K-Orbits on Grassmannians and a PRV Conjecture for Real Groups

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1. Introduction

Schubert varieties are closures of orbits of a Borel subgroup on a generalized flag variety $G/P$ for a complex reductive group $G$. They are known to be normal and to have rational singularities by the work of Anderson, Mehta–Ramanathan, Ramanan, Seshadri, and others. They play an important role in the representation theory of $G$ (cf. [De, Ku2, Math] and others). The purpose of this paper is to study closures of $K$-orbits, where $K$ is the connected component of the fixed point set of an involution $\theta$. The situation for $K$-orbits is considerably more complicated than for $B$-orbits.

Our first main result is the following theorem.

**Theorem A** (See 2.10, 3.7, and 4.8). Let $G/P$ be the Grassmannian of $k$-planes in $\mathbb{C}^n$ and let $\theta$ be any involution of $GL(n)$. Then the closures of $K$-orbits on the Grassmannian are normal, have rational singularities, and are therefore Cohen–Macaulay.

In addition, we determine the sections of an ample line bundle restricted to an orbit closure. For orbits of a Borel subgroup $B$, these results date back to Hodge [Ho] (see also [Se] and others for generalizations). The proof is a case by case argument. It relies on some particularly simple resolutions for $K$-orbit closures and some relatively simple branching laws for certain representations. In each case, there is some small representation which controls the decomposition of all larger representations. Moreover, the decomposition of these small representations is reflected in the geometry of orbits. This fact makes it easy to determine which sections

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vanish on each orbit closure. This observation and a calculation on the resolution enable us to deduce the theorem.

However, for other generalized flag varieties the \( K \)-orbit closures appear to be much more complicated than Schubert varieties. Indeed, they are not always normal (5.7). However, if they are normal, then we can deduce certain results in representation theory. The basic idea is that the study of sections of line bundles over closures of \( K \)-orbits should pick out \( K \)-types in finite dimensional representations of \( G \). We consider a line bundle \( \mathcal{L} \) over the variety of Borel subalgebras \( G/B \) such that \( \mathcal{H}^0(G/B, \mathcal{L}) \cong V(\lambda)^* \), the dual of the finite dimensional representation with highest weight \( \lambda \). To each \( K \)-orbit \( Q \) through a Borel subalgebra \( b_\gamma \), we associate a Weyl group element \( w \) and an associated Borel subalgebra \( b_w = w^{-1} b_\gamma \) such that \( b_w \) is \( \theta \)-stable outside the real roots of \( b_\gamma \). We associate a split Levi subgroup \( L \) to the real roots of \( b_w \). This implies that we can discuss fine \( L \cap K \)-types in this context (see 6.6).

**Theorem B.** Suppose the \( K \)-orbit closure \( \overline{Q} \) is normal, and \( \lambda \) is very dominant. Then any \( K \)-type with extremal weight \( \mu \) occurs in \( V(\lambda) \), provided \( \mu \) is a fine \( L \cap K \)-type occurring in the irreducible representation of \( L \) with extremal weight \( w\lambda \).

By very dominant, we mean \( \lambda - n\rho \) is dominant, where \( n \gg 0 \). In Section 6.7 we will reinterpret this statement in terms of parameters.

Recall that the pair \( (G, K) \) may be regarded as the complexification of a real semisimple Lie group \( G_0 \). When \( G_0 \) is a complex group, the branching law problem above is to decompose the tensor product of two representations. In this case, \( K \)-orbits are normal and \( \lambda \) can be any dominant weight [Ku2, Math, MR2]. The \( K \)-types \( \mu \) we find are just the PRV components which Kumar and Mathieu proved to exist. Given the above geometric results, this allows us to avoid a lemma due to Joseph and to give a simpler proof of the final step of the PRV conjecture. After we give the proof, we remark that our argument can be recast in geometric terms following Chang’s \( D \)-module interpretation of Vogan’s theory of minimal \( K \)-types. In the case when \( G_0 \) is complex, Kostant and Kumar have a complementary interpretation of this result.

We hope that the above two theorems make the following question interesting.

**Problem.** Determine if \( V_K(\mu) \) always occurs in \( V(\lambda) \). If it does not always occur, give some conditions on orbits \( Q \) or weights \( \lambda \) which imply that it occurs.

If the underlying real group \( G_0 \) is complex or \( U(n, 1) \) then \( V_K(\mu) \) always occurs (see also 6.10). The complex case is due to Kumar and Mathieu
while $U(n, 1)$ can be settled either by using explicit branching laws or by a geometric argument. However, for other real groups, some of the geometric facts used by Kumar and Mathieu in the complex case do not hold (see Section 6).

The organization of his paper is as follows. In Sections 2, 3, and 4 we study orbits for the unitary groups, the symplectic group, and the orthogonal group. In Section 5, we show that the orbit closures for the unitary groups always have small resolutions, so that their intersection homology is easily computable. In Section 6, we discuss the general case of $K$-orbits on $G/B$ and prove Theorem B. In addition, we discuss some of the problems associated with extending results concerning $B$-orbits to $K$-orbits.

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2. Orbits for the Unitary Groups

2.1. Let $G = GL(n)$ and $K = GL(p) \times GL(q)$. Fix a decomposition $\mathbb{C}^n = \mathbb{C}^p + \mathbb{C}^q$ such that $GL(p)$ acts on $\mathbb{C}^p$ and $GL(q)$ acts on $\mathbb{C}^q$. We denote the standard coordinate vectors in $\mathbb{C}^n$ by $\{e_1, \ldots, e_p, e_{p+1}, \ldots, e_n\}$.

$K$ acts on the Grassmannian $G(k, n-k)$ of $k$-planes in $\mathbb{C}^n$ by restricting the standard action of $G$. We are interested in the closure of the orbits of $K$, which are analogous to Schubert varieties. If $X \subset Y$ is an inclusion of algebraic varieties, we denote the Zariski closure of $X$ in $Y$ by $\overline{X}$.

For a $k$-plane $U \in G(k, n-k)$, let $s = \dim U \cap \mathbb{C}^p$, $t = \dim U \cap \mathbb{C}^q$, $r = k-s-t$. Let

$$Q(s, t) = \{U \in G(k, n-k) | \dim U \cap \mathbb{C}^p = s, \dim U \cap \mathbb{C}^q = t\}. \quad (2.1.1)$$

The pair $(s, t)$ is an invariant for the action of $K$. The following proposition asserts that the orbit of $U$ is completely determined by this invariant.

**Proposition.** $Q(s, t)$ is a single $K$-orbit. Moreover, $Q(s, t) \subset \overline{Q(s', t')}$ if and only if $s \geq s'$ and $t \geq t'$.

**Proof.** Both assertions follow from the fact that every $Q(s, t)$ contains a plane in standard form

$$U_{s,t} = \text{span}\{e_1, \ldots, e_s, e_{s+1} + e_{p+s+1}, \ldots, e_{s+r} + e_{p+s+r}, e_{p+s+r+1}, \ldots, e_{p+s+r+t}\}. \quad (2.1.2)$$

2.2. Corollary. The closed orbits are the orbits of the form $Q(i, k-i)$, where $0 \leq i \leq \min(p, k)$, $0 \leq k-i \leq \min(q, k)$. 
2.3. **Example.** Consider the action of $GL(2) \times GL(2)$ on $G(2, 2)$. Then the closed orbits are $Q(2, 0)$ and $Q(0, 2)$, which are points, and $Q(1, 1)$, which is a product of two projective lines. There are two three dimensional orbits $Q(1, 0)$ and $Q(0, 1)$. $Q(1, 0)$ (resp. $Q(0, 1)$) contains $Q(1, 1)$ and $Q(2, 0)$ (resp. $Q(1, 1)$ and $Q(0, 2)$) in its closure. It is not difficult to see that both $Q(1, 0)$ and $Q(0, 1)$ are singular. $Q(0, 0)$ is the open orbit.

2.4. To each orbit closure $\overline{Q(s, t)}$, we attach a resolution of singularities $Z$ analogous to the resolution for a single condition Schubert variety.

**Definition.** Let $Z = Z(s, t)$ be the set

$$\{(U, V, W) \in G(k, n-k) \times G(s, p-s) \times G(t, q-t) \mid U \in \overline{Q(s, t)}, V \subset \mathbb{C}^p \cap U, W \subset \mathbb{C}^q \cap U \}.$$

Let

$$\theta: Z \to \overline{Q(s, t)}$$

be the map given by projection on the first factor.

2.5. **Proposition.** $\theta$ is a resolution of $\overline{Q(s, t)}$. In other words, $Z$ is smooth and $\theta$ is birational and proper.

**Proof.** If $U$ is in the orbit $Q(s, t)$, then the definition of $Z$ implies that $V = U \cap \mathbb{C}^p$ and $W = U \cap \mathbb{C}^q$. Hence, $\theta$ is birational. On the other hand, $Z$ is a fiber bundle with fiber $G(r, n-k)$ via

$$pr_2 \times pr_3: Z \to G(s, p-s) \times G(t, q-t).$$

This implies the remaining assertions. \[\square\]

2.6. We now discuss the inclusion $j_{s,t}: \overline{Q(s, t)} \subset G(k, n-k)$. Let $\omega_k$ be the highest weight of the representation of $GL(n)$ on $\wedge^k(C^n)$. Then $\mathcal{H} = \mathcal{L}(-\omega_k)$ is the ample generator for $\text{Pic}(G(k, n-k))$. We have

$$H^0(G(k, n-k), \mathcal{H}) \cong \bigwedge^k (C^n)^*.$$  

(2.6.1)

There is an obvious decomposition into $K$-types,

$$\bigwedge^k (C^n)^* \cong \sum_{\min(p,k) \geq i \geq \max(0,k-q)} \left( \bigwedge^i (C^p) \otimes \bigwedge^{k-i} (C^q) \right)^*.$$  

(2.6.2)
Parametrize the representations of $GL(p)$ by
\[
V_p(a_1, \ldots, a_p) \cong V_p \left( \sum_{1 \leq i < p} (a_i - a_{i+1}) \omega_i \right),
\]
(2.6.3)
where $a_1 \geq \cdots \geq a_p$, with the convention that $a_{p+1} = 0$. Then we can write (2.6.2) as
\[
H^0(G(k, n-k), \mathcal{H}) \cong \bigwedge^k (\mathbb{C}^*)^* \cong V_n(\omega_k)^* \cong \sum (V_p(\omega_i) \otimes V_q(\omega_{k-i}))^*.
\]
(2.6.4)
Note that there is a canonical 1-1 correspondence between closed orbits and $K$-types formed of sections of the ample generator; namely
\[
(V_p(\omega_i) \otimes V_q(\omega_{k-i}))^* \cong \{ f \in H^0(G(k, n-k), \mathcal{H}) \mid f|_{Q/(k-k-j)} \equiv 0, j \neq i \}.
\]
(2.6.5)
This follows by computing the sections of $\mathcal{H}$ for the closed orbits.

**Proposition.** As $K$-modules,
\[
j^*_s(H^0(G(k, n-k), \mathcal{H})) \cong \sum_{s \leq i < k-t} (V_p(\omega_i) \otimes V_q(\omega_{k-i}))^*.
\]
In other words, the $K$-types which "live" on $\overline{Q(s, t)}$ are precisely the ones whose sections are trivial when restricted to the closed orbits not lying in $Q(s, t)$.

**Proof.** Using (2.6.5), it is clear that the right hand side is contained in the left hand side. On the other hand, if $f$ is a section in $\bigwedge^k (\mathbb{C}^*)^*$, with $i < s$ or $i > k - t$, then $f$ vanishes on $Q(s, t)$. Indeed, regard $U \in Q(s, t)$ as an element of $\mathbb{P}(\bigwedge^k(\mathbb{C}^*))$ using the Plücker embedding. Then $U$ can be written as a wedge of $k$-vectors, with exactly $k-t$ having components in $\mathbb{C}^n$, and exactly $k-s$ having components in $\mathbb{C}^q$. Then if $i > k - t$, the associated linear functional clearly vanishes on $U$. The case $i < s$ is similar. \[
\]
2.7. More generally, we have the following result.

**Proposition.** $H^0(G(k, n-k), \mathcal{H}^m) \cong \sum V_p(\sum m_i \omega_i)^* \otimes V_q(\sum m_i \omega_{k-i})^*$, where the sum is taken over sequences $(m_i)$ with $\max(0, k-q) \leq i \leq \min(p, k)$ such that $\sum m_i = m$ and $m_i \geq 0$.

**Proof.** Because $\mathcal{H}^m \cong \otimes^m \mathcal{H}$, there is a map
\[
\otimes^m H^0(G(k, n-k), \mathcal{H}) \to H^0(G(k, n-k), \mathcal{H}^m)
\]
(2.7.1)
given by pointwise multiplication of sections. Using (2.6.4), it is clear that
the representations in the right hand side of the proposition appear in the
image of the above map. The claim of the proposition is that in fact this
is the entire image. Thus it is enough to see that as $\mathcal{K}$-modules,

$$V_n(m\omega_k) \cong \sum_{\sum m_i = n} V_p\left(\sum m_i \omega_i\right) \otimes V_q\left(\sum m_i \omega_{k-i}\right).$$  \hfill (2.7.2)

We may as well assume $p \geq k, q$. Recall that $\mathcal{K}$ is the Levi component of a
parabolic subgroup $P = KN$. Then the Weyl character formula implies that
for any dominant $\lambda$,

$$[V_{\mathcal{K}}(\mu) : V(\lambda)] = \sum_i \sum_{w \in W^+} (-1)^{l(w)} \times \dim \text{Hom}_K[V_{\mathcal{K}}(\mu) \otimes S'(n) : V_{\mathcal{K}}(w(\lambda + \rho) - \rho)],$$  \hfill (2.7.3)

where $W^+$ is the subset of elements in the Weyl group such that
$\omega(\lambda + \rho) - \rho$ is dominant for $\mathcal{K}$.

Write $V_{\mathcal{K}}(\mu) = V_p(\mu_1) \otimes V_q(\mu_2)$. Suppose $\mu$ is a $\mathcal{K}$-type of $V(m\omega_k)$. Then
$\mu$ has to be a weight of $V(m\omega_k)$, so the coordinates of $\mu_1, \mu_2$ are nonnegative. Similarly, the $\mathcal{K}$-types of $S'(n)$ are of the form $V_p(\alpha) \otimes V_q(\beta)^*$, where the highest weights $\alpha$ and $\beta$ have only nonnegative coordinates. It follows that any weights of $V_p(\alpha)$ or $V_q(\beta)$ have only nonnegative coordinates. Thus any $\mathcal{K}$-type occurring in $V_{\mathcal{K}}(\mu) \otimes S'(n)$ has nonnegative coordinates on the first factor. Since these must equal the coordinates of $w(m\omega_k + \rho) - \rho$, and $\omega \in W^+$, this implies that $w = 1$. Write $V_{\mathcal{K}}(m\omega_k) = V_p(m\omega_k) \otimes V_q(0)$. It follows that $\mu_2 = \beta$.

On the other hand, all weights of $V_p(\mu_1)$ and $V_p(\alpha)$ have nonnegative coordinates. A factor of $V_p(\mu_1) \otimes V_p(\alpha)$ has highest weight of the form

$$\eta = \mu_1 + (\text{weight of } V_p(\alpha)) = \alpha + (\text{weight of } V_p(\mu_1)).$$  \hfill (2.7.4)

Thus $V_p(m\omega_k)$ occurs only if $\mu_1$ and $\alpha$ have at most $k$ nonzero coordinates, so $\mu_1, \alpha,$ and $m\omega_k$ define highest weights of $GL(k)$ by deleting zeros. By Weyl's reciprocity formula,

$$[V_p(\mu_1) \otimes V_p(\alpha) : V_p(m\omega_k)] = [V_p(\mu_1) \otimes V_p(\alpha) : V_p(m\omega_k)].$$  \hfill (2.7.5)

The proposition follows by observing that $V_p(m\omega_k)$ is one dimensional and applying the result of [Schmid] which states that $S'(n)$ decomposes with multiplicity one and the $\mathcal{K}$-types that occur are of the form $V_p(\beta) \otimes V_q(\beta)^*$.  \hfill $\square$
2.8. **Proposition.**

\[ j_{*, i}^*(H^0(G(k, n-k), \mathcal{H}^m)) \cong \sum_{\sum m_i = m} V_\rho \left( \sum m_i \omega_i \right)^* \otimes V_\delta \left( \sum m_i \omega_{-i} \right)^*. \]

**Proof.** This follows from 2.7 using 2.6.

2.9. We are now ready to prove our main results for \( GL(p) \times GL(q) \) orbits. For normality, the basic idea is as follows. The map

\[ \theta^*: H^0(\overline{Q(s, t)}, \mathcal{H}^m) \hookrightarrow H^0(Z_{s, t}, \theta^*(\mathcal{H}^m)) \]  

(2.9.1)

is an inclusion. By definition, \( \text{Im}(j^*) \subseteq H^0(\overline{Q(s, t)}, \mathcal{H}^m) \). We will show that \( \text{Im}(j^*) \cong H^0(Z_{s, t}, \theta^*(\mathcal{H}^m)) \), by making an explicit computation on \( Z_{s, t} \). This implies \( \theta^* \) is an isomorphism, which implies normality.

**Proposition.**

\[ H^i(Z_{s, t}, \theta^*(\mathcal{H}^m)) \cong \begin{cases} \text{Im}(j^*) & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases} \]

**Proof.** Let us fix any \( (U_0, V_0, W_0) \in Z_{s, t} \). Let \( P(k) \) be the subgroup of \( GL(n) \) stabilizing \( U_0 \), and let \( P(s + t) \) be the subgroup stabilizing \( V_0 \oplus W_0 \). Then we have the identification

\[ Z_{s, t} \cong K \times_{K \cap P(s + t)} P(s + t)/[P(s + t) \cap P(k)], \]  

(2.9.2)

via \( (k, p) \mapsto kpU_0, kV_0, kW_0 \). Hence,

\[ H^0(Z_{s, t}, \theta^*(\mathcal{H}^m)) \cong H^0(K \times_{K \cap P(s + t)} P(s + t)/[P(s + t) \cap P(k)], \mathcal{L}(-m\omega_k)), \]

\[ \cong H^0(K/K \cap P(s + t), H^0(P(s + t)/P(s + t) \cap P(k), \mathcal{L}(-m\omega_k))), \]

(2.9.3)

where \( \mathcal{L}(-\lambda) \) denotes the line bundle over \( P/Q \) corresponding to the character of \( Q \) induced by a weight \( -\lambda \) on its Cartan subalgebra. Let \( L = L(s + t) \) be the standard maximal reductive subgroup of \( P(s + t) \), so that \( L \cap K \cong GL(s) \times GL(p-s) \times GL(t) \times GL(q-t) \). Then

\[ H^0(P(s + t)/[P(s + t) \cap P(k)], \mathcal{L}(-m\omega_k)) \]

\[ \cong H^0(L/[L \cap P(k)], \mathcal{L}(-m\omega_k)). \]

(2.9.4)
The decomposition of (2.9.4) into $L \cap K$-types is
\[
\sum_{m_1 \geq \cdots \geq m_r} V_i(m \omega_i)^* \otimes V_{\rho - i}(m_1, \ldots, m_r, 0, \ldots, 0)^* \\
\otimes V_j(m \omega_j)^* \otimes V_{\rho - j}(m - m_1, \ldots, m - m_r, 0, \ldots, 0)^*. 
\] (2.9.5)

This result follows by applying Proposition 2.7 to a smaller Grassmannian, after applying a permutation to move $P(k)$ into a standard form. Since
\[
K/[K \cap P(s + t)] \cong G(s, p - s) \times G(t, q - t), 
\] (2.9.6)
we can apply the Borel–Weil–Bott theorem to obtain,
\[
H^0(K/[K \cap P(s + t)], H^0(P(s + t)/P(s + t) \cap P(k), \mathcal{L}(-m \omega_k))) \\
\cong \sum_{s \leq i < k - s, \sum n_i = m} V_p \left(\sum n_i \omega_i\right)^* \otimes V_q \left(\sum n_i \omega_{k - i}\right)^*. 
\] (2.9.7)

By Proposition 2.7 this is isomorphic to $\text{Im}(j^*)$. This establishes the first claim. The vanishing of the higher cohomology follows by using the Leray spectral sequence and the Borel–Weil–Bott theorem.

2.10. Definition. A normal variety $Y$ in characteristic 0 has rational singularities if for some resolution $\theta: Z \to Y$, the higher derived functors of $\theta_*$ vanish [KKMS, p. 50]. If $Y$ is projective, it is projectively normal with respect to an embedding if the associated affine coordinate ring is integrally closed in its function field.

Theorem. The orbit closures $\overline{Q(s, t)}$ are normal and have rational singularities. Moreover, they are projectively normal with respect to the embedding induced by $\mathcal{H}$.

Proof. The proof is standard (see [Ku1, 2.17–2.20], e.g.).

2.11. Corollary. The orbit closures $\overline{Q(s, t)}$ are Cohen–Macaulay.

Proof. This follows by [KKMS, p. 50].

2.12. Remark. It is not difficult to see that the kernel of the differential $\theta_*: T_C(Z) \to T_{H^2}(G(k, n - k))$ is the tangent space to the fiber. This implies that the scheme-theoretic fibers are reduced. One can show that this implies that the orbit closures $\overline{Q(s, t)}$ are normal and have rational singularities. Thus, if one is primarily interested in these two properties, one can give a shorter proof. The same remark applies to the cases of $SO(n)$ and $Sp(n)$. We thank Shrawan Kumar for suggesting this line of reasoning.
3. ORBITS FOR THE SYMPLECTIC GROUP

3.1. In this section we describe the orbits and resolutions for the symplectic group and prove the analogue of Proposition 2.9. This implies corresponding results for orbit closures.

We begin with some generalities about forms, which are also applicable to the orthogonal group. Let $\mathcal{V}$ be a complex $n$-dimensional vector space with a nondegenerate bilinear form $B$, which can be either alternating or symmetric. Let $K$ be the isometry group of $B$. Consider the Grassmannian $G(k, n-k)$ with $k \leq \lfloor n/2 \rfloor$. It is well known that $K$ orbits on $G(k, n-k)$ are $k$-planes with a specified radical $[A_r]$. We record this fact as a proposition.

**Proposition.** Let

$$Q(i) = \{ U \in G(k, n-k) \mid \dim(\text{rad}(U)) = i \}.$$

Then $Q(i)$ is a single $K$-orbit. Moreover, $Q(i) \subset Q(j)$ if and only if $i \geq j$.

3.2. We can construct resolution for $\tilde{Q}(i)$ analogous to the resolutions in 2.3.

**Definition.** Let

$$Z_i = \{(U, V) \in G(k, n-k) \times G(i, n-i) \mid U \in \tilde{Q}(i), V \subset \text{rad}(B|_U)\},$$

and $\theta: Z_i \rightarrow \tilde{Q}(i)$ be the obvious map.

**Proposition.** $\theta$ is a resolution of singularities.

**Proof.** It is obvious that $\theta$ is birational. Moreover, there is a projection

$$Z_i \rightarrow G_0(i, n-i), \quad \text{given by} \quad (U, V) \mapsto V,$$

where $G_0(i, n-i)$ is the set of all isotropic $i$-dimensional subspaces of $\mathbb{C}^n$. The fiber of this map is

$$\{ U \in G(k, n-k) \mid V \subset \text{rad}(B|_U) \}.$$  \hspace{1cm} (3.2.2)

Fix $(U_0, V_0) \in Z_i$. Let $P$ be the subgroup of $GL(n)$ stabilizing $V_0$ and $V_0^\perp$, and $P(k)$ the subgroup of $GL(n)$ leaving $U_0$ invariant. Then the map $(k, p) \mapsto (kpU_0, kV_0)$ gives an isomorphism

$$Z_i \cong K \times_{K \cap P} P/[P \cap P(k)].$$  \hspace{1cm} (3.2.3)

The proof follows.  \hfill \blacksquare
3.3. We now describe the restriction map \( j^* \), where

\[
j: \mathcal{Q}(i) \rightarrow G(k, n-k)
\]

(3.3.1)
is the obvious inclusion. For this, we consider the symplectic and
orthogonal groups separately. For the rest of this section, assume \( B \) is
alternating, let \( G = GL(2n) \) and \( K = Sp(n) \). We take \( i \equiv k \pmod{2} \) so that
the orbits \( \mathcal{Q}(i) \) are nonempty. The required decomposition results are as
follows.

Identify the Cartan subgroup \( T \) with \((\mathbb{C}^*)^n \) and the corresponding Cartan
subalgebra with \( \mathbb{C}^n \). Then choose a positive system so that the simple roots
are, in the standard coordinates,

\[
x_i = e_i - e_{i+1}, \quad \text{for} \quad i = 1, \ldots, n-1, \quad x_n = 2e_n.
\]

Note that the roots of \( T \) in the complementary space to \( Sp(n) \) inside
\( GL(2n) \) are

\[
A(p, t) = \{ \pm e_i \pm e_j, i \neq j \}.
\]

(3.3.3)

Then an irreducible finite dimensional representation of \( Sp(n) \) is
parametrized by its highest weight \( \lambda \), written as

\[
\lambda \cong (a_1, \ldots, a_n) \equiv \sum_{1 \leq i < j \leq n} (a_i - a_{i+1}) \omega_i,
\]

(3.3.4)

where \( a_i \geq a_{i+1}, a_i \in \mathbb{N} \), with the convention that \( a_{n+1} = 0 \) (\( \omega_i \) are the
fundamental weights). In particular, as a representation of \( GL(2n) \),

\[
\bigwedge^k (\mathbb{C}^{2n}) = V_{GL(2n)}(1, \ldots, 1, 0, \ldots, 0) \cong V_{GL(2n)}(\omega_k),
\]

(3.3.5)

and its restriction to \( Sp(n) \) is

\[
V_{GL(2n)}(\omega_k) \mid_{Sp(n)} \cong V_{Sp(n)}(1, \ldots, 1, 0, \ldots, 0) \oplus \bigwedge^{k-2} (\mathbb{C}^n)
\cong V_{Sp(n)}(\omega_k) \oplus \bigwedge^{k-2} (\mathbb{C}^n),
\]

(3.3.6)

where the inclusion of \( \bigwedge^{k-2} \) into \( \bigwedge^k \) is given by wedging with the symplec-
tic form. This decomposition has the following geometric interpretation.

**Proposition.** As a \( K \)-module,

\[
j'^*(H^0(G(k, n-k), \mathcal{M})) \cong \sum_{k \geq s \geq i, s \equiv i(2)} V_{Sp(n)}(1, \ldots, 1, 0, \ldots, 0).
\]
In other words, the $K$-types which "die" on $Q(i)$ are precisely the ones "divisible" by $\omega^{k-\frac{i}{2}+i^2}$ (where $\omega$ is the symplectic form).

Proof. This follows from the fact that $\omega^r | Q(i) \equiv 0$ if and only if $r > (k-i)/2$.

3.4. For multiples of fundamental weights we have the following proposition which can also be derived using branching laws in terms of Littlewood–Richardson coefficients (e.g., [Ha, p. 427, formula (25.39)]).

**Proposition.** If $k = 2l$ is even,

$$V_{GL(2n)}(m\omega_k) \cong \sum_{m > m_1 > \ldots > m_l > 0} V_{Sp(n)}(m_1, m_1, m_2, m_2, \ldots, m_l, 0, \ldots, 0).$$

If $k = 2l + 1$ is odd,

$$V_{GL(2n)}(m\omega_k) \cong \sum_{m > m_1 > \ldots > m_l > 0} V_{Sp(n)}(m, m_1, m_1, m_2, m_2, \ldots, m_l, 0, \ldots, 0).$$

**Proof.** Regard $V(m\omega_k)$ as a representation of the real form $U^*(2n)$.

Assume $k = 2l$. Then $V(m\omega_k)$ is a factor in the induced representation from a character of the parabolic subgroup with Levi component $U^*(2l) \times U^*(2n-2l)$. Thus

$$[V_{Sp(n)}(\mu) : V_{GL(2n)}(m\omega_k)] \leq [V_{Sp(n)}(\mu) : \text{Ind}_{Sp(1) \times Sp(n-1)}^{Sp(n)} (\text{Triv} \otimes \text{Triv})].$$

(3.4.1)

By Helgason’s theorem [He],

$$\text{Ind}_{Sp(n) \times Sp(n-1)}^{Sp(n)} [\text{Triv} \otimes \text{Triv}] \cong \sum_{m_1 > \ldots > m_l > 0} V_{Sp(n)}(m_1, m_1, m_2, m_2, \ldots, m_l, 0, \ldots, 0).$$

(3.4.2)

The statement follows by observing that any $K$-type occurring in $V_{GL(2n)}(m\omega_k)$ has to have coordinates smaller than the coordinates of the highest weight.

Assume $k = 2l + 1$. Then a similar argument to the one in (2.7) (using (3.3.2) and (3.3.3) and Kostant’s multiplicity formula) shows that the only $K$-types occurring must have the first coordinate equal to $m$ and

$$[V_K(m, r_2, \ldots, r_n) : V_{GL(2n)}(m\omega_k)] = [V(r_2, \ldots, r_n) : V_{GL(2n-2)}(m\omega_{k-1})].$$

(3.4.3)

The proof follows from the case $k = 2l$. \[\square\]
The above proposition asserts that the highest weight vectors for $V(m\omega_k)_{Sp(m)}$ are tensor products of the highest weight vectors for $V(\omega_k)_{Sp(m)}$.

3.5. Recall $j$, from (3.3.1).

**Proposition.** As a $K$-module,

$$\text{Im}(j^*) H^0(G(k, n-k), \mathbb{C}H^n)$$

$$\cong \sum_{m \geq m_{i+2} \geq \cdots \geq m_k \geq 0} \times V(m, m, m_{i+2}, m_{i+2}, m_{i+4}, m_{i+4}, \ldots, m_k, m_k, 0, \ldots, 0).$$

**Proof.** The result follows from Proposition 3.3 by interpreting 3.4 in terms of monomials using highest weight vectors.

3.6. Recall $\mathcal{H} \cong Z(-\omega_k)$ from 2.6.

**Proposition.**

$$H^r(Z, \theta^*(\mathcal{H})) = \begin{cases} \text{Im}(j^*) & \text{if } r = 0 \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** The proof follows from a computation on $Z$, using 3.3. It is similar to the proof of 2.9 and will be left to the reader.

Then just as for $GL(p) \times GL(q)$ orbits, the following theorem follows by using the Leray spectral sequence (see 2.10).

3.7. **Theorem.** The orbit closures $\bar{Q}(i)$ are normal and have rational singularities and are therefore Cohen–Macaulay. Moreover, they are projectively normal with respect to the embedding induced by $H$.

4. **Orbits for the Orthogonal Group**

4.1. In this section we study the case of the orthogonal group. For the orthogonal group, the decomposition results we require are slightly more complicated. This is due to the fact that the obvious invariant form lies in $S^2(\wedge \mathbb{C}^n)$.

4.2. We have already described orbits $Q(i)$ for the orthogonal group and resolutions $Z_i$ in 3.1 and 3.2. Let $G = GL(n)$ and $K = O(n)$. The
fact that $O(n)$ is disconnected is not very relevant due to the following proposition.

**Proposition.** All orbits $Q(i)$ for the orthogonal group are connected except for $Q(n/2)$ when $n$ is even. In other words, if $n$ is odd or if $n$ is even and $i < n/2$ all $O(n)$ orbits are connected and if $n$ is even, then only the closed orbit in $G(n/2, n/2)$ is disconnected.

**Proof.** (See [ACGH, p. 102].) The case $i = 0$ is clear. If $i > 0$ let $M = \{ (U, v) \mid v \in U \in Q(i), B(v, v) = 0, v \neq 0 \}$, $N = \{ v \in \mathbb{C}^n \mid B(v, v) = 0, v \neq 0 \}$. It is not difficult to see that $N$ is connected if $n > 2$. $M$ surjects onto $Q(i)$. We will show $M$ is connected unless $i = n/2$ by reducing to the case $i = 0$ or $n = 2$. $M$ fibers over $N$ with fibers $Q(i-1) \subset \mathbb{C}^{n-2}$. Indeed, let $x(v) \in (\mathbb{C}^n)^\ast$ be dual to $v$ using $B$. Then the fiber over $v \in N$ is

$$\{ U \in Q(i) \mid v \in U \} \cong Q(i-1) \subset x(v)^\perp / C_v.$$

Assume $n > 2$. Since $N$ is connected, $M$ is connected if $Q(i-1)$ is connected, so we can reduce by induction to the case $i = 0$ or $i = 1$ and $n = 2$ which can be settled by direct computation.

4.3. The required decomposition results are as follows.

**Proposition.** As $K$-modules,

$$V_{GL(n)}(2, \ldots, 2, 0, \ldots, 0) = \sum_{j=0}^{k} V_{O(n)}(2, \ldots, 2, 0, \ldots, 0)$$

**Proof.** This is elementary. We omit the details.

This result can be interpreted as follows. Let $\phi \in S^2(\mathbb{C}^n)$ represent the nondegenerate symmetric form. Using the obvious injective map, we can regard $\phi \in T^2(\mathbb{C}^n)$. Then there is a map

$$m_\phi : S^2 \left( \bigwedge^i (\mathbb{C}^n) \right) \to S^2 \left( \bigwedge^{i+1} (\mathbb{C}^n) \right)$$

(4.3.1)

given by

$$S^2 \left( \bigwedge^i (\mathbb{C}^n) \right) \to T^2(T^i(\mathbb{C}^n))$$

$$\cong T^{2i}(\mathbb{C}^n) \to T^{2i+2}(\mathbb{C}^n) \to T^2(T^{i+1}(\mathbb{C}^n)) \to S^2 \left( \bigwedge^{i+1} (\mathbb{C}^n) \right)$$

(4.3.2)
where the degree raising map is tensoring with $\phi$, and the degree preserving maps are the above inclusions or symmetrization or anti-symmetrization maps.

4.4. Proposition. $m_{\phi}$ is an $O(n)$ equivariant inclusion if $2i < n$.

Proof. $m_{\phi}$ is obviously $O(n)$-equivariant for any $i$. We claim that $m_{\phi}$, the corresponding interior multiplication $i_{\phi}$ and their bracket define a representation of $sl(2)$ on $S^2(\wedge^i (\mathbb{C}^n))$. The result then follows from the representation theory of $sl(2)$.

We can assume that $\phi$ is the standard form

$$\phi = \sum_{j=1}^{n} x_j^2.$$  \hfill (4.4.1)

Let $x_j = x_{m_1} \wedge \cdots \wedge x_{m_i}$, if $J$ is the multi-index $(m_1, \ldots, m_i)$. Let

$$x_K \cdot x_L \in S^2 \left( \bigwedge^i (\mathbb{C}^n) \right).$$  \hfill (4.4.2)

Then a routine calculation implies that

$$m_{\phi} \circ i_{\phi}(x_K \cdot x_L) = \sum_{j \in K \cap L} \sum_{m \in K \cup L - \{j\}} x_{K \cdot \{j\}, m} x_{L \cdot \{j\}, m},$$  \hfill (4.4.3)

where $K \cdot \{j\} = K \cup \{m\} - \{j\}$ (as an ordered multi-index). In addition,

$$i_{\phi} \circ m_{\phi}(x_K \cdot x_L) = \sum_{j \in K \cap L \cup \{m\}} \sum_{m \in K \cup L} x_{K \cdot \{j\}, m} x_{L \cdot \{j\}, m}.$$  \hfill (4.4.4)

These two calculations imply that

$$[m_{\phi}, i_{\phi}] \vert S^2 \left( \bigwedge^i (\mathbb{C}^n) \right) = (2i - n) \text{Id}$$  \hfill (4.4.5)

so that the claim follows.

For $V_{GL(n)}(2\omega_k)^*$, viewed as a subrepresentation of $S^2(\wedge^i (\mathbb{C}^n))$, the above decomposition result can be interpreted as asserting that

$$V_{GL(n)}(2\omega_k)^* \vert_{O(n)} \cong V_{O(n)}(2, \ldots, 2, 0, \ldots, 0) \oplus m_{\phi} \left[ V_{GL(n)}(2\omega_{k-1}) \right] \vert_{O(n)}.$$  \hfill (4.4.6)

4.5. This result has the following geometric interpretation. Recall $j_k: \mathcal{Q}(i) \subset G(k, n-k)$ from (3.3.1).
Proposition. As a K-module,

\[ j_\ast(\mathcal{H}^0(G(k, n-k)), \mathcal{H}^2) \cong \sum_{i \leq k} V(2, ..., 2, 0, ..., 0). \]

Proof. This is the assertion that if \( V \) is not in the image of \( m_\sigma, m_\sigma'(V) | Q(j) \equiv 0 \) if and only if \( i > k - j \). An easy computation using highest weight vectors implies the nonvanishing result when \( i < k - j \). A somewhat more difficult calculation implies the vanishing when \( i > k - j \). This calculation can be avoided by showing \( H^0(Z_i, \theta^\ast(\mathcal{H}^2)) \) is equal to the right hand side since

\[ H^0(Q(i), \mathcal{H}^2) \cong H^0(Z_i, \theta^\ast(\mathcal{H}^2)). \]

4.6. Proposition. Let

\[ V_{GL(n)}(m, ..., m, 0, ..., 0)^\ast \]

\[ \cong \sum_{0 \leq a_1 \leq \cdots \leq a_k \leq m/2} V_{GL(n)}(m - 2a_1, ..., m - 2a_k, 0, ..., 0). \]

Proof. This is similar to 3.4 and easier. We omit the details. \( \Box \)

4.7. Proposition. As a K-module,

\[ \text{Im}(j_\ast) H^0(G(k, n-k), \mathcal{H}^m) \]

\[ \cong \sum_{0 \leq a_1 + \cdots + a_k \leq m/2} V(m, ..., m, m - 2a_1 + \cdots + m - 2a_k, 0, ..., 0). \]

Proof. When \( m = 1 \), the assertion is easy to check since the sum contains only one term. For \( m \geq 2 \), the result follows from 4.6 using 4.5.

4.8. The following results are analogous to 2.9 and 2.10.

Proposition.

\[ H^r(Z_i, \theta^\ast(\mathcal{H}^m)) = \begin{cases} \text{Im}(j_\ast) & \text{if } r = 0, \\ 0 & \text{otherwise}. \end{cases} \]

Theorem. The orbit closures for \( SO(n) \) acting on \( G(k, n-k) \) are normal and have rational singularities and are therefore Cohen–Macaulay. Moreover, they are projectively normal with respect to the embedding induced by \( H \).
Proof. The proof follows as in 2.10 except for the complication that we are using $O(n)$ to decompose representations and we want to study orbits of $SO(n)$. However, this problem is minor since $SO(n)$ orbits are the same as $O(n)$ orbits unless $k = n/2$. The case when $k = n/2$ causes no difficulty since the corresponding orbits are closed.

5. Intersection Homology of Orbit Closures

5.1. In this section we compute the intersection homology sheaves for the orbit closures $\overline{Q(s, t)}$ for $GL(p) \times GL(q)$-orbits. These groups are easy to compute since the orbit closures always have small resolutions (see 5.2). The small resolutions are the resolutions constructed in 2.4 in roughly half the cases, but a different construction is needed in the other cases. This work is analogous to work of Zelevinsky on Schubert varieties in Grassmannians [Ze]. Its applications to representation theory are less interesting than in the case of Schubert varieties because the orbits $Q(s, t)$ are usually not affinely embedded.

5.2. Definition. Let $\phi: X \to Y$ be a map of complex analytic varieties. Then $\phi$ is small if for $r > 0$,

$$\text{codim}\{y \in Y \mid \dim \phi^{-1}(y) \geq r\} > 2r.$$ 

The importance of this definition is made evident by the following theorem, due to Goresky and MacPherson. Let $IC_X$ denote the intersection sheaf complex of Deligne associated to a space $X$.

Theorem [GM, 6.2]. If $\phi: X \to Y$ is a small resolution, then $\phi_!(C_X) = IC_Y$.

5.3. We recall the setting in Section 2. We are studying $GL(p) \times GL(q)$-orbits on the Grassmannian $G(k, n-k)$ and an orbit closure $\overline{Q(s, t)}$ has a resolution $Z = Z(s, t)$. We may assume that $p \geq q$ and $n-k \geq k$.

Proposition. If $n-k \geq p$, then $\theta: Z(s, t) \to \overline{Q(s, t)}$ is a small resolution.

Proof. Suppose $Q(s', t') = \overline{Q(s, t)}$. If $x \in Q(s', t')$, then

$$\theta^{-1}(x) \cong G(s, s'-s) \times G(t, t'-t).$$

Recall that $r = k - s - t$. We have

$$\dim(\overline{Q(s, t)}) = s(p-s) + t(q-t) + r(n-k),$$
so \( \text{codim}(\overline{Q(s', t')}, \overline{Q(s, t)}) = s(p - s) + t(q - t) + r(n - k) - [s'(p - s') + t'(q - t') + r'(n - k)] \). A routine calculation shows that
\[
2 \cdot \dim(\theta^{-1}(x)) < \text{codim}(\overline{Q(s', t')}, \overline{Q(s, t)})
\]
if \( n - k \geq p \).

**Corollary.** Assume \( n - k \geq p \).

\[
IC^*_{Q(s', t')}((\overline{Q(s, t)})) \cong H^*(G(s, s' - s)) \otimes H^*(G(t, t' - t)).
\]

Let \( IH \) denote the intersection homology of a space \( X \) using the middle perversity. Then

\[
IH(\overline{Q(s, t)}) \cong H^*(G(s, p - s)) \otimes H^*(G(t, q - t)) \otimes H^*(G(k - s - t, n - k)).
\]

**5.4.** If \( n - k < p \), a different resolution is necessary. These resolutions are analogous to some resolutions introduced by Zelevinsky.

**Definition.** Let
\[
\tilde{Z} = \{(U, V, W) \in \overline{Q(s, t)} \times G(k + p - s, n - k - p + s) \\
\times G(t, q - t) \mid W \subset U \cap C^q, U + C^p \subset V\}.
\]

**Proposition.** \( \tilde{\theta}: \tilde{Z} \to \overline{Q(s, t)} \) is a resolution of singularities.

**Proof.** If \( U \in Q(s, t) \), then \( W = U \cap C^q \), and \( V = U + C^p \). This implies the map is birational. There is a fiber bundle \( pr_3: \tilde{Z} \to G(t, q - t) \). The fiber \( F \) is a resolution of the orbit closure \( \overline{Q(s, 0)} \subset G(k - t, n - k) \).

\[
F = \{(\tilde{U}, \tilde{V}) \in G(k - t, n - k) \\
\times G(k + p - t - s, n - k - p + s) \mid \tilde{U} + C^p \cap C^q \subset \tilde{W}\}.
\]

Projection on the second factor makes \( F \) a fiber bundle with base \( G(r, q + s - k) \) and fiber \( G(k - t, p - s) \).

**5.5. Proposition.** Suppose \( p \geq n - k \). Then the resolution
\[
\tilde{\theta}: \tilde{Z} \to \overline{Q(s, t)}
\]
is a small resolution.

**Proof.** The proof is similar to the proof of 5.3. We omit the details.
**Corollary.** Assume \( n - k \leq p \). Then

\[
\mathcal{IC}_{G(s', t)}(Q(s, t)) \cong H_{\bullet}(G(s' - s, t + s' - p)) \otimes H_{\bullet}(G(t, t' - t)) \\
\mathcal{IH}(Q(s, t)) \cong H_{\bullet}(G(k - s - t, q + s - k)) \otimes H_{\bullet}(G(t, q - t)) \\
\otimes H_{\bullet}(G(k - t, p - s)).
\]

5.6. **Remark.** The resolutions for orbits of the symplectic and orthogonal groups constructed in 3.2 are not small in general.

6. **A PRV Conjecture for Real Groups**

6.1. In this section, we motivate our interest in normality of \( K \)-orbits in \( G/B \) by proving that if a \( K \)-orbit closure is normal, then a certain set of \( K \)-types occur in the finite dimensional representation \( V_G(\lambda) \), at least when \( \lambda \) is very dominant. Here \( G \) is a complex reductive group with an algebraic involution \( \theta \) and \( K \) is the connected component of \( G^\theta \). When the group is \( G \times G \) and the involution is \( \theta(x, y) = (y, x) \), then the set of \( K \)-types contains only one element, the PRV component of the tensor product. In general there are several, which correspond roughly to the set of minimal \( K \)-types for a Harish-Chandra module associated to the orbit.

6.2. We begin by reviewing some results about \( K \)-orbits. Let \( \mathcal{B} \cong G/B \) be the flag variety of a complex reductive group \( G \). \( \mathcal{B} \) is identified with the variety of Borel subalgebras of the Lie algebra \( g \) of \( G \), and we denote this identification by \( x \mapsto b_x \). \( b_x \) contains a \( \theta \)-stable Cartan subalgebra \( h_x \). We then associate a positive root system \( \Delta^+ \) to \( x \) by taking \( \alpha \in \Delta^+ \) if \( g_x \supset b_x \). \( \theta \) induces an involution of \( g \) (also denoted \( \theta \)), which determines a Cartan decomposition \( g = \mathfrak{t} + s \) and by restriction a decomposition, \( b_x = \mathfrak{t}_x + a_x \). As usual, we then define imaginary, real, and complex roots as follows: \( \alpha \) is imaginary if \( \theta\alpha = -\alpha \). \( \alpha \) is real if \( \theta\alpha = -\alpha \), \( \alpha \) is complex otherwise. If \( \alpha \) is imaginary, \( \alpha \) is called compact if \( g_x \supset \mathfrak{t} \) and noncompact if \( g_x \supset s \). These definitions are independent of the choice of \( h_x \supset b_x \). Given a choice of \( \Delta^+_x \), we say \( \alpha \in \Delta^+_x \) is \( \theta \)-stable if \( \theta(\alpha) \in \Delta^+_x \). We say \( h_x \) is split if all roots are real and we say \( G \) is split if its Lie algebra contains a split Cartan subalgebra.

For \( x \in \mathcal{B} \), we denote it \( K \)-orbit by \( Q = K \cdot x \) and the closure by \( \overline{Q} \).

6.3. **Definition [Sp, Mats].** An orbit \( Q \) is called distinguished if \( \Delta^+_x \) is \( \theta \)-stable outside the real roots.

Assume \( Q \) is a distinguished orbit. Let \( L \) be the Levi factor defined by the real simple roots and let \( l \) be its Lie algebra. Then there exists a \( \theta \)-stable parabolic subgroup \( P = LU \) such that the derived group \( L_1 = [L, L] \) is
split and \( x \) meets \( l \) in a Borel subalgebra whose orbit under \( (L \cap K) \) is open in \( X_L \), the flag variety of \( L \). Moreover, \( (L \cap K) \cdot x = (L_1 \cap K) \cdot x \) and \( L_1 \cap K \cap B_L = T_1 \) is a finite 2-group [BGG, Lemma 2.1].

Let \( \mathcal{P} \) be the set of simple roots in \( L \) and let \( P = P_{\mathcal{P}} \) be the corresponding parabolic subgroup, with \( \mathcal{X}_{\mathcal{P}} \cong G/P \) the variety of parabolics of type \( S \). Let \( \pi_{\mathcal{P}}: \mathcal{X} \to \mathcal{X}_{\mathcal{P}} \) be the standard projection. Then

\[
\pi_{\mathcal{P}}(Q) \cong K/K \cap P \tag{6.3.1}
\]

is a closed orbit in \( \mathcal{X}_{\mathcal{P}} \) and

\[
\bar{Q} = \pi_{\mathcal{P}}^{-1}(\pi_{\mathcal{P}}(Q)) \cong K \times_{K \cap P} P/B \quad [\text{Sp, Mats}]. \tag{6.3.2}
\]

As a consequence, \( \bar{Q} \) is smooth. \( Q \) is identified with

\[
Q = K \times_{K \cap P} (K \cap L) \cdot x, \tag{6.3.3}
\]

where \( x \) is in the open orbit of \( K \cap L \) on \( x_L \) and \( K \cap P \) acts on \( K \cap L \cdot x \) through its action on \( P/B = \mathcal{X}_L \).

Let \( z \) be a simple root defining a parabolic \( P_z \) and a variety of parabolics \( \mathcal{X}_z \) with projection \( \pi_z: \mathcal{X} \to \mathcal{X}_z \). For \( Y \subset X \) any locally closed subvariety, let

\[
P_z \ast Y = \pi_z^{-1}(Y).
\]

If \( Y = Q = K \cdot x \) is an orbit, then \( P_z \ast Q \) contains a unique dense orbit \( Q' \) and \( Q' = Q \) unless either \( z \) is noncompact imaginary or \( z \) is complex and \( \theta z \in A^+ \) [Vo, 5.1]. For \( Y = Q_b = K \cdot y \) distinguished, let \( z_{i_1}, \ldots, z_{i_k} \) be a sequence of simple roots for \( A^+ \) such that

\[
(1) \quad z_{i_j} \text{ is a complex root for the dense orbit } K \cdot x_{j-1} \text{ in } \\
P_{z_{i_{j-1}}} \cdots \ast P_{z_{i_j}} \ast (Q_b),
\]

\[
(2) \quad \theta z_{i_j} \in A^+_{z_{j-1}}.
\]

Then the orbit closure \( \bar{Q} \) is of the form

\[
\bar{Q} = P_{z_{i_k}} \cdots \ast P_{z_{i_1}} \ast (\bar{Q}_b), \tag{6.3.4}
\]

for some distinguished orbit \( \bar{Q}_b \) and \( z_{i_1}, \ldots, z_{i_k} \) as above [Sp, Mats]. Moreover, \( \dim Q = k + \dim \bar{Q}_b \). Consider the variety

\[
Z(P, i_1, \ldots, i_k) = K \times_{K \cap P} P \times_B P_{z_{i_1}} \times_B \cdots \times_B P_{z_{i_k}} / B. \tag{6.3.5}
\]
We will denote this variety by $Z_k$ when the context is clear. Thus, $Z_k$ is a fiber bundle over $Z_0$ with fiber a Bott–Samelson resolution $Z_w$ with $w = s_{i_1} \cdots s_{i_r}$, the product of the simple reflections corresponding to the $x_i$. The orbit $Q$ can be embedded inside $Z_k$ using the identification

$$Q = K \times_{K \cap B} (L \cap K) \times_{L \cap K \cap B} U.$$  \hspace{1cm} (6.3.6)

Here $U$ is a unipotent group of dimension $k = l(w)$. If $Q$ is associated to a distinguished orbit $Q_b = K \cdot y$ as above, then $x = wy$. There is a map

$$\theta_k : Z_k \to G/B, \quad (k, p, p_1, \ldots, p_k) \mapsto kpp_1 \cdots p_k.$$  \hspace{1cm} (6.3.7)

6.4. **Proposition** [Vo, 6.1]. $\theta_k : Z_k \to Q$ is a resolution of singularities of $Q$.

**Remark.** If $x$ is noncompact imaginary, then

$$\pi_+-1 \pi_+(x) \cap Q$$

consists of one or two points. Following Vogan [Vo], we say $x$ is type I if there is one point and type II if there are two points. The proposition is still true if we replace complex roots by noncompact type I roots in certain places. If some type II roots are used, then $\theta_k$ is a generically finite map.

6.5. Let $\mathcal{L}(-\lambda)$ be an ample line bundle over $\mathcal{X}$ associated to a dominant weight $\lambda$ with respect to some Borel. Let

$$\lambda_+ = 1/2(\lambda + \theta(\lambda)) \in \mathfrak{h}^*.$$ 

Consider $\lambda_+$ as an element in $\mathfrak{h}_+^*$ so that $\lambda$ is dominant with respect to $\mathfrak{b}_+$.

**Proposition.** (See [Ch, 6.10]). For every orbit $Q$, there exists a distinguished orbit $Q_b = K \cdot y$, a resolution $Z_b$, and a $K$-equivariant fiber bundle $\pi_b : Z_b \to \pi_b(Q_b)$ such that $\lambda_+ \in \mathfrak{h}_+^* = \mathfrak{h}_+^*$ is dominant for $\Delta^+_b$.

The positive system $\Delta^+_b$ is not completely determined by $Q$ and $Q_b$, as the following example indicates.

**Example.** If $G_0$ is a complex group and $Q = G_0(B, sB)$ for some simple reflection $s$, then the distinguished orbit is the diagonal in $G_0/B \times G_0/B$. The line bundle $\mathcal{L}(-\lambda) = \mathcal{L}(-\phi, -\mu)$ and the weight $\lambda_+$ associated to $(\lambda, Q)$ is $\phi + s\mu$ when restricted to $t$. The $K$-equivariant projection $Q \to G_0/B$ is projection on the first factor if $\phi + s\mu$ is dominant and projection on the second factor if $s(\phi + s\mu)$ is dominant. Clearly, one of the two must occur.
6.6. We recall the definition of fine $K$-types. Let $G$ be a complex reductive group with involution $\theta$ and let $G_1 = [G, G]$. We say $G$ is split for $\theta$ if we can choose $\theta$-stable Cartan subalgebras $h_1 \subset h$ for $G_1$ and $G$ such that $\theta \mid h_1 = -1$. Let $K = G^\theta$. Choose root vectors so that $\theta(X_\pm) = X_\mp$, $[X_\pm, X_\pm] = -H_\pm$, and $[H_\pm, X_\pm] = \pm 2X_\pm$. Let $Z_\pm = i(X_\pm + \theta(X_\mp))$. Choose some system of positive roots.

**Definition.** A $K$-type $V_K(\mu)$ is fine if $|\mu(Z_\pm)| \leq 1$ for every simple root $\alpha$.

Fine $K$-types can be interpreted as sections on the open $K$-orbit with the slowest growth toward the boundary. If $G$ is connected and semi-simple, the stabilizer groups $K_\alpha$ are finite abelian 2-groups [BGG, Lemma 2.1]. It follows that the $K$-types $V_K(\mu)$ which occur as sections of $\mathcal{L}(\alpha)$ on the open orbit are sections which satisfy a parity condition. If $G$ is only assumed to be reductive, there is an additional condition that $\mu$ and $\pm$ must be equal on the center.

**Theorem [BGG].** Let $K \cdot x$ be the open dense orbit in $G/B$. If $V_K(\mu)$ is fine, and

$$\mu \mid K_\alpha = \pm \mid K_\alpha,$$

then

$$V_K(\mu) \subset V_G(\pm),$$

with multiplicity one.

6.7. We can now state our result. Let $Q$ be an orbit with resolution $Z$ for its orbit closure as in 6.5 with associated split Levi subgroup $L$. Then $Z$ is chosen so that $\lambda_\pm$ is dominant, and $Z$ defines a positive system $A_\pm^+$ for $K$. Let $w$ be given as in (6.3.5).

**Theorem.** Let $\mathcal{L}(\lambda)$ be a line bundle corresponding to a dominant weight $\lambda$ so that $H^0(Z, \mathcal{L}(\lambda)) \cong V(\lambda)^\ast$. Let $\mu$ be the highest weight of a fine $L \cap K$-type occurring in the irreducible representation of $L$ with extremal weight $w\lambda$. Then if $\mu$ is dominant for $A_\pm^+$, then $V_K(\mu)^\ast$ occurs in $H^0(Z, \theta^\ast(\mathcal{L}(\lambda)))$ with multiplicity one.

**Proof.** We have a fiber bundle $Z \to \pi_s(Q_b)$ as in 6.5. Then

$$H^0(Z, \theta^\ast(\mathcal{L}(\lambda))) \cong H^0(K/K \cap P, H^0(P \times B Z_u, \theta^\ast(\mathcal{L}(\lambda))))$$

$$\times H^0(K/K \cap P, (U(b)e^{w_\lambda})^\ast),$$
where \( w_L \) is the long element of the Weyl group of \( L \) and \( e^{n_L w_L} \) is the extremal weight vector of weight \( w_L w \lambda \) in \( V(\lambda) \) (see [Ku1, 2.14] or [An]). By the Borel–Weil–Bott theorem (see [Ko]), for any algebraic \( K \cap P \)-module \( M \), the multiplicity of a \( K \)-type \( V_K(\mu) \) in \( H^b(K/K \cap P, M^*) \) is equal to
\[
\dim(\text{Hom}_{K \cap P}(M, V_K(\mu))).
\]

Hence it suffices to show
\[
\dim(\text{Hom}_{K \cap P}[U(b)e^{n_L w_L}, V_K(\mu)]) = 1.
\]

We are assuming that \( V_K(\mu) \) is a \( K \)-type such that \( V_{L \cap K}(\mu) \) occurs in the finite dimensional representation of \( L \) with extremal weight \( w_L \lambda \). Then the relation \( \mu|_I = \lambda_+ \) must hold.

Let \( p = l + n \) be the Levi decomposition of \( p \). Let \( \xi \) be such that
\[
(\xi, x) = \begin{cases} 
1 & \text{for } x \in A(n), \\
0 & \text{for } x \in A(l).
\end{cases}
\]

There is an exact sequence
\[
0 \to A \to B \to C \to 0,
\]

where \( A = nU(b)e^{n_L w_L}, B = U(b)e^{n_L w_L}, \) and \( C = A/B \). Then \( n \) acts by \( 0 \) on \( C \) and the weights of \( C \) all restrict to \( \lambda_+ \) on \( t \) since roots of \( l \) are \( 0 \) on \( t \). There is an exact sequence
\[
0 \to \text{Hom}_{p \cap t}[C, V_K(\mu)] \to \text{Hom}_{p \cap t}[B, V_K(\mu)] \to \text{Hom}_{p \cap t}[A, V_K(\mu)].
\]

We claim that \( \text{Hom}_{p \cap t}[A, V_K(\mu)] = 0 \). Indeed, any weight \( \eta \) of \( V_K(\mu) \) is of the form
\[
\eta = \lambda_+ - \text{sum of roots in } n \cap t \quad \text{and} \quad 1 \cap t.
\]

But the weights of \( A \) satisfy
\[
\eta = w_L w \lambda + \text{nonempty sum of roots in } A(n).
\]

Then if \( \eta \) occurs in both \( V_K(\mu) \) and \( A \), we would have \((\xi, \eta) > (\xi, \lambda_+)\) from (6.7.7) and \((\xi, \eta) \leq (\xi, \lambda_+)\) from (6.7.6). This is a contradiction.

Finally, \( \text{Hom}_{p \cap t}[C, V_K(\mu)] \) is isomorphic to
\[
\text{Hom}_{t \cap l}[U(l)e^{n_L w_L} : V_{L \cap K}(\mu)].
\]

Our assumptions imply that this multiplicity is one.
We will say $\lambda$ is very dominant if $H^0(G/B, \mathcal{L}(-\lambda))$ surjects onto $H^0(\bar{Q}, \mathcal{L}(-\lambda))$ and

$$(\lambda +, x) > -(\tau, x)$$

is positive for all roots $x \in \mathfrak{n} \cap \mathfrak{t}$ and all fine $L \cap K$-types $\tau$. There are only finitely many choices for $\tau$.

**Corollary.** If $\bar{Q}$ is normal and $\lambda$ is very dominant, then the $K$-type $V_K(\mu)$ occurs in $V_G(\lambda)$.

**Remarks.**

1. A more geometric proof of the above theorem can be given as follows. One chooses $Z$ as above. The Borel–Weil theorem implies that $V_K(\mu)^*$ occurs in $H^0(Q, \mathcal{L}(-\lambda))$. Moreover, $V_K(\mu)^*$ occurs as sections with minimal growth toward the boundary as in [Ch]. In fact, these sections correspond to special $K$-types in [Ch, Sect. 8]. Then by comparing these sections with minimal growth to an arbitrary nonzero section on $H^0(Z, \theta^*(\mathcal{L}(-\lambda)))$, one proves that the sections in $V_K(\mu)^*$ extend to all of $Z$. After a shift, these $K$-types correspond to the special $K$-types defined by Chang. In the complex case, Kostant and Kumar have a complementary interpretation of this result.

2. In terms of parameters our procedure is as follows. Divide the orbits $Q = K \cdot x$ into classes using the equivalence relation $K \cdot x \sim K \cdot y$ if $b_x$ and $b_y$ certain $K$-conjugate $\theta$-stable Cartan subalgebras. This implies that we can choose $b_x$ and $b_y$ so that they contain a common $\theta$-stable Cartan subalgebra. Then in an equivalence class of orbits, fix one orbit $Q_0$, a Borel subalgebra $b_0 \in Q_0$, and a corresponding $\theta$-stable Cartan subalgebra $\mathfrak{h}$. Fix a positive root system $\Delta_0^+$ compatible with $b_0$, as in 6.2 and define $\lambda$ so that $\lambda$ is dominant for $\Delta_0^+$. Choose a positive root system for $K$ compatible with $\Delta_0^+$. Then given an orbit $Q$, pick a Borel subalgebra $b_x$ containing $\mathfrak{h}$. Let $\mathfrak{h} = t + a$ be the Cartan decomposition for $\mathfrak{h}$ and let $b_x = t + a_c$ be a Cartan subalgebra for $K$. Then $\lambda|_{\mathfrak{h}}$ will mean that $\lambda$ is considered as a dominant weight for $b_x$, and then restricted to $\mathfrak{h}$. The same holds for the subalgebra $t$ of $\mathfrak{h}$. We will specify a $K$-type $\mu$ by

$$\mu = (\mu|_t, \mu|_{a_c})$$

Let $W(K)$ be the Weyl group of $K$. If $\phi$ is a weight of $h_c$, denote by $\tilde{\phi}$ the unique dominant $W(K)$-conjugate of $\phi$.

Let

$$\mu = (\nu|_t, \nu|_{a_c})$$

where $\tau$ is a fine $K \cap L$-type of $V_L(\tilde{\nu})$ such that $\mu$ is a dominant weight for $K$. Then $V_K(\mu)$ is the $K$-type identified by our theorem.
In the complex case the argument in 6.7 can be modified trivially to work for the parabolic subalgebra determined by $\lambda_+$. Then we obtain a set of $K$-types occurring, which is potentially larger than the PRV component.

**Conjecture.** If $\lambda$ is dominant and $V_{L \cap \mathcal{K}}(\mu)$ is a fine $L \cap K$-type for $V_L(w\lambda)$ such that $\mu$ is dominant (for the positive system of $K$ determined by $Q_+$), then $V_K(\mu)$ occurs in $V_G(\lambda)$.

6.8. For the case of $GL(n)$ and any involution $\theta$, the results of Sections 2 through 4 provide a wide class of examples of normal orbit closures for the full flag variety. Indeed, let $\pi: \mathcal{A} \to G(k, n-k)$ be the projection from the variety of flags to $G(k, n-k)$. If $\overline{Q}$ is a $K$-orbit closure in $G(k, n-k)$, then $S = \pi^{-1}(\overline{Q})$ is an orbit closure in $\mathcal{A}$. $S$ is normal since $\overline{Q}$ is normal.

**Remark.** The condition on dominance assures that

$$H^0(\mathcal{A}, \mathcal{L}(-\lambda)) \rightarrow H^0(\overline{Q}, \mathcal{L}(-\lambda)) \quad (6.8.1)$$

is surjective. It is likely that it can be weakened considerably. For example, in the case where the real group $G_0$ is complex, the work of Kumar and Matthieu shows that it is sufficient for $\lambda$ to be dominant. There is an example where $G_0 = Sp(2, \mathbb{R})$, where the above map is not surjective for certain singular but dominant $\lambda$ for an orbit of codimension one.

6.9. Moreover, there are examples of orbits which are not normal. We give such an example. Let $G = Sp(8)$. Label the simple roots

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4$$

such that $\alpha_4$ is long and has nonzero inner product with $\alpha_3$.

Consider the distinguished parabolic subgroup $P = P_3$ such that

1. $[L, L]$ is an $SL(2)$ with the real root $\alpha_3$, 
2. $\alpha_1$ is noncompact imaginary.

Consider the orbit $Q$ obtained by applying the roots in order

$$\alpha_2, \alpha_1, \alpha_4, \alpha_3, \alpha_2.$$

Then the resolution of $Q$ is given by

$$\psi: K \times_{K \cap P} P \times_B P_2 \times_B P_1 \times_B P_4 \times_B P_3 \times_B P_2/B \to \overline{Q}.$$
Let $Q'$ be the $K$-orbit whose closure is the image of

$$K \times_{K \cap P} P \times B P_2 \times_B P_4 \times_B P_3 \times P_2 / B$$

under $\Psi$, and $Q''$ be the orbit whose closure is the image of

$$K \times_{K \cap P} P \times B P_2 \times_B P_4 \times_B P_3 / B$$

under $\Psi$. Then $x_2$ is noncompact type II with respect to $Q''$. Also, $\overline{Q}'$ is the open orbit in $P_2 \cdot \overline{Q}'$, and the component group of the centralizer of an element in $Q'$ is order 2. Thus $\Psi$ contains at least two points. Since $\overline{Q}'$ is codimension one, $\psi^{-1}(x)$ cannot be infinite and hence is disconnected. Then Zariski's Main Theorem [Ha, p. 280], implies $Q$ cannot be normal.

6.10. Remark. Several examples we have computed indicate that the $K$-type $\mu$ given in 6.7.1 may occur even if sections on the flag variety do not surject onto sections on the orbit closure. It would be interesting to determine if $\mu$ always occurs in $V_c(\lambda)$. Vanishing theorems concerning the ideal sheaf on $Z$ analogous to [Ku2, 1.4] are false, even if the underlying real group is $U(2, 2)$. We have, however, shown that orbit closures are normal and the restriction map (6.7.5) is surjective in the following cases:

1. Any orbit when the underlying real group is $U(n, 1)$,
2. Any orbit where a Cartan subalgebra is maximally split when the underlying real group is $U(n, 2)$ or $U(n, 3)$,
3. Any orbit obtained by applying $P_a$ operations to a closed orbit where the roots of $K$ are the roots of a Levi factor.

REFERENCES


