Maps preserving the harmonic mean or the parallel sum of positive operators

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ABSTRACT

Let \( H \) be a complex Hilbert space. The symbol \( A!B \) stands for the harmonic mean of the positive bounded linear operators \( A, B \) on \( H \) in the sense of Ando. In this paper we describe the general form of all automorphisms of the set of positive operators with respect to that operation. We prove that any such transformation is implemented by an invertible bounded linear or conjugate-linear operator on \( H \).

Similar results concerning the parallel sum and the arithmetic mean in the place of the harmonic mean are also presented.

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1. Introduction and statement of the results

Let \( H \) be a complex Hilbert space with inner product \( \langle \cdot, \cdot \rangle \). Denote by \( B(H) \) the algebra of all bounded linear operators on \( H \). As usual, an operator \( A \in B(H) \) is called positive if \( \langle Ax, x \rangle \geq 0 \) holds for every \( x \in H \) and in that case we write \( A \geq 0 \). The set of all positive operators on \( H \) is denoted by \( B(H)^+ \).

In the recent paper [12] we described the structure of all bijective maps on \( B(H)^+ \) which preserve the geometric mean \# introduced by Ando in [5]. It turned out that if \( \dim H \geq 2 \), any bijective map \( \phi : B(H)^+ \to B(H)^+ \) satisfying

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Then there is an invertible bounded linear or conjugate-linear operator $S$ on $H$. The geometric mean is well-known to have important applications in operator theory but recently it has found serious applications in other areas, for example, in quantum information theory as well (see [12]). Since $#$ is an operation which makes $B(H)^+$ an algebraic structure and there is general interest in the study of the automorphisms of algebraic structures, this has motivated us in [12] to determine the bijective maps satisfying (1).

The main aim of this paper is to consider the same problem for another important mean, namely for the harmonic mean of positive operators. This concept was introduced in [5] as follows. For arbitrary positive operators $A, B \in B(H)^+$, their harmonic mean $A:B$ is defined by

$$A:B = \max \left\{ X \geq 0 : \begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix} \geq \begin{bmatrix} X & X \\ X & X \end{bmatrix} \right\}.$$ 

Just as in [12] we recall that a general axiomatic theory of operator means was later developed by Kubo and Ando in [10].

For the rest of the paper we list some important properties of the harmonic mean (see [5,10]). In what follows, for arbitrary self-adjoint operators $A, B \in B(H)$ we write $A \leq B$ if and only if $B - A \geq 0$. The set of all nonnegative real numbers is denoted by $\mathbb{R}_+$. All operators appearing in the following list of properties are supposed to belong to $B(H)^+$ with the exception of $S$ appearing in (iv).

(i) $A:B = B:A$.
(ii) For any $\lambda \in \mathbb{R}_+$ we have $(\lambda A)!:(\lambda B) = \lambda (A:B)$.
(iii) If $A \leq C$ and $B \leq D$, then $A:B \leq C:D$.
(iv) (Transfer property) We have $S(A:B)S^* = (SAS^*):(SBS^*)$ for every invertible bounded linear or conjugate-linear operator $S$ on $H$.
(v) Suppose $A_1 \geq A_2 \geq \cdots \geq 0$, $B_1 \geq B_2 \geq \cdots \geq 0$ and $A_n \to A, B_n \to B$ strongly. Then we have that $A_n:B_n \to A:B$ strongly.
(vi) $A!A = A, A!A = 2A(I + A)^{-1}$ and $0!A = 0$.
(vii) $A!B = 2A(A + B)^{-1}B$ if $A$ or $B$ is invertible.
(viii) $A!B = 2(A^{-1} + B^{-1})^{-1}$ if $A$ and $B$ are both invertible.

The transfer property shows that for an arbitrary invertible bounded linear or conjugate-linear operator $S$, the transformation $A \mapsto SAS^*$ is a bijective map of $B(H)^+$ respecting the operation of the harmonic mean. The content of our main result is that the converse is also true: there is no other kind of transformations having this property.

**Theorem.** Let $\phi : B(H)^+ \to B(H)^+$ be a bijective map satisfying

$$\phi(A!B) = \phi(A)!\phi(B) \quad (A, B \in B(H)^+).$$

Then there is an invertible bounded linear or conjugate-linear operator $S$ on $H$ such that $\phi$ is of the form

$$\phi(A) = SAS^* \quad (A \in B(H)^+).$$

There is a concept closely related to the harmonic mean called parallel sum. For arbitrary positive operators $A, B \in B(H)^+$, their parallel sum $A : B$ is expressed as

$$A : B = \frac{1}{2} (A!B).$$

This notion originally defined by Anderson and Duffin [2] in a different way has many important applications in operator theory and in electrical network theory, too. The reason of these latter applications is the following: if $A, B$ are impedance matrices of a resistive $n$-port network, then their parallel sum $A : B$ is just the impedance matrix of the parallel connection [1]. For the most classical results concerning this operation we refer to the papers [1–4]. As an easy corollary of our main result we shall obtain the following description of bijective maps preserving the parallel sum.
Corollary. Let \( \phi : B(H)^+ \to B(H)^+ \) be a bijective map satisfying
\[
\phi(A : B) = \phi(A) : \phi(B) \quad (A, B \in B(H)^+).
\]
Then \( \phi \) respects the operation of the harmonic mean. Consequently, there exists an invertible bounded linear or conjugate-linear operator \( S \) on \( H \) such that \( \phi \) is of the form
\[
\phi(A) = SAS^* \quad (A \in B(H)^+).
\]

Finally, to make our investigation more complete we conclude with the following rather simple result concerning the structure of bijective maps of \( B(H)^+ \) preserving the arithmetic mean \( \nabla \). This operation is defined by
\[
A \nabla B = \frac{1}{2}(A + B)
\]
for all \( A, B \in B(H)^+ \).

Proposition. Let \( \phi : B(H)^+ \to B(H)^+ \) be a bijective map satisfying
\[
\phi(A \nabla B) = \phi(A) \nabla \phi(B) \quad (A, B \in B(H)^+).
\]
Then there exists an invertible bounded linear or conjugate-linear operator \( S \) on \( H \) such that \( \phi \) is of the form
\[
\phi(A) = SAS^* \quad (A \in B(H)^+).
\]

2. Proofs

The proof of the main result is based on the following auxiliary results. First we present a characterization of the invertible elements of \( B(H)^+ \).

Lemma 1. The operator \( A \in B(H)^+ \) is invertible if and only if
\[
\{(...(AT_1)T_2)...T_n) : T_1, \ldots, T_n \in B(H)^+, \quad n \in \mathbb{N}\} = B(H)^+.
\] (2)

Proof. Let \( A \in B(H)^+ \) be invertible. In order to verify (2), observe that by the transfer property (iv) above there is no serious loss of generality in assuming that \( A = I \). Define \( f(t) = 2t/(1 + t), 0 \leq t \in \mathbb{R} \). Then by (vi) we have
\[
IT_1 = 2T_1(1 + T_1)^{-1} = f(T_1) \quad (T_1 \in B(H)^+).
\]
As \( f : [0, \infty[ \to [0, 2] \) is a (strictly increasing) continuous bijective function, it follows that \( IT_1 \) \((T_1 \in B(H)^+)\) runs through the set of all positive operators with spectrum contained in \([0, 2]\). In particular, for any real number \( 0 < \epsilon < 2 \) there exists a \( T \in B(H)^+ \) such that
\[
\epsilon I = IT.
\]
Now, from
\[
(\epsilon IT_1)T_2 = (\epsilon f(T_1))T_2 = \epsilon f((1/\epsilon)T_2)
\]
it follows that \( \epsilon IT_1 \) runs through the set of all positive operators with spectrum contained in \([0, 2\epsilon]\). Consequently, \( \epsilon IT_1 \) runs through the set of all positive operators with spectrum contained in \([0, 4]\). Continuing this process, we see that the operators \( (..., (A!T_1)T_2) ... T_n) \) \((T_1, \ldots, T_n \in B(H)^+, n \in \mathbb{N}\) run through the whole set \( B(H)^+ \).

Next suppose that \( A \) is not invertible. We assert that in that case \( A!T \) is non-invertible for every \( T \in B(H)^+ \). Indeed, if \( T \) is invertible, then by (vii) we have
\[
A!T = 2A(A + T)^{-1}T
\]
from which it is apparent that \( A!T \) is non-invertible. If \( T \) is not invertible, then by (iii) for any positive \( \lambda \in \mathbb{R} \) we have \( A!T \leq A!(T + \lambda I) \). We already know that this latter operator is non-invertible. But this implies
that $A!T$ is non-invertible, too. Consequently, we obtain that in the case when $A$ is non-invertible, the operators $(\ldots((A!T_1)!T_2)\ldots)!T_n(T_1,\ldots,T_n \in B(H)^+, n \in \mathbb{N})$ are all non-invertible. □

In the next lemma we compute the harmonic mean of an arbitrary positive operator $T \in B(H)^+$ and any rank-one projection $P$. To do so, we need the concept of the strength of a positive operator $A$ along a ray represented by any unit vector in $H$. This concept was originally introduced by Busch and Gudder in [6] for the so-called Hilbert space effects in the place of positive operators. Effects play an important role in the mathematical foundations of the theory of quantum measurements. Mathematically, a Hilbert space effect is simply an operator $E \in B(H)$ which satisfies $0 \leq E \leq I$. Although in [6] the authors considered only effects, it is rather obvious that the following definition and result work also for arbitrary positive operators (the reason is simply that any positive operator can be multiplied by a positive scalar to obtain an effect). So, let $A \in B(H)^+$ be a positive operator, consider a unit vector $\varphi$ in $H$ and denote by $P_\varphi$ the rank-one projection onto the subspace generated by $\varphi$. The quantity

$$
\lambda(A, P_\varphi) = \sup\{\lambda \in \mathbb{R} : \lambda P_\varphi \leq A\}
$$

is called the strength of $A$ along the ray represented by $\varphi$. According to [6, Theorem 4] we have the following formula for the strength:

$$
\lambda(A, P_\varphi) = \begin{cases} 
\|A^{-1/2}\varphi\|^{-2} & \text{if } \varphi \in \text{rng}(A^{1/2}); \\
0 & \text{else.}
\end{cases}
$$

(3)

(The symbol rng denotes the range of operators and $A^{-1/2}$ denotes the inverse of $A^{1/2}$ on its range.)

**Lemma 2.** For an arbitrary positive operator $T \in B(H)^+$ and any rank-one projection $P$ on $H$ we have

$$
T!P = \frac{2\lambda(T, P)}{\lambda(T, P) + 1} P.
$$

In particular, $T!P$ is nonzero if and only if $\lambda(T, P) \neq 0$ which is equivalent to $\text{rng}P \subset \text{rng}\sqrt{T}$.

**Proof.** Pick a scalar $1 \leq \lambda \in \mathbb{R}$ such that $T \leq \lambda I$. Using (ii), (iii) and (vi) one can see that

$$
T!P \leq (\lambda I)!P = \lambda(I!P) = \lambda P.
$$

As $P$ is a rank-one operator, it follows that $T!P$ is a scalar multiple of $P$, i.e., we have

$$
T!P = \epsilon P
$$

for some scalar $\epsilon \in \mathbb{R}_+$. We show that $\epsilon$ is necessarily strictly less than 2. Indeed, this follows from the inequality:

$$
T!P \leq T!I = 2T(T + I)^{-1}
$$

referring to the fact that the spectrum of $2T(T + 1)^{-1}$ is contained in $[0, 2]$ (see the proof of Lemma 1).

Let $\delta \in \mathbb{R}_+$ be an arbitrary scalar such that $\delta P \leq T$. We assert that $2\delta/(1 + \delta)P \leq T!P$ holds true. To see this, first observe that $(\delta P)!P \leq T!P$. We next compute $(\delta P)!P$. Applying (vi), for an arbitrary positive $\lambda \in \mathbb{R}$ we have

$$
(\delta P)!P = (P + \lambda I)^{-1}(P + \lambda I)\delta P(P + \lambda I)^{-1} = 2(\delta P)((1 + \delta)P + \lambda P^{-1})^{-1}(P + \lambda P^{-1}) = \frac{2\delta(1 + \delta)}{1 + \delta} P.
$$

Here, $P^\perp$ denotes the projection $I - P$. Letting $\lambda$ tend to 0, by (v) we infer that $(\delta P)!P = 2\delta/(1 + \delta)P$. Therefore, we have $2\delta/(1 + \delta)P \leq T!P$ as asserted.

Conversely, suppose now that $2\delta/(1 + \delta)P \leq T!P$. Then we have

$$
2(\delta P)(P + \lambda I)^{-1} = \frac{2\delta}{1 + \delta} P \leq T!P \leq T!I = 2T(T + I)^{-1}.
$$

Setting $f(t) = 2t/(1 + t), 0 \leq t \in \mathbb{R}$, we can rewrite the above inequality as $f(\delta P) \leq f(T)$. As the inverse of the bijective function $f : [0, \infty] \to [0, 2]$ is the function $g : [0, 2] \to [0, \infty]$ defined by $g(s) = s/(2 - s) = -1 + 2/(2 - s), s \in [0, 2]$ which is clearly operator monotone, it follows that $\delta P \leq T$. 


Therefore, we have proved that for any $\delta \in \mathbb{R}^+_0$

$$\delta P \leq T \iff \frac{2\delta}{1+\delta} P \leq T^T P.$$  

It now easily follows that for $\varepsilon$ in (4) we have $\varepsilon = \frac{2\lambda(TP)}{\lambda(TP)+1}$. □

Let $B(H)^+_1$ denote the set of all invertible positive operators on $H$. In the next lemma we describe the structure of all bijective maps on $B(H)^+_1$ which preserve the arithmetic mean.

**Lemma 3.** Let $\psi : B(H)^+_1 \rightarrow B(H)^+_1$ be a bijective map satisfying

$$\psi((A + B)/2) = (\psi(A) + \psi(B))/2 \quad (A, B \in B(H)^+_1).$$

Then there exists a bounded invertible linear or conjugate-linear operator $S$ on $H$ such that

$$\psi(A) = SAS^* \quad (A \in B(H)^+_1).$$

**Proof.** We first recall that the functional equation (5) above is usually called Jensen equation. We learn from the paper [8] that every function from a nonempty $\mathbb{Q}$-convex subset of a linear space $X$ over $\mathbb{Q}$ into another linear space $Y$ over $\mathbb{Q}$ satisfying the Jensen equation can be written in the form $x \mapsto A_0 + A_1(x)$, where $A_0 \in Y$ and $A_1 : X \rightarrow Y$ is an additive function. (In fact, the main result in [8] concerns more general transformations.)

Let $B_1(H)$ denote the linear space of all self-adjoint operators in $B(H)$. By the above mentioned result it follows that there is an operator $X \in B_1(H)$ and an additive map $L : B_1(H) \rightarrow B_1(H)$ such that

$$\psi(A) = L(A) + X \quad (A \in B(H)^+_1).$$

We assert that $L$ is in fact a continuous linear transformation. First, we know that $L(B) \geq -X$ for every $B \in B(H)^+_1$. It follows that for any operator $A \in B_1(H)$ with $\|A\| \leq 1/2$ we have $L(I + A) \geq -X$ implying that $L(A) \geq -L(I) - X$. Consequently, there is a negative constant $c \in \mathbb{R}$ such that

$$L(A) \geq cl$$

holds whenever $A \in B_1(H)$, $\|A\| \leq 1/2$. Inserting $-A$ in the place of $A$, we get $L(-A) \geq cl$ which yields $L(A) \leq -cl$. Therefore, we obtain that

$$cl \leq L(A) \leq -cl$$

and hence $\|L(A)\| \leq |c|$ holds for every $A \in B_1(H)$, $\|A\| \leq 1/2$. This clearly gives us that the additive map $L$ is continuous and therefore linear.

We next prove that $X = 0$. Let $A \in B(H)^+_1$ be arbitrary. For every $n \in \mathbb{N}$ we have

$$nL(A) + X = L(nA) + X = \psi(nA) \geq 0$$

which gives us that $L(A) + (1/n)X \geq 0$. If $n$ tends to infinity, we obtain $L(A) \geq 0$. Hence we have $\psi(A) = L(A) + X \geq X$. Since the range of $\psi$ is $B(H)^+_1$, it follows that $0 \geq X$. On the other hand, by the continuity of $L$ we deduce

$$X = X + L(0) = X + \lim_{n} L((1/n)I) = \lim_{n} \psi((1/n)I)$$

from which it follows that $X \geq 0$. Consequently, we have $X = 0$ as asserted.

So, there is a continuous linear transformation $L : B_1(H) \rightarrow B_1(H)$ such that $\psi(A) = L(A), A \in B(H)^+_1$. In the same manner there corresponds a continuous linear transformation $L' : B_1(H) \rightarrow B_1(H)$ to the transformation $\psi^{-1}$. Clearly, we have $L'(L(A)) = L(L'(A)) = A$ for every $A \in B(H)^+_1$. Since $B(H)^+_1$ linearly generates $B_1(H)$, it follows that $L'(L(A)) = L(L'(A)) = A$ holds for every $A \in B_1(H)$. This shows that the transformation $L$ is invertible and its inverse is $L'$. Next, it is easy to see that $L$ is a bijective linear transformation of $B_1(H)$ which preserves the positive operators in both directions, i.e., $A \in B(H)^+$ if and only if $L(A) \in B(H)^+$. Indeed, as $L$ coincides with $\psi$ on $B(H)^+_1$, it sends invertible positive operators to invertible positive operators. Using the continuity of $L$ we obtain that $L$ sends positive operators to
positive operators. Applying the same argument for $L'$, it then follows that $L$ preserves the positive operators in both directions. Now, by a well-known result of Kadison [9, Corollary 5] stating that every unital linear bijection between $C^*$-algebras preserving positive elements in both directions is necessarily a Jordan $*$-isomorphism, we infer that $L$ is of the form 
\[ L(A) = SAS^* \ (A \in B_2(H)) \]
with some invertible bounded linear or conjugate-linear operator $S$ on $H$. (We remark that a more general result concerning non-linear bijections of $B_2(H)$ preserving the order in both directions was obtained in [11].) This completes the proof of the lemma. \[ \square \]

**Proof of Proposition.** A simplified version of the argument given above applies to verify the statement. \[ \square \]

Now we are in a position to prove the main result of the paper.

**Proof of Theorem.** Let $\phi : B(H)^+ \to B(H)^+$ be a bijective map preserving the harmonic mean. By Lemma 1, $\phi$ preserves the invertible operators in both directions. This means that $A \in B(H)^+$ is invertible if and only if $\phi(A)$ is invertible. Therefore, $\phi(I)$ is an invertible positive operator. Considering the transformation
\[ A \mapsto \phi(I)^{-1/2} \phi(A) \phi(I)^{-1/2}, \]
by the transfer property we obtain a bijective map on $B(H)^+$ which preserves the harmonic mean and sends $I$ to $I$. Hence, there is no serious loss of generality in assuming that already $\phi$ has the property that $\phi(I) = I$.

We next prove that $\phi$ preserves the projections in both directions. To see this, consider the following simple characterization of projection. The operator $A \in B(H)^+$ is a projection if and only if $A!A = A$. Indeed, by (vi) we know that $I!A = 2(I + A)A^{-1}A$. This implies that $I!A = A$ if and only if $2A = (I + A)A = A + A^2$ which is trivially equivalent to $A = A^2$. By the properties of $\phi$ we infer that $\phi$ indeed preserves the projections in both directions.

Proposition 2 in [7] tells us that for any projections $P, Q$ we have $P!Q = P \wedge Q$. This implies that $\phi$ is a lattice-automorphism on the set of all projections on $H$. Consequently, $\phi$ sends $0$ to $0$ and it preserves the rank-one and also the corank-one projections in both directions.

Our next aim is to show that $\phi$ preserves the rank-one elements of $B(H)^+$ in both directions. Indeed, this follows from the following characterization of rank-one operators. The nonzero element $A \in B(H)^+$ is of rank-one if and only if there exists a corank-one projection $Q$ such that $A!Q = 0$. To see this, first suppose that $A$ is of rank-one. Then there is a rank-one projection $P$ and a positive scalar $\lambda$ such that $A = \lambda P$. By (ii) and Lemma 2, one can see that $A!P = 0$. Conversely, if $A!Q = 0$ for some projection $Q$ of rank-one, then by (iii) we have $A!R = 0$ for every rank-one subprojection of $Q$. But by Lemma 2 again this implies that $\text{rng} Q \cap \text{rng} \sqrt{A} = \{0\}$. We then infer that $\sqrt{A}$ and hence $A$ are both of rank-one.

Now let $P$ be a rank-one projection. We assert that for every $\lambda \in \mathbb{R}_+$ there is a nonnegative scalar $f(\lambda)$ such that $\phi(\lambda P) = f(\lambda) \phi(P)$. In fact, we know that $\phi(\lambda P)$ is of rank-one. Moreover, by Lemma 2 we have $(\lambda P)!P \neq 0$ and hence
\[ \phi(\lambda P)!\phi(P) = \phi((\lambda P)!P) \neq 0. \]
By Lemma 2 again, we deduce that the range of the rank-one projection $\phi(P)$ has nontrivial intersection with the range of the rank-one operator $\phi(P)$. This implies that $\phi((\lambda P)!P)$ is a scalar multiple of $\phi(P)$ and hence there is a nonnegative scalar $f(\lambda)$ such that $\phi(\lambda P) = f(\lambda) \phi(P)$. As $\phi$ and $\phi^{-1}$ share the same properties, it is easy to verify that $f$ is a bijection of $\mathbb{R}_+$ sending $0$ to $0$.

Let $\lambda$ and $\mu$ be positive real numbers. By Lemma 2 we compute
\[ (\lambda P)!((\mu P)P) = \mu!((\lambda/\mu)P)P = \mu \frac{2\lambda}{\lambda/\mu + 1} P = \frac{2\lambda}{\lambda + \mu} P. \]
Similarly, we obtain
\[ (f(\lambda)\phi(P))!(f(\mu)\phi(P)) = \frac{2f(\lambda)}{f(\lambda) + f(\mu)} \phi(P). \]
As
\[ \phi((\lambda P)(\mu P)) = \phi(\lambda P)\phi(\mu P) = (f(\lambda)\phi(P))((\mu)\phi(P)), \]
it follows that
\[ f\left(\frac{2\lambda\mu}{\lambda + \mu}\right)\phi(P) = \frac{2f(\lambda)f(\mu)}{f(\lambda) + f(\mu)}\phi(P). \]

Consequently, we obtain that \( f \) satisfies the functional equation
\[ f\left(\frac{2\lambda\mu}{\lambda + \mu}\right) = \frac{2f(\lambda)f(\mu)}{f(\lambda) + f(\mu)} \quad (0 < \lambda, \mu \in \mathbb{R}). \]

This means simply that \( f \) is a bijection of the set of all positive real numbers satisfying \( f(\lambda!\mu) = f(\lambda)f(\mu) \) for all \( 0 < \lambda, \mu \). Defining \( g(t) = 1/f(1/t) \) for any \( 0 < t \in \mathbb{R} \), it is easy to check that \( g \) is a bijective function of the set of all positive real numbers satisfying the Jensen equation. As a very particular case of Lemma 3 we obtain that \( g \) as well as \( f \) is a positive scalar multiple of the identity. As we have \( f(1) = 1 \), it follows that \( f \) is in fact the identity on \( \mathbb{R}^+ \). Hence we have proved that \( \phi(\lambda P) = \lambda\phi(P) \) holds for every rank-one projection \( P \) and scalar \( \lambda \in \mathbb{R}^+ \).

After these preliminaries now the proof can be completed as follows. Similarly to the case of the scalar function \( f \) above, define a bijective transformations \( \psi \) on \( B(H)^+ \) by
\[ \psi(A) = \phi(A^{-1})^{-1} \quad (A \in B(H)^+). \]

Using the formula \( A!B = 2(A^{-1} + B^{-1})^{-1} \) for invertible operators \( A, B \in B(H)^+ \), it is easy to verify that \( \psi \) satisfies the Jensen equation
\[ \psi((A + B)/2) = (\psi(A) + \psi(B))/2 \quad (A, B \in B(H)^+). \]

Then Lemma 3 applies and we obtain that there exists an invertible bounded linear or conjugate-linear operator \( S \) on \( H \) such that
\[ \psi(A) = SAS^* \quad (A \in B(H)^+). \]

As we have supposed that \( \phi(I) = I \), it follows that \( SS^* = I \), i.e., \( S \) is either a unitary or an antiunitary operator. Denote it by \( U \). We easily obtain that
\[ \phi(A) = UAU^* \quad (A \in B(H)^+). \]

Therefore, considering the transformation
\[ A \mapsto U^*\phi(A)U, \]
we can further assume that \( \phi(A) = A \) holds for every \( A \in B(H)^+ \).

We next prove that \( \phi \) is the identity on the rank-one projections. Let \( P \) be a rank-one projection. Pick an arbitrary operator \( A \in B(H)^+ \). By Lemma 2 we have
\[ \frac{2\lambda(A!P)}{\lambda((A!P)^+)^+} \phi(P) = \phi\left(\frac{2\lambda(A!P)}{\lambda((A!P)^+)^+}\phi(P)\right) = \phi(\lambda(A!P)) = \phi(\lambda)\phi(P) = A\phi(P) = \frac{2\lambda(A!P)}{\lambda(\phi(P)^+)^+} \phi(P). \]

It follows that
\[ \lambda(A,P) = \lambda(A,\phi(P)) \quad (6) \]
holds for every invertible operator \( A \in B(H)^+ \). Moreover, for such operator \( A \) and arbitrary unit vector \( \varphi \in H \), applying (3) we compute
\[ \lambda(A,P_\varphi) = \|A^{-1/2}\varphi\|^{-2} = \frac{1}{\|A^{-1/2}A^{-1/2}\varphi\|} = \frac{1}{\langle A^{-1}\varphi, A^{-1/2}\varphi \rangle} = \frac{1}{\text{tr}A^{-1}P_\varphi}. \]

Therefore, from (6) we deduce that
\[ \text{tr}A^{-1}p = \text{tr}A^{-1}\phi(P) \]

holds for every \( A \in B(H)_{+1} \). As \( B(H)_{+1} \) linearly generates \( B(H) \), we obtain the equality

\[ \text{tr}TP = \text{tr}T\phi(P) \]

for every bounded operator \( T \) on \( H \). Inserting rank-one projections into the place of \( T \) we conclude that

\[ \langle x, Px \rangle = \langle x, \phi(P)x \rangle \]

holds for every unit vector \( x \) in \( H \). This gives us that \( \phi(P) = P \) is valid for any rank-one projection \( P \) implying that \( \phi \) is the identity on the set of all rank-one operators in \( B(H)_{+1} \).

It remains to show that \( \phi(A) = A \) holds for every \( A \in B(H)_{+1} \). To see this, let \( A \in B(H)_{+1} \) be arbitrary. Take any rank-one projection \( P \) on \( H \). Applying Lemma 2, from the equalities

\[ A!P = \phi(A)!P = \phi(A)!\phi(P) = \phi(A)!P \]

we infer that

\[ \frac{2\lambda(A, P)}{\lambda(A, P) + 1} P = \frac{2\lambda(\phi(A), P)}{\lambda(\phi(A), P) + 1} P. \]

This implies that \( \lambda(A, P) = \lambda(\phi(A), P) \) holds for every rank-one projection \( P \) on \( H \). Since every positive operator is uniquely determined by its strength function \([6, \text{Corollary 1}]\), we obtain that \( \phi(A) = A \). This completes the proof of the theorem. \( \square \)

We conclude the paper with the simple proof of the corollary.

**Proof of Corollary.** Let \( A \in B(H)_{+1} \). Using (vi) we get

\[ \phi((1/2)A) = \phi(A : A) = \phi(A) = (1/2)\phi(A). \]

This implies that \( \phi(2A) = 2\phi(A) \) for every \( A \in B(H)_{+1} \). Therefore, we have

\[ \phi(A!B) = \phi(2(A : B)) = 2\phi(A : B) = 2(\phi(A) : \phi(B)) = \phi(A)!\phi(B) \]

for all \( A, B \in B(H)_{+1} \). This shows that \( \phi \) preserves the harmonic mean. An application of Theorem completes the proof. \( \square \)

**References**


