical to) the Gelfand–Cetlin system. This integrable system will sufficiently resemble the toric integrable structure on the special fiber. On the other hand, the complex structures of the fiber over $(\epsilon, \dots, \epsilon)$ and of the special fiber are quite close, too (at least away from the singular locus). Note that the complex structure on the fiber over $(\epsilon, \dots, \epsilon)$ is not the push-forward by the flow, but the natural one as a submanifold of the multiple projective space.

So we want to construct such a flow and this is what we do in the rest of this section (in fact, we will construct a flow over some piecewise linear path, not on $(t, \dots, t)$, since it will suffice for application. However, we can easily modify the construction so that we actually have a flow on $(t, \dots, t)$). One may think that the gradient-Hamiltonian flow along the segment $(t, \dots, t)$, $0 \leqslant t \leqslant 1$ will give such a flow, but this need not be true. The gradient-Hamiltonian flow will produce a structure of an integrable system on the fiber over $(0, \dots, 0)$ by push-forward, but it need not coincide with the toric integrable structure in general (the resulting integrable system may depend on the path on the base). So we will give another construction. Recall that we constructed a family of projective varieties parametrized by $\mathbf{t} = (t_2, \dots, t_n) \in \mathbb{C}^{n-1}$ in Section 4. We write the total space by $\widetilde{\mathfrak{X}} \to \mathbb{C}^{n-1}$. Denote the fiber over $(t_2, \dots, t_n)$ by $X_{(t_2, \dots, t_n)}$ as before. The result is the following.

Proposition 8.1. *There is a toric degeneration* $(\tilde{f}, \gamma, \widetilde{\Phi}, \phi)$ *(in the sense of Definition* 1.1*) of the Gelfand–Cetlin system* (X, ω, Φ) *with the following properties.*

1. $\tilde{f} : \widetilde{\mathfrak{X}} \to \mathbb{C}^{n-1}$ *is the* $(n-1)$*-parameter family constructed in Section* 6.
2. $\gamma : [0,1] \to \mathbb{C}^{n-1}$ *is a piecewise linear path with* $\gamma(0) = (1, \dots, 1)$ *and* $\gamma(1) = (0, \dots, 0)$. *There is a positive small number* ϵ *such that* γ *restricted to the interval* $[1-\epsilon, 1]$ *is given by* $\gamma(t) = (1-t, \dots, 1-t)$.
3. *The map* $\phi_t : X^\circ = X_1^\circ \to X_{1-t}^\circ$, $t \in [0,1]$ *is a diffeomorphism which coincides with the gradient-Hamiltonian flow for* $t \in [0, 1-\epsilon]$, *but (possibly)* not *for* $t \in [1-\epsilon, 1]$. *It preserves the symplectic structures and the completely integrable systems, as required by Definition* 1.1.

Remark 8.2. Contrary to the degenerations in stages, all X_t, $t \neq 0$ are isomorphic to the flag manifold as complex manifolds.

Proof. Let δ be a positive small number. Consider the following piecewise linear path on $\mathbb{C}^{n-1}$, which approximates the path Γ over which we constructed the degeneration of the Gelfand–Cetlin system in stages. We start from $(1, \dots, 1)$ and go straight to $(1, \dots, 1, \delta)$. Then we turn and go to $(1, \dots, 1, \delta^2, \delta)$. Proceeding similarly, we go to $(\delta^{n-1}, \delta^{n-2}, \dots, \delta)$ through the piecewise linear path. Then finally we go to $(\delta^{n-1}, \delta^{n-1}, \dots, \delta^{n-1})$.

Recall that the gradient-Hamiltonian flow can be defined when we have a one parameter family of hypersurfaces in a Kähler manifold. In our case, over the segment between $(1, \dots, 1, \delta^i, \delta^{i-1}, \dots, \delta)$ and $(1, \dots, 1, \delta^{i+1}, \delta^i, \dots, \delta)$, we take the union of the fibers over a holomorphic disk containing this segment as the ambient Kähler manifold (the Kähler structure is induced by the restriction of the Kähler structure on the product of the disk and the multiple projective manifold). The base parameter gives a one parameter family of hypersurfaces, so we have a gradient-Hamiltonian flow along the path between $(1, \dots, 1, \delta^i, \delta^{i-1}, \dots, \delta)$ and $(1, \dots, 1, \delta^{i+1}, \delta^i, \dots, \delta)$ for each i and similarly along the path between $(\delta^{n-1}, \delta^{n-2}, \dots, \delta)$ and $(\delta^{n-1}, \delta^{n-1}, \dots, \delta^{n-1})$.

Since the path from $(1, \dots, 1)$ to $(\delta^{n-1}, \delta^{n-2}, \dots, \delta)$ approximates the path Γ (degeneration in stages), the integrable system on the fiber over $(\delta^{n-1}, \delta^{n-2}, \dots, \delta)$ defined by the push-forward of the Gelfand–Cetlin system by the gradient-Hamiltonian flow approximates the integrable system on the fiber over $(\delta^{n-1}, 0, \dots, 0)$ (which is also defined as the push-forward of the Gelfand–Cetlin system), at least away from the singular locus (when one wants to be more precise, one can say as follows. Consider the path from $(\delta^{n-1}, \delta^{n-2}, \dots, \delta)$ to $(\delta^{n-1}, 0, \dots, 0)$ and the gradient-Hamiltonian flow along this path. Then we have two structures of integrable systems on the fiber over $(\delta^{n-1}, 0, \dots, 0)$, pushing forward the Gelfand–Cetlin system along two paths. One can deform the one integrable structure to the other on a complement of some compact neighborhood (of small measure) of the singular locus by a diffeomorphism of small norm, bounded by $O(\delta)$.).

Since the path between $(\delta^{n-1}, \delta^{n-2}, \dots, \delta)$ and $(\delta^{n-1}, \delta^{n-1}, \dots, \delta^{n-1})$ is very short, one sees that the induced integrable system on the fiber over $(\delta^{n-1}, \dots, \delta^{n-1})$ also approximates the one on the fiber over $(\delta^{n-1}, 0, \dots, 0)$. On the other hand, the structure of the integrable system on the fiber over $(\delta^{n-1}, 0, \dots, 0)$ approximates that of the Gelfand–Cetlin toric variety. Consequently, we constructed an integrable system, which is canonically identified with the Gelfand–Cetlin system, on the fiber over $(\delta^{n-1}, \dots, \delta^{n-1})$, with the property that it approximates well the Gelfand–Cetlin toric integrable system on the central fiber in the sense remarked above.

Now we deform δ to 0. Then for each point p on the (half closed) segment between $(\delta^{n-1}, \dots, \delta^{n-1})$ and $(0, \dots, 0)$ ($(0, \dots, 0)$ is not contained in the segment), we have a diffeomorphism from the fiber over $(1, \dots, 1)$ to the fiber over p, by the composition of the gradient-Hamiltonian flows along the path. Since the path deforms continuously as we deform δ, the diffeomorphism also deforms continuously (with respect to the C^∞-norm), so this defines a flow along the segment between $(\delta^{n-1}, \dots, \delta^{n-1})$ and $(0, \dots, 0)$. By construction (and Corollary 7.3) it is clear that this flow extends to $(0, \dots, 0)$, and the push-forward of the Gelfand–Cetlin system converges to the toric integrable system. This gives the desired flow.

Take γ to be the piecewise linear path

$$(1, \dots, 1) \to (1, \dots, 1, \delta) \to \dots \to \left(\delta^{n-1}, \dots, \delta\right) \to \left(\delta^{n-1}, \dots, \delta^{n-1}\right) \to (0, \dots, 0),$$

with a suitable parametrization, and define the flow ϕ as above, along γ. The claims of the proposition are obvious consequences of the above push-forward construction and the convergence property of the integrable system by the flow. $\square$

9. Holomorphic disks in a flag manifold

Let $(\tilde{f} : \widetilde{\mathfrak{X}} \to \mathbb{C}^{n-1}, \gamma, \widetilde{\Phi}, \phi)$ be a toric degeneration of a Gelfand–Cetlin system as in Section 8. Let ϵ be a positive small number. We consider the subfamily over the closed segment between $(\epsilon, \dots, \epsilon)$ and $(0, \dots, 0)$ with a flow constructed in Section 8. We write the fiber over $(t, \dots, t)$ as X_t. We also fix a point $u \in \operatorname{Int} \Delta_\lambda$ in the interior of the Gelfand–Cetlin polytope and write the Lagrangian fiber $\Phi_t^{-1}(u) \subset X_t$ as L_t.

To compare holomorphic disks in X_0 and X_t, we first construct a map from X_t to X_0. The gradient Hamiltonian flow ϕ_t gives such a map on a dense open subset of X_t, in particular, it induces a diffeomorphism

$$\phi_{t'}|_{L_t} : L_t \xrightarrow{\sim} L_{t-t'},$$

but there is a little problem that the flow is not defined on the whole X_t. So we modify the definition of the flow as follows.

Consider the subfamily of $\widetilde{f} : \widetilde{\mathfrak{X}} \to \mathbb{C}^{n-1}$ over the diagonal

$$\mathbb{C}_\Delta = \left\{(t, \dots, t) \in \mathbb{C}^{n-1} \;\middle|\; t \in \mathbb{C}\right\}.$$

We write it as

$$\widetilde{f}_\Delta : \widetilde{\mathfrak{X}}_\Delta \to \mathbb{C}_\Delta.$$

Note that the total space $\widetilde{\mathfrak{X}}_\Delta$ may not be smooth. However, the base $\mathbb{C}_\Delta$ is taken so that the singularity lies only on the central fiber. So by blowing up along appropriate locus of the central fiber, we have another family

$$\widetilde{f}_{\Delta,sm} : \widetilde{\mathfrak{X}}_{\Delta,sm} \to \mathbb{C}_\Delta$$

such that the total space $\widetilde{\mathfrak{X}}_{\Delta,sm}$ is a smooth variety. The family $\widetilde{f}_\Delta$ is naturally contained in the product

$$\prod \mathbb{P}_{n_j} \times \mathbb{C}_\Delta,$$

so there is a natural metric on $\widetilde{\mathfrak{X}}_\Delta$ induced from the ambient space $\prod \mathbb{P}_{n_j} \times \mathbb{C}_\Delta$, here we take the standard Euclidean metric on $\mathbb{C}_\Delta \cong \mathbb{C}$. Precisely speaking, the metric is defined on the smooth part of $\widetilde{\mathfrak{X}}_\Delta$.

Now we define a metric on $\widetilde{\mathfrak{X}}_{\Delta,sm}$. This is obtained by gluing the above metric on the smooth part of $\widetilde{\mathfrak{X}}_\Delta$ and some fixed metric defined on a neighborhood of the exceptional locus of the blowing-up $\widetilde{\mathfrak{X}}_{\Delta,sm} \to \widetilde{\mathfrak{X}}_\Delta$. For the latter, we take a metric of the form

$$\widetilde{f}^*_{\Delta,sm} g_{\mathbb{C}_\Delta} + \eta g_0,$$

here $g_{\mathbb{C}_\Delta}$ is the standard metric on $\mathbb{C}_\Delta$, g_0 is some fixed metric on the neighborhood of the exceptional locus, and η is a small positive number. Note that in this process, the subset of $\widetilde{\mathfrak{X}}_\Delta$ on which the metric is modified can be taken arbitrary small, the only condition is that the subset must contain the singular locus of $\widetilde{\mathfrak{X}}_\Delta$.

Now let y be the pull-back to $\widetilde{\mathfrak{X}}_{\Delta,sm}$ of the standard coordinate on $\mathbb{C}_\Delta$, and consider the gradient flow

$$\phi_{-|y|^2,s} : \widetilde{\mathfrak{X}}_{\Delta,sm} \to \widetilde{\mathfrak{X}}_{\Delta,sm}, \quad s \in [0,\infty)$$

of the function $-|y|^2$ with respect to the metric constructed above. By the form of the metric we defined, the norm of $\nabla(-|y|^2)$ is bounded by some uniform constant multiple of the norm of $(\widetilde{f}_{\Delta,sm})_*(\nabla(-|y|^2))$. It follows that the length of the integral curve starting from any point of $\widetilde{\mathfrak{X}}_{\Delta,sm}$ is finite.

Let us define

$$D_\Delta = \{(t,\dots,t) \in \mathbb{C}_\Delta \mid |t| < 1\}.$$

By the above observation, $\phi_{-|y|^2}$ extends to the limit $s \to \infty$ and gives a deformation retract of $\widetilde{f}^{-1}_{\Delta,sm}(D_\Delta)$ to the central fiber. By composing with the blowing down of the exceptional locus, we have a deformation retract of $\widetilde{f}^{-1}_\Delta(D_\Delta)$ to X_0. In particular, it gives a map

$$\phi_{-|y|^2,\epsilon} : X_\epsilon \to X_0.$$

Also by construction, the isotopy classes of the restriction of $\phi_{-|y|^2,\epsilon}$,

$$\phi_{-|y|^2,\epsilon} : \phi^{-1}_{-|y|^2,\epsilon}(X_0^\circ) \cap X_\epsilon^\circ \to X_0^\circ$$

and of ϕ_ϵ,

$$\phi_\epsilon : \phi^{-1}_{-|y|^2,\epsilon}(X_0^\circ) \cap X_\epsilon^\circ \to X_0^\circ$$

are the same. It follows that we can modify ϕ_ϵ in a small neighborhood of $X_\epsilon \setminus X_\epsilon^\circ$ so that it extends to a continuous map from X_ϵ to X_0. We write this extension as ϕ'_ϵ.

By construction, we can assume ϕ'_ϵ coincides with ϕ_ϵ on arbitrary large relatively compact subset of X_ϵ°. We summarize the construction so far as follows.

Lemma 9.1. *Let* $\mathrm{Sing}(X_0)$ *be the singular locus of* X_0. *Take an arbitrary small closed neighborhood* W *of* $\mathrm{Sing}(X_0)$. *Then there is a continuous map* $\phi'_\epsilon : X_\epsilon \to X_0$ *such that* ϕ'_ϵ *coincides with* ϕ_ϵ *on* $\phi_\epsilon^{-1}(X_0 \setminus W)$.

In particular, the map

$$\phi'_\epsilon|_{L_\epsilon} : L_\epsilon \xrightarrow{\sim} L_0$$

coincides with $\phi_\epsilon|_{L_\epsilon}$.

Note that the restriction of ϕ'_ϵ to $\phi_\epsilon^{-1}(X_0 \setminus W)$ is a diffeomorphism.

Lemma 9.2. *The map*

$$\phi'_\epsilon : X_\epsilon \to X_0$$

induces an isomorphism

$$\left(\phi'_\epsilon\right)_* : \pi_2(X_\epsilon) \xrightarrow{\sim} \pi_2(X_0)$$

of the homotopy groups.

Proof. Let S be the singular locus of X_0 and

$$\phi'_\epsilon : X_\epsilon \to X_0$$

be the map constructed above. Recall that $\pi_2(X_0)$ is generated by torus-invariant curves. Let

$$p : \widetilde{X}_0 \to X_0$$

be a small resolution. Since the fan for $\widetilde{X}_0$ is obtained from that for X_0 without adding one-dimensional cones, for any torus-invariant curve l in X_0, there is a unique torus-invariant curve $\widetilde{l}$ in $\widetilde{X}_0$ mapped isomorphically to the curve l. We think of l and $\widetilde{l}$ as inclusion maps. Since $\widetilde{X}_0$ is nonsingular and the exceptional locus has real codimension greater than two, one can continuously move $\widetilde{l}$ to a map

$$\widetilde{l}' : S^2 \to \widetilde{X}_0,$$

so that the image is disjoint from $p^{-1}(W)$, here W is the subset defined in Lemma 9.1. Then $l' = p \circ \widetilde{l}'$ is homotopic to l (seen as a map by inclusion) and it can be lifted to X_ϵ by $(\phi'_\epsilon)^{-1}$. This proves that the map

$$\left(\phi'_\epsilon\right)_* : \pi_2(X_\epsilon) \to \pi_2(X_0)$$

is surjective.

On the other hand, recall that $\pi_2(X_\epsilon)$ is generated by classes of rational curves (e.g., by Schubert subvarieties). Let $\beta_1, \beta_2 \in \pi_2(X_\epsilon)$ be any classes represented by (union of) rational curves. Let

$$\varphi_{1,i} : C_{1,i}, \to X_{t_i}, \qquad \varphi_{2,i} : C_{2,i}, \to X_{t_i}$$

where $t_i \to 0$ as $i \to \infty$, be two sequences of holomorphic maps from possibly disjoint unions of prestable rational curves, representing the classes β_1 and β_2 respectively. These sequences can also be seen as sequences of holomorphic curves in the fixed ambient space:

$$C_i \to \prod \mathbb{P}_{n_j}.$$

By Gromov compactness theorem, we can assume there are limits φ_1 and φ_2 of $\varphi_{1,i}$ and $\varphi_{2,i}$ respectively. Again by using the small resolution, we can deform φ_1 and φ_2 so that their images do not intersect (a small neighborhood of) the singular locus of X_0, and we can lift them to X_t. Now assume that φ_1 and φ_2 give the same class in $\pi_2(X_0)$. Then the homotopy classes of their lifts in X_t are also the same. On the other hand, these classes are β_1 and β_2 respectively by construction. Hence $\varphi_{1,i}$ and $\varphi_{2,i}$ must be in the same homotopy class, and the injectivity of the map $(\phi'_\epsilon)_*$ is proved. □

Lemma 9.2 and the long exact sequences of homotopy groups for the pairs (X_ϵ, L_ϵ) and (X_0, L_0) immediately give the following:

Corollary 9.3. *The map*

$$\phi'_\epsilon : X_\epsilon \to X_0$$

induces an isomorphism

$$(\phi'_\epsilon)_* : \pi_2(X_\epsilon, L_\epsilon) \xrightarrow{\sim} \pi_2(X_0, L_0)$$

of the relative homotopy groups.

The Maslov index of holomorphic disks into (X_t, L_t) is a homomorphism

$$\mu : \pi_2(X_t, L_t) \to \mathbb{Z}.$$

Although X_0 is a singular variety, we can define the Maslov index of disks into (X_0, L_0) by using the isomorphism in Corollary 9.3. This is a reasonable definition if any holomorphic disk into (X_0, L_0) can be deformed to avoid the singular locus of X_0, so that they can be lifted to a (not necessarily holomorphic) map into (X_t, L_t). Proposition 9.5 below shows that this is indeed the case. To state it, we recall the notion of toric transversality of holomorphic curves in a toric variety.

Definition 9.4. (See Nishinou and Siebert [20, Definition 4.1].) A holomorphic curve in a toric variety X is said to be *torically transverse* if it is disjoint from all toric strata of codimension greater than one. A stable map $\varphi : C \to X$ is *torically transverse* if $\varphi(C) \subset X$ is torically transverse and $\varphi^{-1}(\operatorname{Int} X) \subset C$ is dense. Here $\operatorname{Int} X$ is the complement of the toric divisors of X.

Note that a torically transverse map does not intersect the singular locus of the toric variety.

Proposition 9.5. *Any disk $\varphi : (D^2, S^1) \to (X_0, L_0)$ can be deformed into a holomorphic disk with the same boundary condition which is torically transverse.*

We need the following to prove Proposition 9.5:

Theorem 9.6. *(See Cho and Oh [6, Theorem 5.3].) Let L be a Lagrangian torus fiber in a smooth projective toric variety*

$$X_\Sigma = \big(\mathbb{C}^m \setminus Z(\Sigma)\big)/K.$$

Here m is the number of one-dimensional cones of a fan Σ, the subset $Z(\Sigma) \subset \mathbb{C}^m$ is defined by the Stanley–Reisner ideal, and K is the kernel of the map $(\mathbb{C}^\times)^m \to (\mathbb{C}^\times)^N$ defined by one-dimensional cones in Σ. Then any holomorphic map

$$\varphi : \big(D^2, \partial D^2\big) \to (X_\Sigma, L)$$

from the unit disk with Lagrangian boundary condition can be lifted to a holomorphic map

$$\widetilde{\varphi} : D^2 \to \mathbb{C}^m \setminus Z(\Sigma)$$

so that the homogeneous coordinate functions $(z_1(\widetilde{\varphi}), \dots, z_m(\widetilde{\varphi}))$ are given by the Blaschke products with constant factors;

$$z_j(\widetilde{\varphi}) = c_j \cdot \prod_{k=1}^{\mu_j} \frac{z - \alpha_{j,k}}{1 - \bar{\alpha}_{j,k} z},$$

where $c_j \in \mathbb{C}^\times$, $\alpha_{j,k} \in \operatorname{Int} D^2$ and μ_j is a non-negative integer for $j = 1, \dots, m$. Moreover, the Maslov index of φ is given by

$$\nu(\varphi) = 2 \sum_{j=1}^{m} \mu_j.$$

Proof of Proposition 9.5. Let $\widetilde{X}_0$ be a small resolution of X_0 and ψ be the proper transform of φ. Since $\widetilde{X}_0$ is smooth, the map ψ has an explicit description

$$z_j(\widetilde{\psi}) = c_j \cdot \prod_{k=1}^{\mu_j} \frac{z - \alpha_{j,k}}{1 - \bar{\alpha}_{j,k} z}$$

by Theorem 9.6. Note that ψ intersects a toric stratum of higher codimension exactly when there are $j_1 \neq j_2$ such that $\alpha_{j_1,k_1} = \alpha_{j_2,k_2}$ for some k_1 and k_2. From this remark, and since $\widetilde{X}_0$ is a small resolution of X_0 so that the exceptional locus has codimension larger than one, we can make ψ torically transverse by perturbing $\alpha_{j,k}$. Since the resolution is small, torically transverse disks in $\widetilde{X}_0$ project to torically transverse disks in X_0. This proves the proposition. □

Corollary 9.7. *Assume that a holomorphic map $\varphi : (D^2, S^1) \to (X_0, L_0)$ intersects the singular locus of X_0. Then the Maslov index of φ is larger than two.*

Proof. In the proof of Proposition 9.5, a disk φ intersecting the singular locus lifts to a disk in $\widetilde{X_0}$ whose description via Theorem 9.6 has at least two non-constant factors. Hence when we deform it into torically transverse disk, it intersects the toric boundary at least at two points. This implies that φ has Maslov index larger than two. □

Corollary 9.8. *Any holomorphic disk $\phi : (D^2, \partial D^2) \to (X_0, L_0)$ with Maslov index two is written as*

$$z_j(\widetilde{\varphi}) = \begin{cases} c_i \cdot \frac{z-\alpha}{1-\bar{\alpha}z} & \text{if } j = i, \\ c_j & \text{otherwise}, \end{cases} \tag{17}$$

for some $i \in \{1, \dots, m\}$, $(c_j)_{j=1}^m \in (\mathbb{C}^\times)^m$ and $\alpha \in \operatorname{Int} D^2$.

The relative homotopy class of the holomorphic disk in Corollary 9.8 will be denoted by β_i. The image of β_i in $\pi_1(L_0)$ under the map in the exact sequence

$$1 \to \pi_2(X_0) \to \pi_2(X_0, L_0) \to \pi_1(L_0) \to 1$$

will be denoted by v_i. If we identify $\pi_1(L_0) \cong \mathbb{Z}^N$ with the lattice of cocharacters of the torus acting on X_0, then the i-th face of the moment polytope of X_0 is defined by

$$\ell_i(u) := \langle v_i, u \rangle - \tau_i = 0$$

for some $\tau_i \in \mathbb{R}$, where $\langle \bullet, \bullet \rangle$ is the standard inner product on $\mathbb{R}^N$ as in Definition 3.11.

Lemma 9.9. *There is a small neighbourhood W_0 of the singular locus $S \subset X_0$ such that any holomorphic disk $\varphi : (D^2, S^1) \to (X_0, L_0)$ of Maslov index two is disjoint from the closure of W_0. We may assume the subset W of Lemma* 9.1 *is contained in W_0.*

Proof. The intersection of the holomorphic disk (17) with the complement of the big torus in X_0 is given by

$$\big[(c_1, \dots, c_{i-1}, 0, c_{i+1}, \dots, c_m)\big] \in X_0.$$

Its image by the moment map is determined by the condition

$$\varphi\big(\partial D^2\big) \subset L_0,$$

so that the image $\varphi(D^2)$ intersects the toric boundary of X_0 only at an interior point of a toric divisor, whose image by the moment map is independent of $(c_j)_j$ and α, and the lemma follows. □

Let us introduce the following notation:

Definition 9.10. Let M be a Kähler manifold, L be its Lagrangian submanifold, and $\beta \in \pi_2(M, L)$ be a relative homotopy class. Then $\overline{\mathcal{M}}_1(M, L; \beta)$ will denote the moduli space of stable maps of degree β from a bordered Riemann surface of genus zero with one marked point and with Lagrangian boundary condition. The open subspace of $\overline{\mathcal{M}}_1(M, L; \beta)$ consisting of maps from a disk will be denoted by $\mathcal{M}_1(M, L; \beta)$.

Theorem 9.11. *(See Cho and Oh [6, Theorem 6.1].) Let $X_\Sigma = (\mathbb{C}^m \setminus Z(\Sigma))/K$ be a projective toric variety and $L \subset X_\Sigma$ be a Lagrangian torus fiber. Assume that a holomorphic disk*

$$\varphi : \left(D^2, S^1\right) \to (X_\Sigma, L)$$

is disjoint from the singular locus of X_Σ and admits a lift

$$\widetilde{\varphi} : \left(D^2, S^1\right) \to \left(\mathbb{C}^r \setminus Z(\Sigma), \pi^{-1}(L)\right)$$

to the homogeneous coordinate space. Then φ is Fredholm regular.

Theorem 9.11 shows that $\mathcal{M}_1(X_0 \setminus W_0, L_0; \beta)$ is a smooth manifold without any virtual structure. Corollary 9.8 and Lemma 9.9 give:

Lemma 9.12. *If $\beta \in \pi_2(X_0, L_0)$ is a class with Maslov index two, then $\mathcal{M}_1(X_0, L_0; \beta) \neq \emptyset$ if and only if $\beta = \beta_i$ for some $1 \leqslant i \leqslant m$, and the evaluation map induces a diffeomorphism*

$$ev : \mathcal{M}_1(X_0 \setminus W_0, L_0; \beta_i) \xrightarrow{\sim} L_0.$$

In terms of the moduli space of holomorphic disks, Lemma 9.9 can be stated as follows:

Lemma 9.13. *For any $1 \leqslant i \leqslant m$, the natural inclusion*

$$\mathcal{M}_1(X_0 \setminus W_0, L_0; \beta_i) \hookrightarrow \mathcal{M}_1(X_0, L_0; \beta_i)$$

is surjective.

The fact that X_0 is a Fano variety implies the following:

Lemma 9.14. *If $\beta \in \pi_2(X_0, L_0)$ is a class with Maslov index two, then the natural inclusion*

$$\mathcal{M}_1(X_0, L_0; \beta) \hookrightarrow \overline{\mathcal{M}}_1(X_0, L_0; \beta)$$

is surjective.

Proof. Let

$$\varphi : D_1 \cup \cdots \cup D_p \cup S_1 \cup \cdots \cup S_q \to X_0$$

be a stable map of genus zero and Maslov index two, where D_i and S_i are disk and sphere components of the domain curve. Then the contribution of each D_i to the Maslov index of φ is greater than one by Theorem 9.6, and that of each S_i is positive since X_0 is a Fano variety. This implies $p = 1$ and $q = 0$ so that the lemma follows. □

The same reasoning as above shows the following:

Lemma 9.15. *The moduli space $\overline{\mathcal{M}}_1(X_0, L_0; \beta)$ is empty if the Maslov index of β is less than two.*

The following is the main result in this section:

Proposition 9.16. *For any relative homotopy class $\beta \in \pi_2(X_0, L_0)$ of Maslov index two, there is a positive real number $0 < t \leqslant 1$ and a diffeomorphism*

$$\psi : \overline{\mathcal{M}}_1(X_0, L_0; \beta) \to \overline{\mathcal{M}}_1(X_t, L_t; \beta)$$

such that the diagram

$$\begin{array}{ccc} H_*(\overline{\mathcal{M}}_1(X_0, L_0; \beta)) & \xrightarrow{ev_*} & H_*(L_0) \\ \downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle (\phi_t^{-1})_*} \\ H_*(\overline{\mathcal{M}}_1(X_t, L_t; \beta)) & \xrightarrow{ev_*} & H_*(L_t) \end{array}$$

is commutative.

The existence of the map ψ comes from the Fredholm regularity:

Proposition 9.17. *For a class $\beta \in \pi_2(X_0, L_0)$ with Maslov index two and a sufficiently small positive number t, there is a map*

$$\psi : \overline{\mathcal{M}}_1(X_0, L_0; \beta) \to \overline{\mathcal{M}}_1(X_t, L_t; \beta)$$

which is a diffeomorphism into a connected component of $\overline{\mathcal{M}}_1(X_t, L_t; \beta)$.

Proof. Let φ_0 be an element of $\overline{\mathcal{M}}_1(X_0, L_0; \beta)$. Lemmas 9.13 and 9.14 imply that φ_0 is a holomorphic map

$$\varphi_0 : (D^2, \partial D^2) \to (X_0 \setminus W_0, L_0),$$

which is Fredholm regular by Theorem 9.11. Then for sufficiently small t, the differential equation for holomorphic maps $\varphi_t : (D^2, \partial D^2) \to (X_t, L_t)$ near $\phi_t^{-1} \circ \varphi_0$ is a small perturbation of the equation on X_0 which has the solution φ_0, so this equation also has a solution and it is Fredholm regular. From this, it follows that the moduli space $\overline{\mathcal{M}}_1(X_t, L_t; \beta)$ contains a connected component diffeomorphic to $\overline{\mathcal{M}}_1(X_0, L_0; \beta) = \mathcal{M}_1(X_0 \setminus W_0, L_0; \beta)$. □

To show the surjectivity of ψ, we use the following version of the Gromov compactness theorem:

Theorem 9.18. *(See Ye [23, Theorem 0.2].) Let (M,ω) be a compact symplectic manifold and assume that we are given the following data:*

- $\{J_t\}_{t\in[0,1]}$ *is a smooth family of tame almost complex structures on M,*
- $\{N_t\}_{t\in[0,1]}$ *is a smooth family of compact totally real submanifolds,*
- $\{t_i\}$ *is a strictly decreasing sequence in $[0,1]$ converging to 0, and*
- $\{\varphi_i\}_{i\in\mathbb{N}}$ *is a sequence of pseudo holomorphic disks in (M, J_{t_i}) with boundary on N_{t_i}.*

Assume further that the area of φ_i is uniformly bounded by a positive constant. Then there is a subsequence of $\{\varphi_{t_i}\}_{i\in\mathbb{N}}$ which converges to a stable J_0-holomorphic map from a bordered Riemannian surface of genus 0 in M with boundary on N_0.

Now we can prove the following:

Corollary 9.19. *For sufficiently small t and a class $\beta\in\pi_2(X_t, L_t)$ of Maslov index two, one has an inclusion*

$$\mathcal{M}_1(X_t\setminus W_t, L_t;\beta)\subset \operatorname{Im}\psi,$$

where $W_t = (\phi'_t)^{-1}(W_0)$.

Proof. Suppose that the statement is false. Then there is a sequence $\{t_i\}_{i\in\mathbb{N}}$ converging to zero and a sequence

$$\varphi_i : \left(D^2, S^1\right)\to (X_{t_i}\setminus W_{t_i}, L_{t_i})$$

of holomorphic disks not contained in $\operatorname{Im}\psi$. By Theorem 9.18, we can assume that φ_i converges to a stable map

$$\varphi : C\to X_0$$

of Maslov index two from a bordered Riemannian surface C of genus 0. Strictly speaking, we need to care about the singularity of X_0. But one can argue as follows. Note that X_0 is equivariantly embedded in the product of projective spaces with a natural torus action. So the Lagrangian torus fiber L_0 extends to a Lagrangian torus $\widetilde{L}_0$ of the product of projective spaces. It is easy to deform $\widetilde{L}_0$ to totally real submanifolds $\widetilde{L}_t$ so that $L_t\subset\widetilde{L}_t$, since totally real condition is an open condition. Now we can apply Theorem 9.18.

By Lemma 9.14, the stable map φ is a holomorphic disk, and Proposition 9.17 implies that φ_{t_i} for sufficiently small t_i are contained in the family constructed there, a contradiction. □

Lemma 9.20. *For sufficiently small t, the Maslov index of any holomorphic disk*

$$\varphi : \left(D^2, \partial D^2\right)\to (X_t, L_t)$$

is greater than or equal to two.

Proof. Assume that the statement is false. Then there is a sequence $\{t_i\}_{i\in\mathbb{N}}$ converging to zero and a sequence

$$\varphi_i : \left(D^2, S^1\right) \to (X_{t_i}, L_{t_i})$$

of holomorphic disks with Maslov index less than two. Then as in the proof of Corollary 9.19, we will have a subsequence of φ_i converging to a stable map

$$\varphi : C \to X_0$$

of Maslov index less than two, which contradicts the fact that X_0 has no such stable maps. □

Lemma 9.21. *For sufficiently small t, the natural inclusion*

$$\mathcal{M}_1(X_t, L_t; \beta) \to \overline{\mathcal{M}}_1(X_t, L_t; \beta)$$

is surjective.

Proof. This follows from Lemma 9.20 and the fact that X_t is Fano in just the same way as in the proof of Lemma 9.14. □

Lemma 9.22. *For sufficiently small t and a class $\beta \in \pi_2(X_t, L_t)$ of Maslov index two, the natural inclusion*

$$\mathcal{M}_1(X_t \setminus W_t, L_t; \beta) \to \overline{\mathcal{M}}_1(X_t, L_t; \beta)$$

is surjective.

Proof. Assume that the statement is false. Then there is a sequence $\{t_i\}_{i\in\mathbb{N}}$ converging to zero and a sequence

$$\varphi_i : \left(D^2, S^1\right) \to (X_{t_i}, L_{t_i})$$

of holomorphic disks intersecting W_{t_i}. As in the proof of Corollary 9.19, we can show that φ_{t_i} converges to a holomorphic map

$$\varphi : D^2 \to X_0$$

from a disk with Maslov index two by taking a suitable subsequence if necessary. Then the image of φ must intersect the closure of W_0, which contradicts our choice of W_0 in Lemma 9.9. □

The commutativity of the diagram in Proposition 9.16 follows from the standard cobordism argument on variations of moduli spaces under perturbations.

10. Potential functions for Gelfand–Cetlin systems

In this section, we recall the definition of the potential function and compute it for Lagrangian torus fibers of the Gelfand–Cetlin system. Since our treatment here follows Fukaya, Oh, Ohta and Ono [9] closely, we only give a sketch of the proof and refer the reader to [9] for further details.

Let

$$\Lambda_0 = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \,\middle|\, a_i \in \mathbb{C},\ \lambda_i \geqslant 0,\ \lim_{i\to\infty} \lambda_i = \infty \right\}$$

be the Novikov ring and

$$\begin{array}{cccc} \mathfrak{v}: & \Lambda_0 & \to & \mathbb{R} \\ & \Psi & & \Psi \\ & \sum_{i=1}^{\infty} a_i T^{\lambda_i} & \mapsto & \min_i \{\lambda_i\}_{i=1}^{\infty} \end{array}$$

be its valuation. The maximal ideal and the quotient field of the local ring Λ_0 will be denoted by Λ_+ and Λ respectively.

For a Lagrangian submanifold L in a symplectic manifold M, Lagrangian intersection Floer theory equips the Λ_0-valued cochain complex of L with the structure of an A_∞-algebra [7,8]. By taking the canonical model, one obtains an A_∞-structure $\{\mathfrak{m}_k\}_{k=0}^{\infty}$ on $H^*(L;\Lambda_0)$. An element $b \in H^1(L;\Lambda_+)$ is called a *weak bounding cochain* if it satisfies the *Maurer–Cartan equation*

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b,\dots,b) \equiv 0 \mod \mathrm{PD}\big([L]\big). \tag{18}$$

The set of weak bounding cochains will be denoted by $\widehat{\mathcal{M}}_{\mathrm{weak}}(L)$. For any $b \in \widehat{\mathcal{M}}_{\mathrm{weak}}(L)$, one can twist the Floer differential as

$$\mathfrak{m}_1^b(x) = \sum_{k,l} \mathfrak{m}_{k+l+1}\big(b^{\otimes k} \otimes x \otimes b^{\otimes l}\big).$$

Maurer–Cartan equation implies $\mathfrak{m}_1^b \circ \mathfrak{m}_1^b = 0$ and the resulting cohomology group

$$HF\big((L;b),(L;b)\big) = \frac{\mathrm{Ker}(\mathfrak{m}_1^b : H^*(L;\Lambda_0) \to H^*(L;\Lambda_0))}{\mathrm{Im}(\mathfrak{m}_1^b : H^*(L;\Lambda_0) \to H^*(L;\Lambda_0))}$$

will be called the *deformed Floer cohomology*. The *potential function*

$$\mathfrak{PO} : \widehat{\mathcal{M}}_{\mathrm{weak}}(L) \to \Lambda_+$$

is defined by

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b,\dots,b) = \mathfrak{PO}(b) \cdot \mathrm{PD}\big([L]\big). \tag{19}$$

Now fix λ as in (1) and let $\Phi_\lambda : F = F(n_1, \dots, n_r, n) \to \Delta_\lambda$ be the Gelfand–Cetlin system. Let $v_i \in \mathbb{R}^N$ be the primitive inward normal vector of the i-th face of Δ_λ and choose $\tau_i \in \mathbb{R}$ so that

$$\ell_i(u) = \langle v_i, u\rangle - \tau_i$$

defines the i-th face of the Gelfand–Cetlin polytope Δ_λ. Here $\langle \bullet, \bullet\rangle$ is the standard inner product on $\mathbb{R}^N$ as in Definition 3.11. The Lagrangian fiber $\Phi_\lambda^{-1}(u)$ over an interior point $u \in \operatorname{Int}\Delta_\lambda$ of the Gelfand–Cetlin polytope will be denoted by $L(u)$. We will identify $H^1(L(u); \Lambda_+)$ with $(\Lambda_+)^N$ using the angle coordinate dual to the standard coordinate on the range $\mathbb{R}^N$ of the Gelfand–Cetlin system. The following theorem is a Gelfand–Cetlin analogue of [9, Proposition 3.2 and Theorem 3.4]:

Theorem 10.1. *For any $u \in \operatorname{Int}\Delta_\lambda$, one has an inclusion*

$$H^1\big(L(u); \Lambda_+\big) \subset \widehat{\mathcal{M}}_{\mathrm{weak}}\big(L(u)\big)$$

and the potential function on $H^1(L(u); \Lambda_+)$ is given by

$$\mathfrak{PO}^u(x) = \sum_{i=1}^m e^{\langle v_i, x\rangle} T^{\ell_i(u)}. \tag{20}$$

Sketch of proof. We use the homotopy-invariance of the A_∞-structure under Hamiltonian isotopy to work with (X_t, L_t) instead of (X_1, L_1), and write the image of $L(u)$ in X_t by the gradient-Hamiltonian flow for sufficiently small t as $L(u)$ by abuse of notation. Recall that the A_∞-structure on $H^*(L(u); \Lambda_+)$ is the canonical model of the A_∞-structure on the cochain complex of $L(u)$ defined by

$$\mathfrak{m}_k(a_1, \dots, a_k) = \sum_{\beta\in\pi_2(F, L(u))} \mathfrak{m}_{k,\beta}(a_1, \dots, a_k),$$

$$\mathfrak{m}_{k,\beta}(a_1, \dots, a_k) = (ev_0)_!^{\mathrm{virt}}\big(ev_1^* a_1 \cup \dots \cup ev_k^* a_k\big) \cdot T^{\beta\cap\omega}$$

where $\overline{\mathcal{M}}_{k+1}(L(u), \beta)$ is the moduli space of stable maps with Lagrangian boundary condition from a bordered Riemann surface of genus zero to F with $k+1$ marked points on the boundary,

$$ev_i : \overline{\mathcal{M}}_{k+1}\big(L(u), \beta\big) \to L(u), \quad i = 0, \dots, k$$

is the evaluation at the i-th marked point, and $(ev_0)_!^{\mathrm{virt}}$ is the integration along the fiber against the virtual fundamental chain. Since

$$\operatorname{virt.dim}\overline{\mathcal{M}}_{k+1}\big(L(u), \beta\big) = \dim L(u) + \mu(\beta) + k - 2,$$

one has

$$\deg \mathfrak{m}_{k,\beta}(b, \dots, b) = 2 - \mu(\beta)$$

if $\deg b = 1$, which is negative if $\mu(\beta) > 2$. Hence only $\beta \in \pi_2(F, L(u))$ with $\mu(\beta) \leqslant 2$ contribute $\mathfrak{m}_k(b, \ldots, b) = \sum_\beta \mathfrak{m}_{k,\beta}(b, \ldots, b)$. Since there is no contribution from a class with Maslov index less than two by Lemma 9.15, $\mathfrak{m}_k(b, \ldots, b)$ must be proportional to $\mathrm{PD}([L])$ for degree reasons, and the inclusion

$$H^1\big(L(u); \Lambda_+\big) \subset \widehat{\mathcal{M}}_{\mathrm{weak}}\big(L(u)\big)$$

follows. It follows from Proposition 9.16 that

$$\begin{aligned}
\mathfrak{m}_{k,\beta_i}(x, \ldots, x) &= (ev_0)_!^{\mathrm{virt}}\big(ev_1^* x \cup \cdots \cup ev_k^* x\big) \cdot T^{\beta \cap \omega} \\
&= \int\limits_{[\overline{\mathcal{M}}_{k+1}(X, L(u); \beta_i)]^{\mathrm{virt}}} \big(ev_0^* \mathrm{PD}\big([pt]\big) \cup \mathrm{ev}_1^* x \cup \cdots \cup ev_k^* x\big) \cdot T^{\beta_i \cap \omega} \\
&= \int\limits_{L(u) \times C_k} \big(ev_0^* \mathrm{PD}\big([pt]\big) \cup \mathrm{ev}_1^* x \cup \cdots \cup ev_k^* x\big) \cdot T^{\beta_i \cap \omega} \\
&= \mathrm{Vol}(C_k)\bigg(\int\limits_{\beta_i} x\bigg)^k \cdot T^{\beta_i \cap \omega} \\
&= \frac{1}{k!}\langle v_i, x\rangle^k T^{\ell_i(u)}
\end{aligned}$$

for $x \in H^1(L(u); \Lambda_+)$ and $i = 1, \ldots, m$. Here, we have used the Fredholm regularity of disks in $\mathcal{M}_{k+1}(X, L(u); \beta_i)$ and the fact that the complement of

$$\mathcal{M}_{k+1}\big(X, L(u); \beta_i\big) = L(u) \times C_k$$

in $\overline{\mathcal{M}}_{k+1}(X, L(u); \beta_i)$ is a measure zero set, where

$$C_k = \big\{(t_1, \ldots, t_k) \bigm| 0 < t_1 < \cdots < t_k < 1\big\}$$

is the configuration space of k points on the unit interval. Now the potential function is given by

$$\begin{aligned}
\mathfrak{PO}^u(x) &= \sum_{k=0}^{\infty} \mathfrak{m}_k(x, \ldots, x) \\
&= \sum_{i=1}^{m} \sum_{k=0}^{\infty} \mathfrak{m}_{k,\beta_i}(x, \ldots, x) \\
&= \sum_{i=1}^{m} \sum_{k=0}^{\infty} \frac{1}{k!}\langle v_i, x\rangle^k T^{\ell_i(u)} \\
&= \sum_{i=1}^{m} e^{\langle v_i, x\rangle} T^{\ell_i(u)}.
\end{aligned}$$

This concludes the proof of Theorem 10.1. □

The following is an immediate corollary of Theorem 10.1:

Corollary 10.2. *The potential function*

$$\mathfrak{PO}^u : H^1\big(L(u); \Lambda_+\big) \to \Lambda_+$$

can be regarded as a Laurent polynomial

$$\mathfrak{PO}^u \in \mathbb{Q}\big[Q_1^{\pm 1}, \ldots, Q_{r+1}^{\pm 1}\big]\big[y_1^{\pm 1}, \ldots, y_N^{\pm 1}\big]$$

where

$$y_k = e^{x_k} T^{u_k}, \quad k = 1, \ldots, N$$

are combinations of the variable $x \in H^1(L(u); \Lambda_+)$ with the parameter $u \in \Delta_\lambda$ for the position of the fiber and

$$Q_j = T^{\lambda_{n_j}}, \quad j = 1, \ldots, r+1$$

is the parameter for the symplectic structure on F.

11. Examples

In this section, we study the critical points of the potential function for the full flag manifold $F(1,2,3)$ and the Grassmannian $Gr(2,4)$. In the latter case, we will see that the number of critical points is strictly smaller than the rank of the cohomology group.

Let us first discuss the case of $F(1,2,3)$. Fix $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ satisfying

$$\lambda_1 > \lambda_2 > \lambda_3,$$

so that the corresponding coadjoint (or adjoint) orbit $\mathcal{O}_\lambda$ is the full flag manifold of dimension three. The Gelfand–Cetlin pattern in this case is given by

$$\begin{array}{ccccccccc} \lambda_1 & & & & \lambda_2 & & & & \lambda_3 \\ & \rotatebox{-45}{$\geqslant$} & & \rotatebox{45}{$\geqslant$} & & \rotatebox{-45}{$\geqslant$} & & \rotatebox{45}{$\geqslant$} & \\ & & u_1 & & & & u_2 & & \\ & & & \rotatebox{-45}{$\geqslant$} & & \rotatebox{45}{$\geqslant$} & & & \\ & & & & u_3 & & & & \end{array}$$

and the Gelfand–Cetlin polytope Δ_λ is defined by six inequalities

$$\Delta_\lambda = \big\{ u = (u_1, u_2, u_3) \in \mathbb{R}^3 \;\big|\; \ell_i(u) = \langle v_i, u \rangle - \tau_i \geqslant 0, \; i = 1, \ldots, 6 \big\}$$

where

$$\begin{aligned}
\ell_1(u) &= \langle(-1,0,0),u\rangle + \lambda_1,\\
\ell_2(u) &= \langle(1,0,0),u\rangle - \lambda_2,\\
\ell_3(u) &= \langle(0,-1,0),u\rangle + \lambda_2,\\
\ell_4(u) &= \langle(0,1,0),u\rangle - \lambda_3,\\
\ell_5(u) &= \langle(1,0,-1),u\rangle,\\
\ell_6(u) &= \langle(0,-1,1),u\rangle.
\end{aligned}$$

The potential function is given by

$$\begin{aligned}
\mathfrak{PO} &= e^{-x_1}T^{-u_1+\lambda_1} + e^{x_1}T^{u_1-\lambda_2} + e^{-x_2}T^{-u_2+\lambda_2}\\
&\quad + e^{x_2}T^{u_2-\lambda_3} + e^{x_1-x_3}T^{u_1-u_3} + e^{-x_2+x_3}T^{-u_2+u_3}\\
&= \frac{Q_1}{y_1} + \frac{y_1}{Q_2} + \frac{Q_2}{y_2} + \frac{y_2}{Q_3} + \frac{y_1}{y_3} + \frac{y_3}{y_2}.
\end{aligned}$$

By equating the partial derivatives

$$\begin{aligned}
\frac{\partial \mathfrak{PO}}{\partial y_1} &= -\frac{Q_1}{y_1^2} + \frac{1}{Q_2} + \frac{1}{y_3},\\
\frac{\partial \mathfrak{PO}}{\partial y_2} &= -\frac{Q_2}{y_2^2} + \frac{1}{Q_3} - \frac{y_3}{y_2^2},\\
\frac{\partial \mathfrak{PO}}{\partial y_3} &= -\frac{y_1}{y_3^2} + \frac{1}{y_2}
\end{aligned}$$

with zero, one obtains

$$\begin{aligned}
Q_1Q_2y_3 &= y_1^2(y_3+Q_2),\\
Q_3(y_3+Q_2) &= y_2^2,\\
y_1y_2 &= y_3^2,
\end{aligned}$$

whose solutions are given by

$$\begin{aligned}
y_1 &= \frac{y_3^2}{y_2},\\
y_2 &= \pm\sqrt{Q_3(y_3+Q_2)},\\
y_3 &= \sqrt[3]{Q_1Q_2Q_3},\ \omega\sqrt[3]{Q_1Q_2Q_3},\ \omega^2\sqrt[3]{Q_1Q_2Q_3},
\end{aligned}$$

where $\omega = \exp(2\pi\sqrt{-1}/3)$ is a primitive cubic root of unity. Since $\dim H^*(F(1,2,3),\Lambda)$ is six, one has as many critical point as $\dim H^*(F(1,2,3),\Lambda)$ in this case. One can show that all these critical points are non-degenerate by computing the Hessian. The valuations of the critical points are given by

$$
\begin{aligned}
u_1 &= \mathfrak{v}(y_1) = \mathfrak{v}\big(y_3^2/y_2\big) \\
&= -u_2 + 2u_3, \\
u_2 &= \mathfrak{v}(y_2) = \frac{1}{2}\mathfrak{v}\big(Q_3(Q_2 + y_3)\big) \\
&= \frac{1}{2}\big(\lambda_3 + \min\{\lambda_2, u_3\}\big), \\
u_3 &= \mathfrak{v}(y_3) = \frac{1}{3}\mathfrak{v}(Q_1 Q_2 Q_3) \\
&= \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3),
\end{aligned}
$$

so that $u = (u_1, u_2, u_3)$ is unique for any $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and always lies in the interior of the Gelfand–Cetlin polytope.

Next we discuss the case of $Gr(2,4)$. Fix $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^3$ satisfying

$$\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4,$$

so that $\mathcal{O}_\lambda$ is the Grassmannian of two-planes in a four-space. The Gelfand–Cetlin pattern in this case is given by

$$
\begin{array}{ccccccccccccc}
\lambda_1 & & & & \lambda_1 & & & & \lambda_3 & & & & \lambda_3 \\
& \rotatebox{-45}{$=$} & & \rotatebox{45}{$=$} & & \rotatebox{-45}{$\geqslant$} & & \rotatebox{45}{$\leqslant$} & & \rotatebox{-45}{$=$} & & \rotatebox{45}{$=$} & \\
& & \lambda_1 & & & & u_1 & & & & \lambda_3 & & \\
& & & \rotatebox{-45}{$\geqslant$} & & \rotatebox{45}{$\leqslant$} & & \rotatebox{-45}{$\geqslant$} & & \rotatebox{45}{$\leqslant$} & & & \\
& & & & u_2 & & & & u_3 & & & & \\
& & & & & \rotatebox{-45}{$\geqslant$} & & \rotatebox{45}{$\leqslant$} & & & & & \\
& & & & & & u_4 & & & & & &
\end{array}
$$

so that the Gelfand–Cetlin polytope Δ_λ is defined by six inequalities

$$\Delta_\lambda = \big\{ u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 \;\big|\; \ell_i(u) = \langle v_i, u\rangle - \tau_i \geqslant 0,\ i = 1, \dots, 6 \big\}$$

where

$$
\begin{aligned}
\ell_1(u) &= \big\langle (0, -1, 0, 0), u \big\rangle + \lambda_1, \\
\ell_2(u) &= \big\langle (-1, 1, 0, 0), u \big\rangle, \\
\ell_3(u) &= \big\langle (1, 0, -1, 0), u \big\rangle, \\
\ell_4(u) &= \big\langle (0, 0, 1, 0), u \big\rangle - \lambda_3, \\
\ell_5(u) &= \big\langle (0, 1, 0, -1), u \big\rangle, \\
\ell_6(u) &= \big\langle (0, 0, -1, 1), u \big\rangle,
\end{aligned}
$$

and the potential function is given by

$$\begin{aligned}
\mathfrak{PO} &= e^{-x_2}T^{-u_2+\lambda_1} + e^{-x_1+x_2}T^{-u_1+u_2} + e^{x_1-x_3}T^{u_1-u_3} \\
&\quad + e^{x_3}T^{u_3-\lambda_3} + e^{x_2-x_4}T^{u_2-u_4} + e^{-x_3+x_4}T^{-u_3+u_4} \\
&= \frac{Q_1}{y_2} + \frac{y_2}{y_1} + \frac{y_1}{y_3} + \frac{y_3}{Q_3} + \frac{y_2}{y_4} + \frac{y_4}{y_3}.
\end{aligned}$$

By equating the partial derivatives

$$\begin{aligned}
\frac{\partial \mathfrak{PO}}{\partial y_1} &= -\frac{y_2}{y_1^2} + \frac{1}{y_3}, \\
\frac{\partial \mathfrak{PO}}{\partial y_2} &= -\frac{Q_1}{y_2^2} + \frac{1}{y_1} + \frac{1}{y_4}, \\
\frac{\partial \mathfrak{PO}}{\partial y_3} &= -\frac{y_1}{y_3^2} + \frac{1}{Q_3} - \frac{y_4}{y_3^2}, \\
\frac{\partial \mathfrak{PO}}{\partial y_4} &= -\frac{y_2}{y_4^2} + \frac{1}{y_3}
\end{aligned}$$

with zero, one obtains

$$\begin{aligned}
y_1^2 &= y_2 y_3, \\
Q_1 y_1 y_4 &= y_2^2(y_1 + y_4), \\
y_3^2 &= Q_3(y_1 + y_4), \\
y_4^2 &= y_2 y_3,
\end{aligned}$$

whose solutions are given by

$$\begin{aligned}
y_1 &= \pm\sqrt{Q_1 Q_3}, \\
y_2 &= Q_1 Q_3 / y_3, \\
y_3 &= \pm\sqrt{2 Q_3 y_1}, \\
y_4 &= y_1.
\end{aligned}$$

Since $\dim H^*(Gr(2,4), \Lambda)$ is six, one has less critical points than $\dim H^*(Gr(2,4), \Lambda)$ in this case, in contrast to the case of $F(1,2,3)$. All these critical points are non-degenerate and one can see that

$$\begin{aligned}
u_1 &= \frac{1}{2}(\lambda_1 + \lambda_3), \\
u_2 &= \frac{1}{4}(3\lambda_1 + \lambda_3), \\
u_3 &= \frac{1}{4}(u_1 + 3\lambda_3), \\
u_4 &= \frac{1}{2}(\lambda_1 + \lambda_3),
\end{aligned}$$

so that $u = (u_1, u_2, u_3, u_4)$ is unique for any $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and always lies in the interior of the Gelfand–Cetlin polytope.

12. Non-displaceable Lagrangian torus fibers

We give a proof of the following theorem in this section:

Theorem 12.1. *Let $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n)$ be a non-increasing sequence of real numbers and $\Phi_\lambda : F \to \Delta_\lambda$ be the corresponding Gelfand–Cetlin system. Then there exists $u \in \operatorname{Int} \Delta_\lambda$ such that the Lagrangian torus fiber $L(u) = \Phi_\lambda^{-1}(u)$ satisfies*

$$\psi\big(L(u)\big) \cap L(u) \neq \emptyset$$

for any Hamiltonian diffeomorphism $\psi : \mathcal{O}_\lambda \to \mathcal{O}_\lambda$. If $\psi(L(u))$ is transversal to $L(u)$ in addition, then

$$\#\big(\psi\big(L(u)\big) \cap L(u)\big) \geqslant 2^N.$$

This theorem is an analogue of [9, Theorem 1.5] for flag manifolds, and follows immediately from Theorem 12.2 below and the Hamiltonian isotopy invariance in Lagrangian intersection Floer theory [8, Theorem J].

Theorem 12.2. *For any λ, there exists $u \in \operatorname{Int} \Delta_\lambda$ and $\mathfrak{x} \in H^1(L(u); \Lambda_0)$ such that the deformed Floer cohomology is isomorphic to the ordinary cohomology:*

$$HF\big(\big(L(u), \mathfrak{x}\big), \big(L(u), \mathfrak{x}\big); \Lambda_0\big) \cong H\big(L(u); \Lambda_0\big).$$

Note that $\mathfrak{x}$ above is taken from $H^1(L(u); \Lambda_0)$ whereas the bounding cochain b appearing in the definition of the deformed Floer cohomology in Section 10 is taken from $H^1(L(u); \Lambda_+)$. To define the deformed Floer cohomology twisted by $\mathfrak{x} \in H^1(L(u); \Lambda_0)$, one divide $\mathfrak{x}$ into the constant part and the positive part

$$\begin{aligned} \mathfrak{x} &= \mathfrak{x}_0 + \mathfrak{x}_+, \\ \mathfrak{x}_0 &\in H^1\big(L(u); \mathbb{C}\big), \\ \mathfrak{x}_+ &\in H^1\big(L(u); \Lambda_+\big), \end{aligned}$$

take the flat non-unitary line bundle $\mathcal{L}_\rho$ whose holonomy representation is given by

$$\rho = \exp(\mathfrak{x}_0) : H_1\big(L(u); \mathbb{Z}\big) \to \mathbb{C}^\times,$$

consider the A_∞-operation $\{\mathfrak{m}_k^\rho\}_{k=0}^\infty$ twisted by the flat non-unitary line bundle $\mathcal{L}_\rho$ as in Cho [5], and define the deformed Floer differential $\mathfrak{m}_1^{\mathfrak{x}}$ by

$$\mathfrak{m}_1^{\mathfrak{x}}(x) = \sum_{k,l} \mathfrak{m}_k^\rho\big(\mathfrak{x}_+^{\otimes k} \otimes x \otimes \mathfrak{x}_+^{\otimes l}\big).$$

Let $\{\mathbf{e}_i\}_{i=1}^N$ be the basis of $H_1(L(u);\mathbb{Z})$ corresponding to the angle coordinate of $L(u)$. Then for any $i = 1, \dots, N$, one has

$$\begin{aligned}\mathfrak{m}_1^{\mathfrak{x}}(\mathbf{e}_i) \cap \bigl[L(u)\bigr] &= \sum_{k,l} \mathfrak{m}_k^{\rho}\bigl(\mathfrak{x}_+^{\otimes l} \otimes \mathbf{e}_i \otimes \mathfrak{x}_+^{\otimes(k-l-1)}\bigr) \cap \bigl[L(u)\bigr] \\ &= y_i \frac{\partial \mathfrak{PO}^u(y)}{\partial y_i}\bigg|_{y=\exp(\mathfrak{x})}.\end{aligned}$$

This shows that $\mathfrak{m}_1^{\mathfrak{x}} = 0$ on $H^1(L(u);\Lambda_0)$ if $\mathfrak{y} = \exp(\mathfrak{x})$ is a critical point of $\mathfrak{PO}^u(y)$. If this is the case, the induction argument of [9, Lemma 12.1] on the degree and the Maslov index $\mu(\beta)$ using the A_∞-relation

$$\begin{aligned}\mathfrak{m}_{1,\beta}^{\rho,b}(\mathbf{f}_1 \cup \mathbf{f}_2) = &\sum_{\beta_1+\beta_2=\beta} \pm \mathfrak{m}_{2,\beta_1}^{\rho,b}\bigl(\mathfrak{m}_{1,\beta_2}^{\rho,b}(\mathbf{f}_1) \otimes \mathbf{f}_2\bigr) \\ &+ \sum_{\beta_1+\beta_2=\beta} \pm \mathfrak{m}_{2,\beta_1}^{\rho,b}\bigl(\mathbf{f}_1 \otimes \mathfrak{m}_{1,\beta_2}^{\rho,b}(\mathbf{f}_2)\bigr) \\ &+ \sum_{\beta_1+\beta_2=\beta} \pm \mathfrak{m}_{1,\beta_1}^{\rho,b}\bigl(\mathfrak{m}_{2,\beta_2}^{\rho,b}(\mathbf{f}_1 \otimes \mathbf{f}_2)\bigr)\end{aligned}$$

shows that $\mathfrak{m}_1^{\mathfrak{x}} = 0$ on $H^*(L(u);\Lambda_0)$, so that the deformed Floer cohomology is isomorphic to the ordinary cohomology;

$$HF^*\bigl(\bigl(L(u),\mathfrak{x}\bigr), \bigl(L(u),\mathfrak{x}\bigr); \Lambda_0\bigr) \cong H^*\bigl(L(u);\Lambda_0\bigr).$$

Note that

$$\mathfrak{v}\bigl(\exp(\mathfrak{x})\bigr) = 0.$$

Hence one can find a twisting cochain $\mathfrak{x} \in H^1(L(u);\Lambda_0)$ such that the deformed Floer cohomology $HF^*((L(u),\mathfrak{x}),(L(u),\mathfrak{x});\Lambda_0)$ is isomorphic to the ordinary cohomology, if there is a critical point

$$\mathfrak{y} = (\mathfrak{y}_1, \dots, \mathfrak{y}_N) \in \Lambda^N$$

of the Laurent polynomial

$$\mathfrak{P}(y) = \sum_{i=1}^m y^{v_i} T^{-\tau_i}$$

such that

$$\mathfrak{v}(\mathfrak{y}) = \bigl(\mathfrak{v}(\mathfrak{y}_1), \dots, \mathfrak{v}(\mathfrak{y}_N)\bigr) \in \Delta_\lambda \subset \mathbb{R}^N.$$

The existence of such a critical point follows from Proposition 12.3, which we learned from Hiroshi Iritani. See also [9, Proposition 3.6].

Proposition 12.3. *For a convex polytope*

$$\Delta = \{u \in \mathbb{R}^N \mid \ell_i(u) \geqslant 0,\ i = 1, \dots, m\}$$

where

$$\ell_i(u) = \langle v_i, u \rangle - \tau_i,$$

define a Laurent polynomial $\mathfrak{P} \in \Lambda[y_1^{\pm 1}, \dots, y_N^{\pm 1}]$ *by*

$$\mathfrak{P} = \sum_{i=1}^{m} y^{v_i} T^{-\tau_i}.$$

Then $\mathfrak{P}$ *has at least one critical point whose valuation lies in the interior of* Δ.

We divide the proof into three steps:

Step 1. *Let* $P \in \mathbb{R}[y_1^{\pm 1}, \dots, y_N^{\pm 1}]$ *be a Laurent polynomial over the field of real numbers such that every non-zero coefficient is positive and the origin is in the interior of the Newton polytope. Then* P *has a critical point in* $(\mathbb{R}^{>0})^N$.

Proof. The set

$$\{y \in (\mathbb{R}^{>0})^N \mid P(y) \leqslant c\}$$

is compact for any $c \in \mathbb{R}$, so that P has a global minimum in $(\mathbb{R}^{>0})^N$. □

Step 2. *The Laurent polynomial* $\mathfrak{P}$ *has a critical point* $\mathfrak{y}$ *in* $(\Lambda^\times)^N$.

Proof. The set of critical points is defined as the common zero of partial derivatives of the potential function, which always exists in the compactification $\mathbb{P}^N(\Lambda)$ of the torus $(\Lambda^\times)^N$. If all the critical points lie at infinity and none of them lies on the torus, then it remains so after substituting any real number into T. However $\mathfrak{P}$ has a critical point in $(\mathbb{R}^{>0})^N$ after substituting any positive real number in T by Step 1, which shows that $\mathfrak{P}$ also have a critical point $\mathfrak{y}$ on the torus $(\Lambda^\times)^N$. □

Step 3. *The valuation of* $\mathfrak{y}$ *lies in the interior of* Δ.

Proof. Let Γ be the convex hull of the set

$$\{(v_i, z) \in \mathbb{R}^N \times \mathbb{R} \mid z \geqslant -\tau_i\}_i \cup \{(0, z) \in \mathbb{R}^N \times \mathbb{R} \mid z \geqslant 0\}$$

and $\phi : \mathbb{R}^N \to \mathbb{R}$ be the piecewise-linear map such that the union of faces of Γ containing the origin is a part of the graph of ϕ. A subset of $\mathbb{R}^N$ where ϕ is linear forms a maximal-dimensional cone of a complete fan Σ in $\mathbb{R}^N$. For each cone σ in Σ, define $u_\sigma \in \mathbb{R}^N$ by

$$\phi|_\sigma(v) = -\langle u_\sigma, v \rangle.$$

It follows from the construction of ϕ that

$$-\tau_i \geqslant \phi(v_i) \geqslant -\langle u_\sigma, v_i \rangle$$

for any i, so that $u_\sigma \in \Delta$. We will write $\Sigma^{(N)}$ for the set of N-dimensional cones of the fan Σ.

Let u be the valuation of $\mathfrak{y}$ and put $\tau_u = \min_i\{\langle v_i, u\rangle - \tau_i\}$. The leading term $\mathfrak{P}_u$ of $\mathfrak{P}$ is defined as

$$\mathfrak{P}_u = \sum_{i:\ \langle v_i, u\rangle - \tau_i = \tau_u} y^{v_i} T^{-\tau_i},$$

which has the leading term $\mathfrak{y}_0$ of $\mathfrak{y}$ as its critical point. Assume that u is not in the interior of the convex hull of $\{u_\sigma\}_{\sigma \in \Sigma^{(N)}}$. Then the Newton polytope of $\mathfrak{P}_u$ will not contain the origin in its interior, and one can choose a coordinate of the torus so that $\mathfrak{P}_u$ contains only non-negative powers of y_1. This shows that the coefficient of any term in $\partial\mathfrak{P}_u/\partial y_1$ is positive if one substitutes a positive real number into T. Recall from Step 2 that $\mathfrak{y}$ gives positive real numbers if one substitutes a positive real number into T. It follows that $\mathfrak{y}_0$ gives positive real numbers after substituting a sufficiently small positive real number ϵ into T and hence one has

$$\left.\frac{\partial \mathfrak{P}_u}{\partial y_1}(\mathfrak{y}_0)\right|_{T=\epsilon} > 0.$$

This contradicts the fact that $\mathfrak{y}_0$ is a critical point of $\mathfrak{P}_u$ so that u must be contained in the interior of the convex hull of $\{u_\sigma\}_{\sigma \in \Sigma^{(N)}}$, which in turn is contained in Δ. $\square$

13. A relation with Toda lattice

In this section, we discuss the potential function for the full flag manifold after substituting e^{-1} into the indeterminate element T in the Novikov ring, and its relation with quantum cohomology and the quantum Toda lattice. Although the potential function is no longer invariant under Hamiltonian isotopy after this substitution and hence unfit for application to symplectic topology, it is the potential function after this substitution which appears as the Landau–Ginzburg potential of the mirror of Fano manifolds, studied by string theorists such as Hori and Vafa [16]. The main result in this section is Theorem 13.3, which is an immediate consequence of Theorem 10.1 and Givental's integral representation in Theorem 13.2.

Let us first recall the definition of quantum cohomology and Givental's J-function. For a projective manifold X with its Kähler class ω, the quantum product $\circ$ on $H^*(X;\Lambda)$ is defined by

$$\langle A \circ B, C\rangle = \sum_{\beta \in H_2(X;\mathbb{Z})} T^{\beta\cap\omega} \int_{[\overline{\mathcal{M}}_{0,3}(X,\beta)]^{\mathrm{virt}}} ev_1^*(A) \cup ev_2^*(B) \cup ev_3^*(C)$$

where $\langle \bullet, \bullet\rangle$ is the Poincaré pairing, $[\overline{\mathcal{M}}_{0,3}(X,\beta)]^{\mathrm{virt}}$ is the virtual fundamental class of the moduli space of stable maps of genus zero and degree β with three marked points into X, and

$$ev_i : \overline{\mathcal{M}}_{0,3}(X,\beta) \to X, \quad i = 1, 2, 3,$$

is the evaluation map at the i-th marked point. The quantum product equips the cohomology group $H^*(X;\Lambda)$ with the structure of a Frobenius algebra. Now consider the substitution $T = e^{-1}$, although it may not make sense since the definition of quantum product involves an infinite sum. When this sum converges, the quantum cohomology ring $\circ$ can be regarded as a family of Frobenius algebras, parametrized by (an open subset of) $H^2(X;\mathbb{R})$ considered as the moduli space of symplectic structures.

Now choose a basis $\{T_i\}_{i=1}^h$ of $H^*(X;\mathbb{R})$ such that $\{T_i\}_{i=1}^r$ is a basis of $H^2(X;\mathbb{R})$. Let $(t_i)_{i=1}^r$ be the coordinate of $H^2(X;\mathbb{R})$ dual to the basis $\{T_i\}_{i=1}^r$, so that the symplectic form ω of X is represented as $\omega = \sum_{i=1}^r t_i T_i$. Then quantum product is an infinite series in

$$q = (q_i)_{i=1}^r = \bigl(\exp(-t_i)\bigr)_{i=1}^r,$$

which, in the case of the flag manifold, is known to be convergent for sufficiently small q. One can also let q take values in the complexification $H^2(X;\mathbb{C})$ of $H^2(X;\mathbb{R})$.

The Givental's (small) J-function is defined by

$$J_j = \sum_{\beta\in H_2(F^{(n)};\mathbb{Z})} q^\beta \int_{[\overline{\mathcal{M}}_{0,1}(X);\beta)]^{\mathrm{virt}}} \frac{ev^*(T_j \wedge \exp(\sum_{i=1}^r p_i t_i/\hbar))}{\hbar(\hbar-\psi)}, \quad j = 1,\dots,h.$$

It is known that

$$J_j = \langle s_j, 1\rangle,$$

where $(s_j)_{j=1}^h$ is a basis of flat sections of the *Givental connection*, which is a connection on the trivial vector bundle on $H^2(X;\mathbb{C})$ with fiber $H^*(X;\mathbb{C})$ defined by

$$\nabla_{\frac{\partial}{\partial t_i}} = \hbar\frac{\partial}{\partial t_i} - T_i \circ .$$

The $\mathcal{D}$-module on $H^2(X;\mathbb{C})$ generated by the J-function is called the *quantum $\mathcal{D}$-module*, whose characteristic variety is the spectrum of the quantum cohomology ring.

For the full flag manifold $F^{(n)}$, let $\mathcal{V}_i \to F^{(n)}$ be the universal subbundle of rank i and

$$p_i = c_1(\mathcal{V}_{i+1}/\mathcal{V}_i) \in H^2\bigl(F^{(n)};\mathbb{Z}\bigr), \quad i = 0,\dots,n-1$$

be the first Chern class of the i-th quotient line bundle $\mathcal{V}_{i+1}/\mathcal{V}_i$. The set $\{p_i\}_{i=0}^{n-1}$ generates $H^*(F^{(n)};\mathbb{Z})$ and the complete set of relations is given by

$$(\lambda + p_0)\cdots(\lambda + p_{n-1}) = \lambda^n.$$

We introduce a redundant parameter $(t_i)_{i=0}^{n-1}$ for $H^2(F^{(n)};\mathbb{C})$ and define the J-function by

$$J_j = \sum_{\beta\in H_2(F^{(n)};\mathbb{Z})} q^\beta \int_{[\overline{\mathcal{M}}_{0,1}(F^{(n)});\beta)]^{\mathrm{virt}}} \frac{ev^*(T_j \wedge \exp(\sum_{i=0}^{n-1} p_i t_i/\hbar))}{\hbar(\hbar-\psi)}$$

where j runs from 1 to $\dim H^*(F^{(n)};\mathbb{C}) = n!$.

Now we recall the quantum Toda lattice and its relation with the J-function of the full flag manifold following Givental and Kim [12] (see also Kim [18] and Joe and Kim [17]). The quantum Toda Hamiltonian is defined by

$$H = \frac{\hbar^2}{2}\sum_{i=0}^{n-1}\frac{\partial^2}{\partial t_i^2} - \sum_{i=1}^{n-1} e^{t_i - t_{i-1}}.$$

It commutes with n mutually commutative differential operators

$$D_i\left(\hbar\frac{\partial}{\partial t_0}, \dots, \hbar\frac{\partial}{\partial t_{n-1}}, q_1, \dots, q_{n-1}\right), \quad i = 1, \dots, n \tag{21}$$

where $q_i = \exp(t_i - t_{i-1})$,

$$\det(A + xI) = x^{n+1} + \sum_{i=1}^{n} D_i(p_0, \dots, p_{n-1}, q_1, \dots, q_{n-1})x^{n-i},$$

and

$$A = \begin{pmatrix} p_0 & q_1 & 0 & \cdots & 0 & 0 \\ -1 & p_1 & q_2 & \cdots & 0 & 0 \\ 0 & -1 & p_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-2} & q_{n-1} \\ 0 & 0 & 0 & \cdots & -1 & p_{n-1} \end{pmatrix}.$$

The following theorem gives an astonishing relation between the quantum cohomology ring of the full flag manifold and the quantum Toda lattice:

Theorem 13.1. *(See Givental and Kim [12], Kim [18].) The J-function of the full flag manifold $F^{(n)}$ is an eigenfunction of the quantum Toda lattice:*

$$D_i J_j = 0, \quad i = 1, \dots, n,\ j = 1, \dots, n!.$$

It follows from this theorem that the quantum cohomology ring of the full flag manifold is isomorphic to the coordinate ring of the Lagrangian level set of the classical Toda Hamiltonians.

Now we recall the stationary-phase integral representation of the eigenfunction of the quantum Toda lattice due to Givental [11]. Consider $n(n-1)$ variables

$$\{X_{ij}, Y_{ij} \mid i = 1, \dots, n-1,\ j = 1, \dots, n-i\}$$

and the $n(n-1)/2$-dimensional torus Y_q cut out from $\operatorname{Spec}\mathbb{C}[X_{ij}^{\pm 1}, Y_{ij}^{\pm 1}]_{i,j}$ by the equations

$$Y_{i,j}X_{i,j} = X_{i+1,j}Y_{i,j+1}, \quad i = 1, \dots, n-2,\ j = 1, \dots, n-i-1,$$

and

$$X_{i,n-i} Y_{i,n-i} = q_i, \quad i = 1, \dots, n-1,$$

where $q = (q_1, \dots, q_{n-1}) \in (\mathbb{C}^\times)^{n-1}$. These relations imply that X_{ij} and Y_{ij} can be expressed by $n(n-1)/2$ variables

$$\{T_{ij} \mid i = 1, \dots, n-1, \ j = 1, \dots, n-i\}$$

as

$$X_{ij} = \exp(T_{ij} - T_{i,j+1})$$

and

$$Y_{ij} = \exp(T_{i+1,j} - T_{ij}),$$

where

$$q_i = \exp(T_{i+1,n-i} - T_{i,n-i+1}), \quad i = 1, \dots, n-1.$$

Define the phase function f_q and the holomorphic volume form ω on Y_q by

$$f_q = \sum_{i,j} (X_{ij} + Y_{ij})$$

and

$$\omega = \bigwedge_{i,j} dT_{ij}.$$

Fix $\hbar \in \mathbb{C}^\times$ and a complete Kähler metric on Y_q. A *Lefschetz thimble* is the unstable manifold of $\Re(f_q/\hbar)$ starting from a critical point of f_q. The following theorem is due to Givental:

Theorem 13.2. *(See Givental [11].) The phase function has* $\dim H^*(F^{(n)}) = n!$ *critical points, and the stationary-phase integrals*

$$I_a = \int_{\Gamma_a} e^{f_q/\hbar} \omega$$

for the corresponding Lefschetz thimbles $\{\Gamma_a\}_{a=1}^{n!}$ *gives the component* J_a *of the* J*-function for a suitable choice of a basis of* $H^*(F^{(n)}; \mathbb{C})$.

Now it is obvious that the potential function $\mathfrak{PO}|_{T=e^{-1}}$ and the phase function f_q are related by

$$T_{ij} = x_i^{(i+j-1)} + \lambda_i^{(i+j-1)}, \quad i = 1, \dots, n-1, \ j = 1, \dots, n-i,$$

and

$$T_{i,n-i+1} = \lambda_i, \quad i = 1, \dots, n.$$

This results in the following striking relation between the Gelfand–Cetlin system and the quantum Toda lattice:

Theorem 13.3. *The potential function for Lagrangian torus fibers of the classical Gelfand–Cetlin system on the full flag manifold $F^{(n)}$, considered as a Laurent polynomial in $n(n-1)/2$ variables with n parameters after substituting e^{-1} to T, is the phase function for an integral representation of the solution to the quantum Toda lattice.*

Now let us discuss the classical limit of the above story. The classical Toda lattice is a completely integrable system whose Hamiltonians are the classical limits

$$D_i(p_0, \dots, p_{n-1}, q_1, \dots, q_{n-1}), \quad i = 0, \dots, n-1$$

of the differential operators (21). The level set of $\{D_i\}_{i=0}^{n-1}$ is a Lagrangian subvariety of $(\operatorname{Spec} \mathbb{C}[p_1, \dots, p_{n-1}, q_1, \dots, q_{n-1}], \omega)$, where p_0 is determined by

$$D_1(p_0, \dots, p_{n-1}) = p_0 + \dots + p_{n-1} = 0,$$

and the symplectic form is given by

$$\omega = \sum_{i=1}^{n-1} p_i \wedge \frac{dq_i}{q_i}.$$

The classical limit of the stationary-phase integral is controlled by the Jacobi ring

$$J(f_q) = \mathbb{Q}[q_1, \dots, q_{n-1}]\big[y_1^{\pm 1}, \dots, y_N^{\pm 1}\big] \Big/ \left(\frac{\partial f_q}{y_i \partial y_i}\right)_{i=1}^{N}$$

whose spectrum is the set $\operatorname{Cr}(f_q)$ of critical points of f_q, in that there is a birational map

$$\begin{array}{ccc} \operatorname{Cr}(f_q) & \to & \operatorname{Spec} \mathbb{C}[p_1, \dots, p_{n-1}, q_1, \dots, q_{n-1}] \\ \Psi & & \Psi \\ (y, q) & \mapsto & \left(q \dfrac{\partial f_q}{\partial q}(y),\ q\right), \end{array}$$

into the characteristic variety of the D-module generated by the stationary phase integrals. On the other hand, the characteristic variety of the quantum D-module is the spectrum of the quantum cohomology ring. By putting them together, we obtain the following:

Corollary 13.4. *The Jacobi ring of the potential function for Lagrangian torus fibers of the classical Gelfand–Cetlin system on the full flag manifold $F^{(n)}$ is isomorphic to the ring of functions on the level set of the classical Toda Hamiltonians, and hence to the quantum cohomology ring of $F^{(n)}$.*

Note that the isomorphism between the Jacobi ring of the potential function for Lagrangian torus fibers of the classical Gelfand–Cetlin system and the quantum cohomology ring cannot hold for general partial flag manifolds; the simplest example is the Grassmannian $Gr(2,4)$ where the number of critical points of the potential function for general q is four, which is strictly smaller than the rank of the cohomology ring.

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