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Integrability and asymptotics of positive solutions of a γ-Laplace system

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ABSTRACT

In this paper, we use the potential analysis to study the properties of the positive solutions of a γ-Laplace system in R^n

$$-div\big(|\nabla u|^{\gamma-2}\nabla u\big) = u^p v^q,$$
$$-div\big(|\nabla v|^{\gamma-2}\nabla v\big) = v^p u^q.$$

Here $1 < \gamma \leqslant 2$, $p, q > 0$ satisfy the critical condition $p + q = \gamma^* - 1$. First, the positive solutions u and v satisfy an integral system involving the Wolff potentials. We then use the method of regularity lifting to obtain an optimal integrability for this Wolff type integral system. Different from the case of $\gamma = 2$, it is more difficult to handle the asymptotics since u and v have not radial structures. We overcome this difficulty by a new method and obtain the decay rates of u and v as $|x| \to \infty$. We believe that this new method is appropriate to deal with the asymptotics of other decaying solutions without the radial structures.

1. Introduction

In [27], Li and Ma studied a stationary Schrödinger system with critical exponents for the Bose–Einstein condensate (cf. [2,29,30])

$$-\Delta u = u^p v^q, \quad \text{in } R^n,$$
$$-\Delta v = v^p u^q, \quad \text{in } R^n. \tag{1.1}$$

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Here $p, q > 0$ satisfy $p + q = \frac{n+2}{n-2}$. Clearly, this system is equivalent to an integral system involving the Newton potentials

$$u(x) = \int_{R^n} \frac{u^p(y)v^q(y)\,dy}{|x-y|^{n-2}},$$
$$v(x) = \int_{R^n} \frac{u^q(y)v^p(y)\,dy}{|x-y|^{n-2}}. \tag{1.2}$$

The authors used the method of moving planes of integral forms (cf. [8]) to obtain that the positive solutions are radially symmetric.

If we replace the Newton potentials by the Riesz potentials, (1.2) becomes

$$u(x) = \int_{R^n} \frac{u^p(y)v^q(y)\,dy}{|x-y|^{n-\alpha}},$$
$$v(x) = \int_{R^n} \frac{u^q(y)v^p(y)\,dy}{|x-y|^{n-\alpha}}. \tag{1.3}$$

Here $\alpha \in (0, n)$, $p, q > 0$ and $p + q = \frac{n+\alpha}{n-\alpha}$. By the ideas in [17], we can also obtain the integrability result:

(R1) If $u, v \in L^{\frac{2n}{n-\alpha}}(R^n)$, then $u, v \in L^s(R^n)$ for all $s > \frac{n}{n-\alpha}$.

Based on this integrability and the radial symmetry, one can estimate the decay rate (cf. [24]):

(R2) $u, v = O(|x|^{\alpha-n})$ when $|x| \to \infty$.

Naturally, we conjecture that the positive solutions of a γ-Laplace system

$$-div\big(|\nabla u|^{\gamma-2}\nabla u\big) = u^p v^q, \quad \text{in } R^n,$$
$$-div\big(|\nabla v|^{\gamma-2}\nabla v\big) = v^p u^q, \quad \text{in } R^n, \tag{1.4}$$

should have the analogous integrability and asymptotics. Here $1 < \gamma \leqslant 2$, $n > 2$, $p, q > 0$ and $p+q \geqslant 1$.

When $u = v$ in (1.4), the system reduces to a single equation

$$-div\big(|\nabla u|^{\gamma-2}\nabla u\big) = u^{p+q}, \quad \text{in } R^n. \tag{1.5}$$

Let $p + q$ be equal to the critical exponent $\frac{n\gamma}{n-\gamma} - 1$. According to the existence results of Serrin and Zou, (1.5) has bounded classical positive solutions (cf. [36]). For a radial solution of (1.5), [14] obtained the estimates of the decay rates when $|x| \to \infty$.

However, different from the result of (1.1), not all solutions of (1.4) have the radial structure. Let $\Omega \subset R^n$ be a radially symmetric and bounded domain. The paper [1] shows that the positive solutions of (1.4) in Ω are radially symmetric by using the ideas of Damascelli, Pacella and Ramaswamy (cf. [12] and [13]). When $\Omega = R^n$, it is unknown whether there exists a non-radial solution or not. In [3], Byeon, Jeanjean and Maris proved that a class of important solutions, the least energy solutions, are radially symmetric and monotony decreasing about some point $x_0 \in R^n$. Besides the radial symmetry, there is another distinction between the two systems (1.1) and (1.4). Eq. (1.1) has an equivalent integral system (1.2), and (1.4) has not.

Fortunately, the positive solutions of (1.4) satisfy another integral system involving the Wolff potentials. The Wolff potential of a positive function $f \in L^1_{loc}(R^n)$ is defined as (cf. [16])

$$W_{\beta,\gamma}(f)(x) = \int_0^\infty \left[\frac{\int_{B_t(x)} f(y)\,dy}{t^{n-\beta\gamma}} \right]^{\frac{1}{\gamma-1}} \frac{dt}{t},$$

where $1 < \gamma < \infty$, $\beta > 0$, $\beta\gamma < n$, and $B_t(x)$ is a ball of radius t centered at x. It is not difficult to see that $W_{1,2}(f)$ is Newton's potential, and $W_{\frac{\alpha}{2},2}(f)$ is Reisz's potential.

The Wolff potentials are helpful to well understand the nonlinear PDEs (cf. [18–20,22,34]), and other nonlinear problems (cf. [10] and [33]). For example, $W_{1,\gamma}(\omega)$ can be used to estimate the solutions u of γ-Laplace equation

$$-div\big(|\nabla u|^{\gamma-2}\nabla u\big) = \omega.$$

If $\inf_{R^n} u = 0$, then there exist positive constants C_1 and C_2 such that (cf. [19])

$$C_1 W_{1,\gamma}(\omega)(x) \leqslant u(x) \leqslant C_2 W_{1,\gamma}(\omega)(x), \quad x \in R^n. \tag{1.6}$$

Now, we introduce the integral system involving the Wolff potentials

$$\begin{aligned} u(x) &= R_1(x) W_{\beta,\gamma}\big(u^p v^q\big)(x), \\ v(x) &= R_2(x) W_{\beta,\gamma}\big(v^p u^q\big)(x). \end{aligned} \tag{1.7}$$

Here $x \in R^n$ and there exist c and C such that

$$0 < c \leqslant R_1(x), R_2(x) \leqslant C. \tag{1.8}$$

The corresponding critical case is

$$p + q = \gamma^* - 1 := \frac{n\gamma}{n - \beta\gamma} - 1. \tag{1.9}$$

To investigate the integrability and the regularity of positive solutions of (1.7), we consider $w = u + v$. It is not difficult to see that w satisfies

$$w(x) = R(x) W_{\beta,\gamma}\big(w^{\gamma^*-1}\big)(x) \quad \text{in } R^n, \tag{1.10}$$

where $0 < R(x) \leqslant C$ (see Proposition 2.1). We use a regularity lifting lemma (Lemma 2.1 in [17]) to obtain an integrability interval of positive solutions. Based on this, we can extend the interval to an optimal one by means of the Hardy–Littlewood–Sobolev inequality and the Wolff type inequality (cf. [31] and [32]).

Recently, Li, Chen and Ma consider a pair of positive solutions $(u, v) \in L^{q_1+\gamma-1}(R^n) \times L^{q_2+\gamma-1}(R^n)$ of another Wolff type system (cf. [7] and [32])

$$\begin{aligned} u(x) &= W_{\beta,\gamma}\big(v^{q_2}\big)(x), \\ v(x) &= W_{\beta,\gamma}\big(u^{q_1}\big)(x). \end{aligned} \tag{1.11}$$

The critical condition is

$$\frac{\gamma-1}{q_1+\gamma-1}+\frac{\gamma-1}{q_2+\gamma-1}=1-\frac{\beta\gamma}{n}. \tag{1.12}$$

The system (1.11) is a generalization of Hardy–Littlewood–Sobolev type integral system. Namely, when $\gamma=2$ and $\beta=\alpha/2$ in (1.11), then

$$\begin{aligned} u(x) &= \int_{R^n} \frac{v(y)^{q_2}}{|x-y|^{n-\alpha}}\,dy, \\ v(x) &= \int_{R^n} \frac{u(y)^{q_1}}{|x-y|^{n-\alpha}}\,dy. \end{aligned} \tag{1.13}$$

The pair of solutions $(u,v)=(\lambda_1 f^{r-1},\lambda_2 g^{s-1})$ is a critical point of the functional

$$J(f,g)=\int_{R^n}\int_{R^n}\frac{f(x)g(y)\,dx\,dy}{|x-y|^{n-\alpha}}$$

with the constraint $\|f\|_r=\|g\|_s=1$, where $r=\frac{q_1+1}{q_1}$ and $s=\frac{q_2+1}{q_2}$. One can maximize this functional to find the best constant $C(n,s,\alpha)$ in the following Hardy–Littlewood–Sobolev inequality (cf. [37])

$$\int_{R^n}\int_{R^n}\frac{f(x)g(y)\,dx\,dy}{|x-y|^{n-\alpha}}\leqslant C(n,s,\alpha)\|f\|_r\|g\|_s,$$

where $s,r>1$, $\frac{1}{r}+\frac{1}{s}=\frac{n+\alpha}{n}$, $f\in L^r(R^n)$, $g\in L^s(R^n)$ (cf. [9] and [28]). The classification of positive solutions of (1.13) and its corresponding PDEs were studied rather thoroughly (cf. [4–6,8,11,15,25,35] and the references therein).

The integrability for positive solutions of the integral system (1.13) was well studied by Jin and Li (cf. [17]). Following those ideas, Chen, Li and Ma obtained the integrability intervals and the regularity results for the Wolff type system (1.11) (cf. [32]). Afterwards, the asymptotic behavior of positive solutions is estimated (cf. [21]).

It should be pointed out that the system (1.11) with the critical condition (1.12) is not appropriate to study γ-Laplace equations without ratio coefficients. For simplicity, we consider (1.11) with $q_1=q_2$ and $u\equiv v$, i.e.

$$u=W_{\beta,\gamma}\big(u^{q_1}\big), \tag{1.14}$$

then (1.12) implies

$$q_1=\frac{n+\beta\gamma}{n-\beta\gamma}(\gamma-1). \tag{1.15}$$

Clearly, when $\gamma\neq 2$, this is different from the critical condition (1.9).

For the γ-Laplace equation (1.5), the critical condition (1.9) makes sense and (1.12) does not. In fact, Corollary II in [36] shows that

$$-div\big(|\nabla u|^{\gamma-2}\nabla u\big)=u^{q_1} \tag{1.16}$$

has positive solutions if and only if

$$q_1 \geqslant \frac{n\gamma}{n-\gamma} - 1.$$

Thus, (1.9) with $\beta = 1$ is the critical condition. On the contrary, $q_1 = q_2$ in (1.12) with $\beta = 1$ implies

$$q_1 = \frac{n+\gamma}{n-\gamma}(\gamma - 1) < \gamma^* - 1.$$

Now, (1.16) has no positive solution.

In the following, we state the main results in this paper. For the solution of (1.7), we have

Theorem 1.1. *Assume $u, v \in L^{\gamma^*}(R^n)$ solve (1.7) with (1.8) and (1.9). Then,*

i) *u and v belong to $L^s(R^n)$ for any $s \in (\frac{n(\gamma-1)}{n-\beta\gamma}, \infty)$. The left end point $\frac{n(\gamma-1)}{n-\beta\gamma}$ is optimal. In addition, u and v are bounded in R^n.*
ii) *$u(x)$ and $v(x)$ are equal to $O(|x|^{\frac{\beta\gamma-n}{\gamma-1}})$ when $|x| \to \infty$.*

Remark 1.1. (i) When $\beta = \alpha/2$, $\gamma = 2$, i) and ii) of Theorem 1.1 are the same as (R1) and (R2).

(ii) For the system (1.14) with another critical condition (1.15), the results of the integrability and asymptotics in [32] and [21] are the same as i) and ii) of Theorem 1.1 (cf. §5).

Remark 1.2. Papers [21,23,24,26] studied the asymptotics of the Wolff type and the weighted Hardy–Littlewood–Sobolev type systems. Papers [21] and [24] used the radial symmetry of the positive solutions to estimate the decay rates. Papers [23] and [26] investigated the singularity of the positive solutions near the origin for the weighted Hardy–Littlewood–Sobolev type systems, and then obtained the decay rates when $|x| \to \infty$ by means of the Kelvin transform. However, (1.7) has neither radial structures nor the invariability under the Kelvin transform. Therefore, the techniques in those papers cannot be used in this paper. We have to find a new method to obtain the asymptotic estimates.

For the solution of (1.4), we have

Corollary 1.2. *Let $u, v \in L^{\gamma^*}(R^n)$ be a pair of positive solutions of (1.4) and (1.9) with $\beta = 1$. Then*

i) *u and v belong to $L^s(R^n)$ for any $s \in (\frac{n(\gamma-1)}{n-\gamma}, \infty)$. The left end point $\frac{n(\gamma-1)}{n-\gamma}$ is optimal. In addition, u and v are bounded in R^n.*
ii) *$u(x)$ and $v(x)$ are equal to $O(|x|^{\frac{\gamma-n}{\gamma-1}})$ when $|x| \to \infty$.*

Remark 1.3. Result ii) of Corollary 1.2 shows that the decay rate of those positive solutions is the same as the fast decay rate in [14] even if the solutions have not radial structures. Another slow decay rate $O(|x|^{\frac{\gamma}{\gamma-\gamma^*}})$ in [14] is the asymptotic behavior of the singular solutions instead of the regular solutions, since such a rate shows that the solutions do not belong to $L^{\gamma^*}(R^n)$.

2. Integrability interval

In this section, we prove conclusion i) in Theorem 1.1.

Proposition 2.1. *Assume $u, v \in L^{\gamma^*}(R^n)$ solve (1.7) with (1.8) and (1.9). Then we can find a positive bounded function $R(x)$ such that $w = u + v$ satisfies (1.10).*

Proof. First there exists a constant $C > 0$ such that

$$u^p v^q \leqslant C w^{\gamma^*-1}, \qquad v^p u^q \leqslant C w^{\gamma^*-1}.$$

Hence,

$$w(x) \leqslant C W_{\beta,\gamma}\big(w^{\gamma^*-1}\big)(x).$$

Thus, $R(x) := \frac{w(x)}{W_{\beta,\gamma}(w^{\gamma^*-1})(x)}$ satisfies

$$0 < R(x) \leqslant C, \quad \text{for } x \in R^n. \tag{2.1}$$

Then w satisfies (1.10). Proposition 2.1 is proved. □

Theorem 2.2. *Assume $w \in L^{\gamma^*}(R^n)$ solves* (1.10) *with* (1.9) *and* (2.1)*. Then*

$$w \in L^s\big(R^n\big), \quad \forall \frac{1}{s} \in \left(0, \frac{n-\beta\gamma}{n(\gamma-1)}\right). \tag{2.2}$$

Proof. *Step 1.* For $A > 0$, set

$$\begin{aligned} &w_A(x) = w(x), \quad \text{if } w(x) > A \text{ or } |x| > A;\\ &w_A(x) = 0, \qquad \text{otherwise,} \end{aligned}$$

and $w_B(x) = w(x) - w_A(x)$. Let σ satisfy

$$\frac{2-\gamma}{\gamma^*} < \frac{1}{\sigma} < \frac{2-\gamma}{\gamma^*} + \frac{n-\beta\gamma}{n}. \tag{2.3}$$

For $g \in L^\sigma(R^n)$, define operators T and S,

$$Tg := R(x) \int_0^\infty \left(\frac{\int_{B_t(x)} w^{\gamma^*-1}(y)\,dy}{t^{n-\beta\gamma}} \right)^{\frac{2-\gamma}{\gamma-1}} \frac{\int_{B_t(x)} w_A^{\gamma^*-2}(y) g(y)\,dy}{t^{n-\beta\gamma}} \frac{dt}{t},$$

$$Sg := \int_0^\infty \left(\frac{\int_{B_t(x)} w_A^{\gamma^*-2}(y) g(y)\,dy}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t}$$

and write

$$F := R(x) \int_0^\infty \left(\frac{\int_{B_t(x)} w^{\gamma^*-1}(y)\,dy}{t^{n-\beta\gamma}} \right)^{\frac{2-\gamma}{\gamma-1}} \frac{\int_{B_t(x)} w_B^{\gamma^*-1}(y)\,dy}{t^{n-\beta\gamma}} \frac{dt}{t}.$$

Clearly, w is a solution of the following equation

$$g = Tg + F.$$

Step 2. T is a contraction map from $L^\sigma(r^n)$ into itself.
In fact, by the Hölder inequality and (2.1), there holds

$$|Tg| \leqslant C w^{2-\gamma} |Sg|^{\gamma-1}.$$

Using the Hölder inequality, we get

$$\|Tg\|_\sigma \leqslant C \|w\|_{\gamma^*}^{2-\gamma} \|Sg\|_t^{\gamma-1} \tag{2.4}$$

where $t > 0$ satisfies

$$\frac{1}{\sigma} = \frac{2-\gamma}{\gamma^*} + \frac{\gamma-1}{t}. \tag{2.5}$$

By (2.3) and (2.5),

$$0 < \frac{\gamma-1}{t} < 1 - \frac{\beta\gamma}{n}. \tag{2.6}$$

Therefore, we can use the Hardy–Littlewood–Sobolev inequality and the Wolff type inequality to obtain

$$\|Sg\|_t \leqslant C \left\| w_A^{\gamma^*-2} g \right\|_{\frac{nt}{n(\gamma-1)+t\beta\gamma}}^{\frac{1}{\gamma-1}}. \tag{2.7}$$

Since (2.5) leads to

$$\frac{\gamma-1}{t} - \frac{1}{\sigma} = \frac{\gamma^*-2}{\gamma^*} - \frac{\beta\gamma}{n},$$

it follows from (2.7) and the Hölder inequality that

$$\|Sg\|_t^{\gamma-1} \leqslant C \|w_A\|_{\gamma^*}^{\gamma^*-2} \|g\|_\sigma.$$

Inserting this into (2.4) yields

$$\|Tg\|_\sigma \leqslant C \|w\|_{\gamma^*}^{2-\gamma} \|w_A\|_{\gamma^*}^{\gamma^*-2} \|g\|_\sigma. \tag{2.8}$$

By virtue of $w \in L^{\gamma^*}(R^n)$,

$$C \|w\|_{\gamma^*}^{2-\gamma} \|w_A\|_{\gamma^*}^{\gamma^*-2} \leqslant \frac{1}{2}$$

when A is sufficiently large. Then T is a shrinking operator. Noticing that T is linear, we know that T is a contraction map from $L^\sigma(R^n)$ to itself as long as σ satisfies (2.3).

Step 3. Estimating F to lift the regularity.
Similarly to (2.4) and (2.7), for all σ satisfying (2.3), there holds

$$\|F\|_\sigma \leqslant C \|w\|_{\gamma^*}^{2-\gamma} \left\| w_B^{\gamma^*-1} \right\|_{\frac{nt}{n(\gamma-1)+t\beta\gamma}}$$

where t satisfies (2.6). Noting $w \in L^{\gamma^*}(R^n)$ and the definition of w_B, we see that

$$F \in L^{\sigma}\left(R^n\right)$$

as long as σ satisfies (2.3). Taking $X = L^{\gamma^*}(R^n)$, $Y = L^{\sigma}(R^n)$ and $Z = L^{\gamma^*}(R^n) \cap L^{\sigma}(R^n)$ in Lemma 2.1 of [17], we have

$$w \in L^{\sigma}\left(R^n\right),$$

for all σ satisfying (2.3).

Step 4. Extend the interval from (2.3).

Let

$$\frac{1}{s} \in \left(0, \frac{n-\beta\gamma}{n(\gamma-1)}\right). \tag{2.9}$$

Thus, we can use the Hardy–Littlewood–Sobolev inequality and the Wolff type inequality to deduce that

$$\|w\|_s \leqslant C \left\| w^{\gamma^*-1} \right\|^{\frac{1}{\gamma-1}}_{\frac{ns}{n(\gamma-1)+s\beta\gamma}} \leqslant C \|w\|^{\frac{\gamma^*-1}{\gamma-1}}_{\frac{ns(\gamma^*-1)}{n(\gamma-1)+s\beta\gamma}}. \tag{2.10}$$

Noting (2.3), from (2.10) we see that

$$\|w\|_s < \infty$$

as long as s satisfies

$$\frac{2-\gamma}{\gamma^*} < \frac{n(\gamma-1)+s\beta\gamma}{ns(\gamma^*-1)} < \frac{2-\gamma}{\gamma^*} + \frac{n-\beta\gamma}{n}. \tag{2.11}$$

Next, we will prove that (2.11) is true as long as (2.9) holds.

In fact, $\frac{1}{\gamma^*} = \frac{n-\beta\gamma}{n\gamma}$ leads to $\frac{2-\gamma}{\gamma^*} = \frac{n-\beta\gamma}{n}(\frac{2}{\gamma} - 1)$. Then (2.11) is equivalent to

$$\frac{n-\beta\gamma}{n}\left(\frac{2}{\gamma} - 1\right) < \frac{\gamma-1}{s(\gamma^*-1)} + \frac{\beta\gamma}{n(\gamma^*-1)} < \frac{2(n-\beta\gamma)}{n\gamma}$$

or

$$\frac{\gamma^*-1}{\gamma-1}\left[\frac{n-\beta\gamma}{n}\left(\frac{2}{\gamma} - 1\right) - \frac{\beta\gamma}{n(\gamma^*-1)}\right] < \frac{1}{s} < \frac{\gamma^*-1}{\gamma-1}\left[\frac{2(n-\beta\gamma)}{n\gamma} - \frac{\beta\gamma}{n(\gamma^*-1)}\right].$$

Thus, we need to verify two conclusions:

$$\frac{n-\beta\gamma}{n}\left(\frac{2}{\gamma} - 1\right) - \frac{\beta\gamma}{n(\gamma^*-1)} \leqslant 0 \tag{2.12}$$

and

$$\frac{\gamma^*-1}{\gamma-1}\left[\frac{2(n-\beta\gamma)}{n\gamma} - \frac{\beta\gamma}{n(\gamma^*-1)}\right] \geqslant \frac{n-\beta\gamma}{n(\gamma-1)}. \tag{2.13}$$

Verify (2.12). By $p+q \geqslant 1$, we have $\gamma \geqslant \frac{2n}{n+2\beta}$, which implies $(n+2\beta)\gamma^2 - (3n+2\beta)\gamma + 2n \geqslant 0$. So $(\gamma^* - 1)(n-\beta\gamma)(2-\gamma) \leqslant \beta\gamma^2$, and hence

$$\frac{(n-\beta\gamma)(2-\gamma)}{n\gamma} - \frac{\beta\gamma}{n(\gamma^*-1)} \leqslant 0.$$

Verify (2.13). By $\gamma \geqslant \frac{2n}{n+2\beta}$, we have $2n\gamma - 2n + 2\beta\gamma - \beta\gamma^2 \geqslant n\gamma - \beta\gamma^2$. This means

$$\frac{2n\gamma - 2n + 2\beta\gamma - \beta\gamma^2}{n\gamma(\gamma-1)} \geqslant \frac{n-\beta\gamma}{n(\gamma-1)},$$

which implies (2.13).

Eqs. (2.12) and (2.13) show that the integrability interval is (2.9). Theorem 2.2 is proved. □

Theorem 2.3. *Assume $u, v \in L^{\gamma^*}(R^n)$ solve* (1.7) *with* (1.8) *and* (1.9). *Then*

$$u \in L^s\big(R^n\big), \qquad v \in L^s\big(R^n\big), \quad \forall \frac{1}{s} \in \left(0, \frac{n-\beta\gamma}{n(\gamma-1)}\right). \tag{2.14}$$

In addition, the right end point $\frac{n-\beta\gamma}{n(\gamma-1)}$ of the integrability interval is optimal.

Proof. As a direct corollary of Proposition 2.1 and Theorem 2.2, (2.14) is easy to be obtained. Next, we prove that $\frac{n-\beta\gamma}{n(\gamma-1)}$ is optimal. In fact, for sufficiently large $|x|$, from (1.8) we deduce that

$$\begin{aligned} u(x) &\geqslant c \int_{2|x|}^{4|x|} \left(\frac{\int_{B_1(0)} u^p(y) v^q(y)\,dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ &\geqslant c \int_{2|x|}^{4|x|} \left(\frac{1}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \geqslant c|x|^{\frac{\beta\gamma-n}{\gamma-1}}. \end{aligned} \tag{2.15}$$

If $\frac{1}{s} \geqslant \frac{n-\beta\gamma}{n(\gamma-1)}$, then for some large constant $d > 0$,

$$\|u\|^s_{L^s(R^n \setminus B_d(0))} \geqslant c \int_d^\infty r^{n - s\frac{n-\beta\gamma}{\gamma-1}} \frac{dr}{r} = \infty.$$

For v, we have the same conclusion. Theorem 2.3 is proved. □

Theorem 2.4. *Assume $u, v \in L^{\gamma^*}(R^n)$ solve* (1.7) *with* (1.8) *and* (1.9). *Then u and v are bounded in R^n.*

Proof. In view of (1.8),

$$\begin{aligned} u(x) &\leqslant C\left(\int_0^1 \left[\frac{\int_{B_t(x)} u^p(y) v^q(y)\,dy}{t^{n-\beta\gamma}}\right]^{\frac{1}{\gamma-1}} \frac{dt}{t} + \int_1^\infty \left[\frac{\int_{B_t(x)} u^p(y) v^q(y)\,dy}{t^{n-\beta\gamma}}\right]^{\frac{1}{\gamma-1}} \frac{dt}{t}\right) \\ &:= C(H_1 + H_2). \end{aligned}$$

By Hölder's inequality, for any $l > 1$,

$$\int_{B_t(x)} u^p(y)v^q(y)\,dy \leqslant C\left\|w^{\gamma^*-1}\right\|_l \left|B_t(x)\right|^{1-1/l}.$$

Take l sufficiently large such that $\frac{1}{(\gamma^*-1)l} = \varepsilon \in (0, \min\{\frac{\beta\gamma}{n(\gamma^*-1)}, \frac{n-\beta\gamma}{n(\gamma-1)}\})$ sufficiently small. According to Theorem 2.2, $\|w^{\gamma^*-1}\|_l < \infty$. Therefore,

$$H_1 \leqslant C\int_0^1 \left(\frac{|B_t(x)|^{1-(\gamma^*-1)\varepsilon}}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \leqslant C\int_0^1 t^{\frac{\beta\gamma-n(\gamma^*-1)\varepsilon}{\gamma-1}} \frac{dt}{t} \leqslant C.$$

If $z \in B_\delta(x)$, then $B_t(x) \subset B_{t+\delta}(z)$. For $\delta \in (0,1)$ and $z \in B_\delta(x)$, by (1.8),

$$\begin{aligned} H_2 &= \int_1^\infty \left[\frac{\int_{B_t(x)} u^p(y)v^q(y)\,dy}{t^{n-\beta\gamma}}\right]^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ &\leqslant \int_1^\infty \left(\frac{\int_{B_{t+\delta}(z)} u^p(y)v^q(y)\,dy}{(t+\delta)^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \left(\frac{t+\delta}{t}\right)^{\frac{n-\beta\gamma}{\gamma-1}+1} \frac{d(t+\delta)}{t+\delta} \\ &\leqslant (1+\delta)^{\frac{n-\beta\gamma}{\gamma-1}+1} \int_{1+\delta}^\infty \left(\frac{\int_{B_t(z)} u^p(y)v^q(y)\,dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \leqslant Cu(z). \end{aligned} \tag{2.16}$$

Combining the estimates of H_1 and H_2, we have

$$u(x) \leqslant C + Cu(z), \quad \text{for } z \in B_\delta(x),$$

where $\delta \in (0,1)$. Integrating on $B_\delta(x)$, we get

$$\begin{aligned} \left|B_\delta(x)\right|u(x) &\leqslant C + C\int_{B_\delta(x)} u(z)\,dz \\ &\leqslant C + C\|u\|_{\gamma^*}\left|B_\delta(x)\right|^{1-\frac{1}{\gamma^*}} \leqslant C. \end{aligned}$$

This shows u is bounded in R^n. Similarly, v is also bounded. Theorem 2.4 is proved. □

3. Decay rates

Consider a special case at first. Let $R(x) \equiv 1$ in (1.10). Then

$$w(x) = W_{\beta,\gamma}\left(w^{\gamma^*-1}\right)(x), \quad \text{in } R^n. \tag{3.1}$$

Theorem 1.3 in [31] shows that w is radially symmetric and decreasing about some $x_0 \in R^n$. Then we can write

$$w(x) = \omega(r), \quad \text{where } r = |x - x_0|.$$

Furthermore, we have

Proposition 3.1. *Assume $w \in L^{\gamma^*}(R^n)$ solves* (3.1) *with* (1.9)*. Then $w(x) = O(|x|^{\frac{\beta\gamma-n}{\gamma-1}})$ as $|x| \to \infty$.*

Proof. *Step 1.* For any $\frac{1}{s} \in (0, \frac{n-\beta\gamma}{n(\gamma-1)})$, there exists $C > 0$, such that for any $R > 0$,

$$\omega(R) \leqslant CR^{-n/s}. \tag{3.2}$$

In fact, according to Theorem 2.2, we know $w \in L^s(R^n)$ when $\frac{1}{s}$ is in the integrability interval of (2.2). Thus, we can denote $\int_{R^n} w^s(y)\,dy$ by a constant C_s. In view of the monotonicity of $\omega(r)$, we deduce that

$$\begin{aligned}\left|S^{n-1}\right|\omega^s(R)\left(\frac{R}{2}\right)^n \ln 2 &\leqslant \left|S^{n-1}\right| \int_{R/2}^{R} \omega^s(r) r^n \frac{dr}{r} \\ &= \int_{B_R(x_0)\setminus B_{R/2}(x_0)} w^s(y)\,dy \leqslant C_s.\end{aligned}$$

Thus, (3.2) is verified.

Step 2. There exists $c > 0$ such that for large $|x|$, $w(x) \geqslant c|x|^{-\frac{n-\beta\gamma}{\gamma-1}}$.

In fact, if $|y - x_0| < 1$, then for large $|x|$ and $t \in (2|x|, 4|x|)$, we have $|y - x| \leqslant |y - x_0| + |x - x_0| < 1 + |x_0| + |x| < t$. This means $B_1(x_0) \subset B_t(x)$. Therefore, by the monotonicity of $\omega(r)$,

$$\int_{B_t(x)} w^{\gamma^*-1}(y)\,dy \geqslant \int_{B_1(x_0)} w^{\gamma^*-1}(y)\,dy \geqslant \omega^{\gamma^*-1}(1)\left|B_1(x_0)\right|,$$

when $t \in (2|x|, 4|x|)$. Thus,

$$w(x) \geqslant \int_{2|x|}^{4|x|} \left[\frac{\int_{B_t(x)} w^{\gamma^*-1}(y)\,dy}{t^{n-\beta\gamma}}\right]^{\frac{1}{\gamma-1}} \frac{dt}{t} \geqslant c \int_{2|x|}^{4|x|} \frac{1}{t^{\frac{n-\beta\gamma}{\gamma-1}}} \frac{dt}{t} \geqslant \frac{c}{|x|^{\frac{n-\beta\gamma}{\gamma-1}}}.$$

Step 3. Next, we estimate the upper bound of $w(x)$ for large $|x|$.

Clearly, we have

$$\begin{aligned} w(x) &\leqslant C\left(\int_0^{\frac{|x|}{2}} \left[\frac{\int_{B_t(x)} w^{\gamma^*-1}(y)\,dy}{t^{n-\beta\gamma}}\right]^{\frac{1}{\gamma-1}} \frac{dt}{t} + \int_{\frac{|x|}{2}}^{\infty} \left[\frac{\int_{B_t(x)} w^{\gamma^*-1}(y)\,dy}{t^{n-\beta\gamma}}\right]^{\frac{1}{\gamma-1}} \frac{dt}{t}\right) \\ &:= C(I_1 + I_2).\end{aligned}$$

Claim 1. *There exists $C > 0$ such that for large $|x|$, $I_1 \leqslant C|x|^{-\frac{n-\beta\gamma}{\gamma-1}}$.*

In fact, $\frac{|x|}{2} \leqslant |y| \leqslant \frac{3|x|}{2}$ when $y \in B_t(x) \subset B_{\frac{|x|}{2}}(x)$. By virtue of the monotonicity of $\omega(r)$, for large $|x|$ there holds $w(y) = \omega(|y - x_0|) \leqslant C\omega(\frac{|x|}{3})$. Therefore, by (3.2) we can deduce that

$$|x|^{\frac{n-\beta\gamma}{\gamma-1}} I_1 \leqslant C|x|^{\frac{n-\beta\gamma}{\gamma-1}} \int_0^{\frac{|x|}{2}} \left[\frac{\int_0^t \omega^{\gamma^*-1}(|x|/3)r^{n-1}dr}{t^{n-\beta\gamma}}\right]^{\frac{1}{\gamma-1}} \frac{dt}{t}$$

$$\leqslant C|x|^{\frac{n-\beta\gamma}{\gamma-1}-\frac{n(\gamma^*-1)}{s(\gamma-1)}} \int_0^{\frac{|x|}{2}} t^{\frac{\beta\gamma}{\gamma-1}} \frac{dt}{t} \leqslant C|x|^{\frac{n}{\gamma-1}-\frac{n(\gamma^*-1)}{s(\gamma-1)}}. \tag{3.3}$$

We choose $\frac{1}{s}$ approaching the right end points of the intervals of (2.2) such that

$$1-\frac{\gamma^*-1}{s}<0. \tag{3.4}$$

Inserting it into (3.3), we can prove the claim.

Claim 2. *There exists $C>0$ such that for large $|x|$, $I_2 \leqslant C|x|^{-\frac{n-\beta\gamma}{\gamma-1}}$.*

In fact, by $\frac{1}{\gamma-1} \geqslant 1$ and the Jensen inequality, we get

$$I_2 \leqslant C\left(\int_{\frac{|x|}{2}}^{\infty}\left[\frac{\int_{B_{2|x_0|}(0)\cap B_t(x)} w^{\gamma^*-1}(y)\,dy}{t^{n-\beta\gamma}}\right]^{\frac{1}{\gamma-1}}\frac{dt}{t}+\int_{\frac{|x|}{2}}^{\infty}\left[\frac{\int_{B_t(x)\setminus B_{2|x_0|}(0)} w^{\gamma^*-1}(y)\,dy}{t^{n-\beta\gamma}}\right]^{\frac{1}{\gamma-1}}\frac{dt}{t}\right)$$

$$:= C(I_{21}+I_{22}).$$

By virtue of $\int_{B_{2|x_0|}(0)} w^{\gamma^*-1}(y)\,dy \leqslant C$, we have

$$|x|^{\frac{n-\beta\gamma}{\gamma-1}} I_{21} \leqslant C|x|^{\frac{n-\beta\gamma}{\gamma-1}} \int_{\frac{|x|}{2}}^{\infty}\left[\frac{1}{t^{n-\beta\gamma}}\right]^{\frac{1}{\gamma-1}}\frac{dt}{t} \leqslant C.$$

On the other hand, when $|y| \geqslant 2|x_0|$, there holds $|y-x_0| \geqslant |y|-|x_0| \geqslant \frac{|y|}{2}$. Applying the monotonicity of $\omega(r)$ and (3.2), we get

$$w^{\gamma^*-1}(y)=\omega^{\gamma^*-1}\big(|y-x_0|\big) \leqslant \omega^{\gamma^*-1}\left(\frac{|y|}{2}\right) \leqslant C\left(\frac{|y|}{2}\right)^{-n(\gamma^*-1)/s}.$$

Thus

$$\int_{B_t(x)\setminus B_{2|x_0|}(0)} w^{\gamma^*-1}(y)\,dy \leqslant C\int_{2|x_0|}^{|x|+t} r^{n-\frac{n(\gamma^*-1)}{s}}\frac{dr}{r}.$$

Eq. (3.4) implies $n-\frac{n(\gamma^*-1)}{s}<0$, so

$$\int_{B_t(x)\setminus B_{2|x_0|}(0)} w^{\gamma^*-1}(y)\,dy \leqslant C.$$

Inserting this into I_{22} yields

$$|x|^{\frac{n-\beta\gamma}{\gamma-1}} I_{22} \leqslant C|x|^{\frac{n-\beta\gamma}{\gamma-1}} \int_{\frac{|x|}{2}}^{\infty} t^{-\frac{n-\beta\gamma}{\gamma-1}} \frac{dt}{t} \leqslant C.$$

Claim 2 is complete and Proposition 3.1 is proved. □

Next, we prove conclusion ii) in Theorem 1.1. Now, $R(x) \not\equiv 1$ and hence the radial symmetry is lost. The estimate (3.2) seems difficult to be deduced. We have to find another way to estimate the decay rate.

Theorem 3.2. *Assume $u, v \in L^{\gamma^*}(R^n)$ solve (1.7) with (1.8) and (1.9). Then, $u(x), v(x) = O(|x|^{\frac{\beta\gamma-n}{\gamma-1}})$ when $|x| \to \infty$.*

Proof. *Step 1.* There exists a positive constant c, such that for large $|x|$,

$$u(x), v(x) \geqslant c|x|^{-\frac{n-\beta\gamma}{\gamma-1}}. \tag{3.5}$$

In fact, similarly to the derivation of (2.15), from $\int_{B_1(0)} u^p(y)v^q(y)\,dy \geqslant c > 0$ and (1.8), it follows that

$$u(x) \geqslant c \int_{|x|+1}^{\infty} \left[\frac{\int_{B_1(0)} u^p(y)v^q(y)\,dy}{t^{n-\beta\gamma}}\right]^{\frac{1}{\gamma-1}} \frac{dt}{t}$$
$$\geqslant c \int_{|x|+1}^{\infty} t^{-\frac{n-\beta\gamma}{\gamma-1}} \frac{dt}{t} \geqslant c|x|^{-\frac{n-\beta\gamma}{\gamma-1}}.$$

Similarly, v has the same lower estimate.

Write $w = u + v$. In the following, we prove that for large $|x|$,

$$w(x) \leqslant C|x|^{-\frac{n-\beta\gamma}{\gamma-1}}. \tag{3.6}$$

Step 2. We claim that u, v are decaying. Namely,

$$\lim_{|x|\to\infty} u(x) = 0, \qquad \lim_{|x|\to\infty} v(x) = 0. \tag{3.7}$$

Take $x_0 \in R^n$. By Theorem 2.4, $\|w\|_\infty < \infty$. Thus, $\forall \varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\int_0^\delta \left[\frac{\int_{B_t(x_0)} u^p(z)v^q(z)\,dz}{t^{n-\beta\gamma}}\right]^{\frac{1}{\gamma-1}} \frac{dt}{t} \leqslant C\|w\|_\infty^{\frac{\gamma^*-1}{\gamma-1}} \int_0^\delta t^{\frac{\beta\gamma}{\gamma-1}} \frac{dt}{t} < \varepsilon.$$

On the other hand, similarly to the derivation of (2.16), as $|x - x_0| < \delta$,

$$\int_\delta^\infty \left[\frac{\int_{B_t(x_0)} u^p(z)v^q(z)\,dz}{t^{n-\beta\gamma}}\right]^{\frac{1}{\gamma-1}} \frac{dt}{t} \leqslant \int_\delta^\infty \left[\frac{\int_{B_{t+\delta}(x)} u^p(z)v^q(z)\,dz}{(t+\delta)^{n-\beta\gamma}}\right]^{\frac{1}{\gamma-1}} \left(\frac{t+\delta}{t}\right)^{\frac{n-\beta\gamma}{\gamma-1}+1} \frac{d(t+\delta)}{t+\delta}$$
$$\leqslant C\int_0^\infty \left[\frac{\int_{B_t(x)} u^q(z)v^q(z)\,dz}{t^{n-\beta\gamma}}\right]^{\frac{1}{\gamma-1}} \frac{dt}{t}$$
$$\leqslant Cu(x).$$

Combining these estimates with (1.8), we get

$$u(x_0) < \varepsilon + Cu(x), \quad \text{for } |x-x_0| < \delta.$$

Since $u \in L^{\gamma^*}(R^n)$, there holds $\lim_{|x_0|\to\infty} \int_{B_\delta(x_0)} u^{\gamma^*}(x)\,dx = 0$. Thus, we have

$$u^{\gamma^*}(x_0) = \left|B_\delta(x_0)\right|^{-1} \int_{B_\delta(x_0)} u^{\gamma^*}(x_0)\,dx$$
$$\leqslant C\varepsilon^{\gamma^*} + C\left|B_\delta(x_0)\right|^{-1} \int_{B_\delta(x_0)} u^{\gamma^*}(x)\,dx \to 0 \tag{3.8}$$

when $|x_0| \to \infty$ and $\varepsilon \to 0$. Similarly, v has the same result. Thus, (3.7) holds.

Step 3. Take a cutting-off function $\psi(x) \in C_0^\infty(B_2 \setminus B_1)$ satisfying

$$0 \leqslant \psi(x) \leqslant 1, \quad \text{for } 1 \leqslant |x| \leqslant 2;$$
$$\psi(x) = 1, \quad \text{for } \frac{5}{4} \leqslant |x| \leqslant \frac{7}{4}.$$

For any $\rho > 0$, set $\psi_\rho(x) = \psi(\frac{x}{\rho})$. Define

$$h(x) = w(x)|x|^{n/(\gamma^*-1)}\psi_\rho(x).$$

Then, one of the following two cases must be true:

(1) There exists a positive constant C (independent of ρ) such that

$$h(x) \leqslant C, \quad \forall x; \tag{3.9}$$

(2) There exists an increasing sequence $\{\rho_j\}_{j=1}^\infty$ satisfying $\lim_{j\to\infty} \rho_j = \infty$, such that as $x_{\rho_j} \in B_{2\rho_j} \setminus B_{\rho_j}$,

$$\lim_{j\to\infty} h(x_{\rho_j}) = \infty. \tag{3.10}$$

If (3.9) is true, then for large $|x|$,

$$w(x) \leqslant C|x|^{-n/(\gamma^*-1)}. \tag{3.11}$$

When $t \in (0, |x|/2)$, $y \in B_t(x)$ implies $|x|/2 \leqslant |y| \leqslant 3|x|/2$. Then by (2.1) and (3.11), we have

$$R(x) \int_0^{\frac{|x|}{2}} \left(\frac{\int_{B_t(x)} w^{\gamma^*-1}(y)\,dy}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \leqslant \frac{C}{|x|^{\frac{n}{\gamma-1}}} \int_0^{\frac{|x|}{2}} t^{\frac{\beta\gamma}{\gamma-1}} \frac{dt}{t} \leqslant \frac{C}{|x|^{\frac{n-\beta\gamma}{\gamma-1}}}. \tag{3.12}$$

On the other hand, Theorem 2.2 shows $w \in L^{\gamma^*-1}(R^n)$. Then

$$R(x) \int_{|x|/2}^{\infty} \left(\frac{\int_{B_t(x)} w^{\gamma^*-1}(y)\,dy}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \leqslant C \int_{|x|/2}^{\infty} t^{\frac{\beta\gamma-n}{\gamma-1}} \frac{dt}{t} \leqslant C|x|^{-\frac{n-\beta\gamma}{\gamma-1}}.$$

Combining this result with (3.12), we obtain

$$w(x) = R(x) W_{\beta,\gamma}\big(w^{\gamma^*-1}\big)(x) \leqslant C|x|^{-\frac{n-\beta\gamma}{\gamma-1}}.$$

This is (3.6).

Next, we prove case (2) does not happen.

Step 4. Let x_ρ be the maximum point of $h(x)$ in $B_{2\rho} \setminus B_\rho$. It follows from (3.10) that

$$w(x_{\rho_j}) = \frac{h(x_{\rho_j})}{\psi_{\rho_j}(x_{\rho_j})|x_{\rho_j}|^{n/(\gamma^*-1)}} \geqslant \frac{c}{\rho_j^{n/(\gamma^*-1)}}. \tag{3.13}$$

For convenience, we denote ρ_j by ρ.

We claim that

$$\psi_\rho(x_\rho) > \delta \tag{3.14}$$

for some $\delta > 0$ (independent of ρ). Otherwise, for any $\delta > 0$, there exists $R_0 > 0$ such that as $\rho > R_0$,

$$\psi_\rho(x_\rho) \leqslant \delta. \tag{3.15}$$

Let $\bar{x}_\rho$ be the maximum point of $w(x)$ in $B_{2\rho} \setminus B_\rho$. Namely,

$$M(\rho) := w(\bar{x}_\rho) = \sup\big\{w(x);\, x \in B_{2\rho} \setminus B_\rho\big\}.$$

Take $\bar{\rho} > R_0$ such that $\bar{\rho} = \frac{2}{3}|\bar{x}_\rho|$. By (3.15), we deduce that

$$\begin{aligned} M(\rho)|\bar{x}_\rho|^{n/(\gamma^*-1)} &= w(\bar{x}_\rho)|\bar{x}_\rho|^{n/(\gamma^*-1)} \leqslant h(x_{\bar{\rho}}) \\ &\leqslant \delta w(x_{\bar{\rho}})|x_{\bar{\rho}}|^{n/(\gamma^*-1)} \leqslant C\delta M\big(|x_{\bar{\rho}}|\big)|\bar{x}_\rho|^{n/(\gamma^*-1)}. \end{aligned}$$

Denote $C\delta$ by $\bar{\delta}$, then

$$\begin{aligned} M(\rho) &\leqslant \bar{\delta} M\big(|x_{\bar{\rho}}|\big) \leqslant \bar{\delta} \sup\left\{M(l);\, \frac{\rho}{2} \leqslant l \leqslant 2\rho\right\} \\ &\leqslant \bar{\delta} \sup\left\{M(l);\, \frac{\rho}{2} \leqslant l < \infty\right\} := \bar{\delta} \bar{M}\left(\frac{\rho}{2}\right). \end{aligned}$$

Taking the supremum to both sides of the inequality above, we have

$$\bar{M}(\rho) \leqslant \bar{\delta}\bar{M}\left(\frac{\rho}{2}\right).$$

Let $\bar{\delta} \leqslant 2^{-2n/(\gamma^*-1)}$. By induction we obtain

$$\bar{M}\left(2^k R_*\right) = \bar{\delta}^k \bar{M}(R_*) \leqslant 2^{-2kn/(\gamma^*-1)} \bar{M}(R_*)$$

for some fixed $R_* > 0$. Therefore,

$$\bar{M}(r) \leqslant Cr^{-2n/(\gamma^*-1)}.$$

This result contradicts (3.13). Thus, (3.14) is true.

Step 5. Using (3.14) and the smoothness of ψ, we can find a suitably small positive constant σ, such that $\psi_\rho(y) > \delta/2$ for $|y - x_\rho| < \sigma|x_\rho|$. Hence, by $h(y) \leqslant h(x_\rho)$, we get

$$w(y) \leqslant C\frac{w(x_\rho)}{\psi_\rho(y)} \leqslant C(\delta)w(x_\rho), \quad \text{as } |y - x_\rho| < \sigma|x_\rho|. \tag{3.16}$$

In view of (1.10) and (2.1), there holds

$$\begin{aligned} w(x_\rho) &\leqslant C\left[\int_0^{\sigma|x_\rho|}\left(\frac{\int_{B_t(x_\rho)} w^{\gamma^*-1}(y)\,dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t} + \int_{\sigma|x_\rho|}^{\infty}\left(\frac{\int_{B_t(x_\rho)} w^{\gamma^*-1}(y)\,dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t}\right] \\ &:= C(J_1 + J_2). \end{aligned} \tag{3.17}$$

Clearly, from $w \in L^{\gamma^*-1}(R^n)$ it follows

$$J_2 \leqslant C\int_{\sigma|x_\rho|}^{\infty} t^{-\frac{n-\beta\gamma}{\gamma-1}}\frac{dt}{t} \leqslant C|x_\rho|^{-\frac{n-\beta\gamma}{\gamma-1}}. \tag{3.18}$$

Using (3.16), we obtain that, for $r \in (0, \sigma|x_\rho|)$,

$$\begin{aligned} J_1 &\leqslant Cw(x_\rho)\left[\int_0^{r}\left(\frac{\int_{B_t(x_\rho)} w^{\gamma^*-\gamma}(y)\,dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t} + \int_{r}^{\sigma|x_\rho|}\left(\frac{\int_{B_t(x_\rho)} w^{\gamma^*-\gamma}(y)\,dy}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t}\right] \\ &:= Cw(x_\rho)(J_{11} + J_{12}). \end{aligned} \tag{3.19}$$

By (3.7), for any $\varepsilon \in (0, 1)$, there holds

$$J_{11} \leqslant \|w\|_{L^\infty(B_{\sigma|x_\rho|}(x_\rho))}^{\frac{\gamma^*-\gamma}{\gamma-1}} \int_0^r t^{\frac{\beta\gamma}{\gamma-1}}\frac{dt}{t} \leqslant \varepsilon$$

as long as ρ is sufficiently large. On the other hand, by Hölder's inequality and Theorem 2.4,

$$\int_{B_t(x_\rho)} w^{\gamma^*-\gamma}(y)\,dy \leqslant \|w\|_s^{\gamma^*-\gamma} \left|B_t(x_\rho)\right|^{1-\frac{\gamma^*-\gamma}{s}} \leqslant Ct^{n-n(\gamma^*-\gamma)/s}$$

for any $\frac{1}{s} \in (0, \frac{n-\beta\gamma}{n(\gamma-1)})$. Hence,

$$J_{12} \leqslant C \int_r^{\sigma|x_\rho|} t^{\frac{\beta\gamma-n(\gamma^*-\gamma)/s}{\gamma-1}} \frac{dt}{t}.$$

Taking $\frac{1}{s}$ close to $\frac{n-\beta\gamma}{n(\gamma-1)}$, we get $\beta\gamma - n(\gamma^* - \gamma)/s < 0$. Therefore, if ρ is sufficiently large and r is chosen suitably large, then

$$J_{12} \leqslant \varepsilon.$$

Substituting the estimates of J_{11} and J_{12} into (3.19), we obtain

$$J_1 \leqslant C\varepsilon w(x_\rho)$$

when ρ is sufficiently large. Inserting this result and (3.18) into (3.17), and choosing ε sufficiently small, we get

$$w(x_\rho) \leqslant C|x_\rho|^{-\frac{n-\beta\gamma}{\gamma-1}}.$$

By (3.16), we obtain that as $|x - x_\rho| < \sigma|x_\rho|$,

$$w(x) \leqslant Cw(x_\rho) \leqslant C|x_\rho|^{-\frac{n-\beta\gamma}{\gamma-1}} \leqslant C|x|^{-\frac{n-\beta\gamma}{\gamma-1}}.$$

Since ρ is arbitrary, the result above still holds for all x as long as $|x|$ is large. This result contradicts (3.10). Thus case (2) does not happen. □

4. Proof of Corollary 1.2

Proposition 4.1. *Let $u, v \in L^{\gamma^*}(R^n)$ be a pair of positive solutions of* (1.4)*. Then there exist two functions $R_1(x)$ and $R_2(x)$, such that*

$$u(x) = R_1(x)W_{1,\gamma}\left(u^p v^q\right)(x), \quad in\ R^n;$$
$$v(x) = R_2(x)W_{1,\gamma}\left(v^p u^q\right)(x), \quad in\ R^n. \tag{4.1}$$

Moreover, we can find positive constants C_1 and C_2 such that

$$C_1 \leqslant R_1(x), R_2(x) \leqslant C_2. \tag{4.2}$$

Proof. By virtue of $u, v \in L^{\gamma^*}(R^n)$, we get $\inf_{R^n} u = \inf_{R^n} v = 0$. By Corollary 4.13 in [19], there exist $C_1, C_2 > 0$ such that

$$C_1 W_{1,\gamma}\big(u^p v^q\big)(x) \leqslant u(x) \leqslant C_2 W_{1,\gamma}\big(u^p v^q\big)(x), \quad x \in R^n;$$
$$C_1 W_{1,\gamma}\big(u^q v^p\big)(x) \leqslant v(x) \leqslant C_2 W_{1,\gamma}\big(u^q v^p\big)(x), \quad x \in R^n. \tag{4.3}$$

Set

$$R_1(x) = \frac{u(x)}{W_{1,\gamma}(u^p v^q)(x)}, \qquad R_2(x) = \frac{v(x)}{W_{1,\gamma}(u^q v^p)(x)}. \tag{4.4}$$

Then the solutions u and v of γ-Laplace system (1.4) satisfy (4.1). At the same time, (4.3) leads to (4.2). Proposition 4.1 is proved. □

Proof of Corollary 1.2. As a direct corollary of Theorem 1.1 and Proposition 4.1, we can see conclusions i) and ii) of Corollary 1.2. □

5. Remarks on integrability intervals

Theorem 2.2 shows that the solution w of (1.10) belongs to $L^s(R^n)$ for all $s \in (\frac{n(\gamma-1)}{n-\beta\gamma}, \infty)$. According to Theorem 1 in [32], if $u \in L^{q_1+\gamma-1}(R^n)$ solves (1.14) with (1.15), we also have $u \in L^s(R^n)$ for all $s \in (\frac{n(\gamma-1)}{n-\beta\gamma}, \infty)$.

In this section, we explain briefly why the integrability intervals of the positive solutions of (1.7) and (1.14) are concordant under the different critical conditions (1.9) and (1.15).

To do this, we recall the derivation the integrability of (1.11) in [32]. Different from $u, v \in L^{\gamma^*}(R^n)$ for (1.7), for (1.11)

$$(u, v) \in L^{q_1+\gamma-1}\big(R^n\big) \times L^{q_2+\gamma-1}\big(R^n\big). \tag{5.1}$$

We write a more general initial integrability assumption: $(u, v) \in L^{r_0}(R^n) \times L^{s_0}(R^n)$. The most essential idea in [32] is obtaining a contraction map by means of inequalities of the Wolff type and the Hardy–Littlewood–Sobolev type. In order to use those inequalities, it is sufficient to take $r, s > 1$ satisfying

$$\frac{1}{r} - \frac{1}{s} = \frac{2-\gamma}{r_0} + \frac{q_2-1}{s_0} - \frac{\beta\gamma}{n}, \tag{5.2}$$
$$\frac{1}{s} - \frac{1}{r} = \frac{2-\gamma}{s_0} + \frac{q_1-1}{r_0} - \frac{\beta\gamma}{n}, \tag{5.3}$$

and

$$\frac{1}{r} - \frac{1}{s} = \frac{1}{r_0} - \frac{1}{s_0}. \tag{5.4}$$

(These are (32), (35) and (39) in [32].)

Adding (5.2) and (5.3) together, and subtracting each other, we respectively obtain that

$$\frac{q_1-\gamma+1}{r_0} + \frac{q_2-\gamma+1}{s_0} = \frac{2\beta\gamma}{n}, \tag{5.5}$$

and

$$\frac{1}{r} - \frac{1}{s} = \frac{3-q_1-\gamma}{2r_0} + \frac{q_2+\gamma-3}{2s_0}. \tag{5.6}$$

Combining (5.4) with (5.6), we see that r_0 and s_0 satisfy

$$\frac{r_0 - q_1 - (\gamma - 1)}{r_0} = \frac{s_0 - q_2 - (\gamma - 1)}{s_0}. \tag{5.7}$$

To understand well the relation between (1.9) and (1.12), we particularly take

$$r_0 = q_1 + t, \qquad s_0 = q_2 + t,$$

where $t > 0$ will be determined later. Therefore, from (5.5) and (5.7) we deduce that

$$\frac{q_1 - (\gamma - 1)}{q_1 + t} + \frac{q_2 - (\gamma - 1)}{q_2 + t} = \frac{2\beta\gamma}{n}, \tag{5.8}$$

and

$$\big[t - (\gamma - 1)\big]\left(\frac{1}{q_1 + t} - \frac{1}{q_2 + t}\right) = 0, \tag{5.9}$$

respectively. Clearly, (5.9) is true if and only if

$$t = \gamma - 1, \tag{5.10}$$

or

$$q_1 = q_2. \tag{5.11}$$

Case I. If (5.10) holds, then

$$r_0 = q_1 + \gamma - 1, \quad s_0 = q_2 + \gamma - 1.$$

This is (5.1). At the same time, (5.8) is equivalent to the critical condition (1.12). It is the natural generalization of the Hardy–Littlewood–Sobolev type integral equations which are helpful to understand the best constant of the Hardy–Littlewood–Sobolev inequality.

Case II. If (5.11) is true, then (1.11) is reduced to a single equation. By Proposition 2.1, the single equation is similar to the system (1.7) with the homogeneous exponents. Since the positive solutions of (1.7) belong to $L^{\gamma^*}(R^n)$, it follows $r_0 = s_0 = q_1 + t = q_2 + t = \gamma^*$. This implies that (5.8) is equivalent to the critical condition (1.9).

Thus, the Wolff and the Hardy–Littlewood–Sobolev inequalities work in both cases I and II. We can also obtain the contraction map, and the rest proof makes sense if we argue by the same way as in [32].

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References

[1] C. Azizieh, Symmetry and monotonicity results for positive solutions of p-Laplace systems, Rend. Istit. Mat. Univ. Trieste 34 (2002) 67–98.
[2] J. Bourgain, Global Solutions of Nonlinear Schrödinger Equations, Amer. Math. Soc. Colloq. Publ., vol. 46, Amer. Math. Soc., Providence, RI, 1999.
[3] J. Byeon, L. Jeanjean, M. Maris, Symmetry and monotonicity of least energy solutions, Calc. Var. Partial Differential Equations 36 (2009) 481–492.
[4] L. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Comm. Pure Appl. Math. 42 (1989) 271–297.
[5] W. Chen, C. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J. 63 (1991) 615–622.
[6] W. Chen, C. Li, A priori estimates for prescribing scalar curvature equations, Ann. of Math. 145 (1997) 547–564.
[7] W. Chen, C. Li, Radial symmetry of solutions for some integral systems of Wolff type, Discrete Contin. Dyn. Syst. 30 (2011) 1083–1093.
[8] W. Chen, C. Li, B. Ou, Classification of solutions for a system of integral equations, Comm. Partial Differential Equations 30 (2005) 59–65.
[9] W. Chen, C. Li, B. Ou, Classification of solutions for an integral equation, Comm. Pure Appl. Math. 59 (2006) 330–343.
[10] C. Cascante, J. Ortega, I. Verbitsky, Wolff's inequality for radially nonincreasing kernels and applications to trace inequalities, Potential Anal. 16 (2002) 347–372.
[11] A. Chang, P. Yang, On uniqueness of an n-th order differential equation in conformal geometry, Math. Res. Lett. 4 (1997) 1–12.
[12] L. Damascelli, F. Pacella, Monotonicity and symmetry of solutions of p-Laplace equations, $1 < p < 2$, via the moving plane method, Ann. Sc. Norm. Super. Pisa Cl. Sci. 27 (1998) 689–707.
[13] L. Damascelli, F. Pacella, M. Ramaswamy, Symmetry of ground states of p-Laplace equations via the moving plane method, Arch. Ration. Mech. Anal. 148 (1999) 291–308.
[14] M. Franca, Classification of positive solutions of p-Laplace equation with a growth term, Arch. Math. 40 (2004) 415–434.
[15] B. Gidas, W.M. Ni, L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in R^n, in: Mathematical Analysis and Applications, in: Adv. in Math. Suppl. Stud., vol. 7a, Academic Press, New York, 1981.
[16] L.I. Hedberg, T. Wolff, Thin sets in nonlinear potential theory, Ann. Inst. Fourier (Grenoble) 33 (1983) 161–187.
[17] C. Jin, C. Li, Qualitative analysis of some systems of integral equations, Calc. Var. Partial Differential Equations 26 (2006) 447–457.
[18] T. Kilpelaiinen, J. Maly, Degenerate elliptic equations with measure data and nonlinear potentials, Ann. Sc. Norm. Super. Pisa Cl. Sci. 19 (1992) 591–613.
[19] T. Kilpelaiinen, J. Maly, The Wiener test and potential estimates for quasilinear elliptic equations, Acta Math. 172 (1994) 137–161.
[20] D. Labutin, Potential estimates for a class of fully nonlinear elliptic equations, Duke Math. J. 111 (2002) 1–49.
[21] Y. Lei, Decay rates for solutions of an integral system of Wolff type, Potential Anal. 35 (2011) 387–402.
[22] Y. Lei, C. Li, C. Ma, Decay estimation for positive solutions of a γ-Laplace equation, Discrete Contin. Dyn. Syst. 30 (2011) 547–558.
[23] Y. Lei, C. Li, C. Ma, Asymptotic radial symmetry and growth estimates of positive solutions to weighted Hardy–Littlewood–Sobolev system, Calc. Var. Partial Differential Equations, doi:10.1007/s00526-011-0450-7.
[24] Y. Lei, C. Ma, Asymptotic behavior for solutions of some integral equations, Commun. Pure Appl. Anal. 10 (2011) 193–207.
[25] C. Li, Local asymptotic symmetry of singular solutions to nonlinear elliptic equations, Invent. Math. 123 (1996) 221–231.
[26] C. Li, J. Lim, The singularity analysis of solutions to some integral equations, Commun. Pure Appl. Anal. 6 (2007) 453–464.
[27] C. Li, L. Ma, Uniqueness of positive bound states to Schrödinger systems with critical exponents, SIAM J. Math. Anal. 40 (2008) 1049–1057.
[28] E. Lieb, Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities, Ann. of Math. 118 (1983) 349–374.
[29] T. Lin, J. Wei, Ground state of N coupled nonlinear Schrödinger equations in R^n, $n \leqslant 3$, Comm. Math. Phys. 255 (2005) 629–653.
[30] T. Lin, J. Wei, Spikes in two coupled nonlinear Schrödinger equations, Ann. Inst. H. Poincare Anal. Non Lineaire 22 (2005) 403–439.
[31] S. Liu, Regularity, symmetry, and uniqueness of some integral type quasilinear equations, Nonlinear Anal. 71 (2009) 1796–1806.
[32] C. Ma, W. Chen, C. Li, Regularity of solutions for an integral system of Wolff type, Adv. Math. 226 (2011) 2676–2699.
[33] J. Maly, Wolff potential estimates of superminimizers of Orlicz type Dirichlet integrals, Manuscripta Math. 110 (2003) 513–525.
[34] N. Phuc, I. Verbitsky, Quasilinear and Hessian equations of Lane–Emden type, Ann. of Math. 168 (2008) 859–914.
[35] J. Serrin, A symmetry problem in potential theory, Arch. Ration. Mech. Anal. 43 (1971) 304–318.
[36] J. Serrin, H. Zou, Cauchy–Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities, Acta Math. 189 (2002) 79–142.
[37] E.M. Stein, G. Weiss, Fractional integrals in n-dimensional Euclidean space, J. Math. Mech. 7 (1958) 503–514.