On properties of $h$-homogeneous spaces of first category

S.V. Medvedev

Faculty of Mechanics and Mathematics, South-Ural State University, pr. Lenina, 76, Chelyabinsk, Russia 454080

**A R T I C L E   I N F O**

Article history:
Received 2 December 2009
Received in revised form 27 August 2010
Accepted 30 August 2010

Keywords:
h-Homogeneous space
Set of first category
Extended Borel set
Canonical element

**A B S T R A C T**

A metric space $X$ is called $h$-homogeneous if $\text{Ind } X = 0$ and each nonempty open-closed subset of $X$ is homeomorphic to $X$. We describe how to assign an $h$-homogeneous space of first category and of weight $k$ to any strongly zero-dimensional metric space of weight $\leq k$. We investigate the properties of such spaces. We show that if $Q$ is the space of rational numbers and $Y$ is a strongly zero-dimensional metric space, then $Q \times Y^{\omega}$ is an $h$-homogeneous space and $F \times Q \times Y^{\omega}$ is homeomorphic to $Q \times Y^{\omega}$ for any $F_\sigma$-subset $F$ of $Q \times Y^{\omega}$. L. Keldysh proved that any two canonical elements of the Borel class $\alpha$ are homeomorphic. The last theorem is generalized for the nonseparable case.

© 2010 Elsevier B.V. All rights reserved.

---

0. Introduction

All topological spaces under discussion are metrizable and strongly zero-dimensional (i.e., $\text{Ind } X = 0$).

The aim of this paper is to describe how to assign an $h$-homogeneous space $h(X, k)$ of first category and of weight $k$ to any strongly zero-dimensional metric space $X$ of weight $\leq k$. Using this construction, we generalize for the nonseparable case the L. Keldysh theorem [4] stating that any two canonical elements of class $\alpha$ are homeomorphic.

Let $X$ be a space of weight $\leq k$. We shall show (see Theorems 6 and 8) that the family $\mathcal{H}_k(X) = \{Y: w(Y) = k, Y$ is an $h$-homogeneous space of first category, and $Y$ contains a closed copy of $X\}$ has a unique (up to homeomorphism) element $h(X, k)$ such that every $Y \in \mathcal{H}_k(X)$ contains a closed copy of $(h(X, k))$. This element is called the $h$-homogeneous extension of the space $X$ of weight $k$ with respect to spaces of first category. Briefly, $h(X, k)$ is the $h$-homogeneous extension of $X$. The proof of Theorem 6 gives a direct construction of the space $h(X, k)$ for any space $X$.

Theorem 7 states that if $X$ is an $h$-homogeneous space of weight $k$, then $h(X, k) \approx Q \times X$. This case is the simplest.

A topological characterization of $Q \times X$ for an $h$-homogeneous space $X$ was studied by van Mill [9] (in the separable case) and by Ostrovsky [10]. For an arbitrary space $X$ the product $Q \times X$ might not be $h$-homogeneous.

Theorem 4 implies that every $h$-homogeneous space $X$ of first category is homeomorphic to the product $Q \times X$. This result suggests the following

**Question.** Let $X$ be an $h$-homogeneous space of first category and $Q$ be the space of rational numbers. Is there an $h$-homogeneous space $Y$ such that $X$ is homeomorphic to $Q \times Y$ and $Y$ is not of first category?

Theorem 10 shows that $h(X, k)$ is homeomorphic to $h(Y, k)$ if and only if $X \in \sigma FL(Y)$ and $Y \in \sigma FL(X)$. We also obtain (see Theorem 11) that the Cartesian product of the $h$-homogeneous extensions of two spaces is homeomorphic to the $h$-homogeneous extension of the product of these spaces.

Let us recall the L. Keldysh definition [4] of a canonical element of the Borel class $\alpha$, where $\alpha$ is a countable ordinal. Put $\mathcal{M}_\alpha = \{X \subseteq B(\omega): X$ is of multiplicative class $\alpha\}$. A set $U \in \mathcal{M}_\alpha$ is called universal for $\mathcal{M}_\alpha$ if for each $X \in \mathcal{M}_\alpha$ there

---

E-mail address: medv@is74.ru.

0166-8641/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.topol.2010.08.021
exists a perfect set \( P \subseteq B(k) \) with \( P \cap U = X \). A set \( E \in \mathcal{M}_\alpha \) of first category is called a canonical element of class \( \alpha \) if it is everywhere universal for \( \mathcal{M}_\alpha \). L. Keldysh proved [4] that any two canonical elements of class \( \alpha \) are homeomorphic. Using the last theorem, Ostrovsky (see [11,12]) established that each canonical element of class \( \alpha \) is homeomorphic to \( Q \times M_{\alpha-1} \) (3 \( \leq \alpha < \omega \)) or \( Q \times M_\alpha \) (\( \omega \leq \alpha < \omega_1 \)). Note that the Ostrovsky sets \( M_\alpha \) are slightly differed from our sets \( M_\alpha(\omega) \) (see below). Of course, the sets \( M_\alpha \) and \( M_\alpha(\omega) \) are similar to the Siksorski sets [13].

Now, we generalize the definition of a canonical element. Let \( \mathcal{U} \) be a family of subsets of the Baire space \( B(k) \). A set \( U \in \mathcal{U} \) is called universal for \( \mathcal{U} \) if for each \( X \in \mathcal{U} \) there exists a perfect set \( P \subseteq B(k) \) with \( P \cap U = X \). A set \( E \in \mathcal{U} \) of first category is called a canonical element for \( \mathcal{U} \) if it is everywhere universal for \( \mathcal{U} \).

Let us analyze the last definition. Firstly, according to Theorem 1, it appears that \( E \) is an \( h \)-homogeneous space of first category. From Theorem 4 it follows that if we take another canonical element \( E_1 \) for \( \mathcal{U} \), then \( E \approx E_1 \). So, the uniqueness of a canonical element does not depend on the structure of members of \( \mathcal{U} \). Secondly, the existence of a canonical element for \( \mathcal{U} \) is included in the definition. However, the family \( \{ \text{point, point} \oplus \text{point} \} \) is an example of a family without a canonical element. Thirdly, Theorem 12 shows that we always can find a necessary perfect set \( P \). This allows us to consider Theorem 12 as a general theorem for canonical elements. Roughly, it states that if a subset \( E \subseteq B(k) \) of first category is a canonical element for \( \mathcal{U} \), then \( \mathcal{U} \subseteq \{ T \subseteq B(k): T \text{ is homeomorphic to a closed subset of } E \} \) and \( E \in \mathcal{U} \).

L. Keldysh aspired to represent any separable Borel set as a countable union of (the simplest) canonical elements (including the one-point set and \( B(\omega) \) into the list of canonical elements). Therefore, she did not consider sets strictly of additive class \( \alpha \) as canonical elements. However, our aim is another. So, we deal with canonical elements of multiplicative class \( \alpha \) and of additive class \( \alpha \). Take an ordinal \( \alpha \) and a cardinal \( k \) with \( 3 \leq \alpha < k^+ \). We show that for any cardinal \( \tau \) such that \( \alpha < \tau^+ \) and \( \tau \leq k \) there exists a unique (up to homeomorphism) canonical element of multiplicative (of additive) class \( \alpha \) and of type \((k, \tau)\) (see Section 5). Of course, these elements are not homeomorphic if \( \tau_1 \neq \tau_2 \). To give concrete expression to a canonical element of multiplicative class \( \alpha \) we study the product \( X \times Y^\omega \) with an \( h \)-homogeneous space \( X \) of first category. Using Theorem 5, we observe that any canonical element of multiplicative class \( \alpha \) and of type \((k, \tau)\) is homeomorphic to \( Q(k) \times M_\alpha(\tau) \).

The paper is organized as follows: in Section 1 we establish notation and give some lemmas concerning the family \( \sigma LF(X) \) for a space \( X \). In Section 2 we recall some properties of \( h \)-homogeneous spaces of first category and investigate the product \( X \times Y^\omega \) with an \( h \)-homogeneous space \( X \) of first category. In Section 3 we study the properties of \( h \)-homogeneous extensions of spaces. Section 4 is devoted to general canonical elements. In Section 5 we give a brief introduction in the theory of extended Borel sets and describe canonical elements for the family of extended Borel sets.

1. Notation and some lemmas

For all undefined terms and notation see [2]. \( X \approx Y \) means that \( X \) and \( Y \) are homeomorphic spaces. Let \( \mathcal{P} \) be a topological property. Then a space \( X \) is nowhere \( \mathcal{P} \) if no nonempty open subset of \( X \) has property \( \mathcal{P} \). A space \( X \) is everywhere \( \mathcal{P} \) if each nonempty open subset of \( X \) has property \( \mathcal{P} \).

We identify cardinals with initial ordinals; in particular, \( \omega = \aleph_0 = \{ 0, 1, 2, \ldots \} \) and \( N = \{ 1, 2, \ldots \} \). For every infinite cardinal \( k \) let \( B(k) = k^\omega \) be the Baire space of weight \( k \) (see [2]). For a point \( x = (x_0, x_1, \ldots) \in k^\omega \) we denote by \( x \cap n \) the \( n \)-tuple \((x_0, x_1, \ldots, x_{n-1})\). Let \( k^0 = 0 \) and \( k^1 = k \). If \( t \in k^0 \) and \( t_n \in k \), then \( t^t \in k^0 \). For an \( n \)-tuple \( t \in k^0 \) we denote by \( B_1(k) \) the Baire interval \( \{ x \in k^0: x \cap n = t \} \).

A space \( X \) is called of first category (or meager) if \( X = \bigcup \{ X_i: i \in \omega \} \), where each \( X_i \) is a nowhere dense set. We can assume without loss of generality that each \( X_i \) is a nowhere dense closed subset of \( X \).

A clopen set is a set which is both closed and open. A strongly zero-dimensional space \( X \) is called an \( h \)-homogeneous (or strongly homogeneous [9]) if every nonempty clopen subset of \( X \) is homeomorphic to \( X \). Each \( h \)-homogeneous space is homeomorphic, but the converse is false. For example, if we delete a point from the Cantor set \( C \), we obtain a homogeneous, not \( h \)-homogeneous space. For a space \( X \) let \( u(X) = \{ Y: \text{Ind } Y = 0 \} \) and every nonempty clopen subset \( V \subseteq Y \) contains a closed copy of \( X \). A space \( Y \) is called an \( u \)-homogeneous with respect to the space \( X \) if \( u(Y) \subseteq u(X) \). A space \( X \) is said to be \( u \)-homogeneous if \( X \) is \( u \)-homogeneous with respect to itself. It is clear that each \( h \)-homogeneous space is a \( u \)-homogeneous space. For a cardinal \( k \) put \( \mathcal{E}_k = \{ X: \text{Ind } X = 0 \} \) and every nonempty open subset of \( X \) has weight \( k \). Evidently, if \( X \) is a \( u \)-homogeneous space, then \( X \in \mathcal{E}_k \) for \( k = \text{w}(X) \).

For a family \( \mathcal{U} \) of subsets of \( X \) let \( \bigcup \mathcal{U} = \bigcup \{ U: U \in \mathcal{U} \} \) and mesh \( \mathcal{U} = \sup \{ \text{diam}(U): U \in \mathcal{U} \} \). Next, if \( f: X \rightarrow Y \) is a mapping, then we write \( f(\mathcal{U}) = \{ f(U): U \in \mathcal{U} \} \).

For a space \( X \) define a family \( LF(X) = \{ Y: \) each point \( y \in Y \) lies in some clopen neighborhood that is homeomorphic to a closed subset of \( Y \} \). A space \( Y \in \sigma LF(X) \) if \( Y = \bigcup \{ Y_i: i \in \omega \} \), where each \( Y_i \in LF(X) \) and each \( Y_i \) is a closed subset of \( Y \). From the locally finite sum theorem and the Katetov–Morita theorem (see [2]) we obtain the following lemma.

**Lemma 1.** If \( \text{Ind } X = 0 \) and \( Y \in \sigma LF(X) \), then \( \text{Ind } Y = 0 \).

**Lemma 2.** Suppose \( \text{Ind } X = 0 \), \( Y \in \sigma LF(X) \), and \( Y \) is a space of first category. Then \( Y = \bigcup \{ Y_i: i \in \omega \} \), where each \( Y_i \in LF(X) \) and each \( Y_i \) is a nowhere dense closed set of \( Y \).
Proof. $Y$ is a space of first category. Hence, $Y = \bigcup \{A_i : i \in \omega\}$, where each $A_i$ is a nowhere dense closed subset of $Y$. On the other hand, $Y = \bigcup \{B_j : j \in \omega\}$, where each $B_j$ is closed in $Y$ and $B_j \in \mathcal{L}(X)$. The nonempty intersections $A_i \cap B_j$ form the required cover of $Y$. \hfill \square

Lemma 3. Let $\text{Ind} \, X = 0$. If $Y \in \sigma \mathcal{L}(X)$ and $Z \in \sigma \mathcal{L}(Y)$, then $Z \in \sigma \mathcal{L}(X)$.

Proof. By the definition, $Z = \bigcup \{Z_n : n \in \omega\}$, where each $Z_n$ is closed in $Z$ and $Z_n \in \mathcal{L}(Y)$. Since $\text{Ind}(Z_n) = 0$ for any $n \in \omega$, we obtain $Z = \bigcup \{Z_n : n \in \omega\}$, where every $Z_n$ is closed in $Z$ and every $Z_n$ is homeomorphic to a closed subset of $Y$. Hence, $Z = \bigcup \{Z_n : n \in \omega\}$, where every $Z_n \in \mathcal{L}(X)$. The closed set $Z_n = \bigcup \{Z_n,i : i \in \omega\}$, where every $Z_n,i \in \mathcal{L}(X)$. The closed set $Z_n = \bigcup \{Z_n,i : i \in \omega\}$ belongs to the family $\mathcal{L}(X)$; thus, $Z = \bigcup \{Z_n,i : n \in \omega, \, i \in \omega\} \in \sigma \mathcal{L}(X)$. \hfill \square

2. Some properties of $h$-homogeneous spaces

In the next sections we shall use the following statements.

Theorem 1. Every $u$-homogeneous space of first category is $h$-homogeneous.

In [6] we proposed to consider the space $Q(k) = \{(x_0, x_1, \ldots) : x_i \in k^\omega, \forall i \in \omega) \}$ as a nonseparable collection of weight $k$ for the space $Q$ of rational numbers. A topological description of the space $Q(k)$ is given by the following theorem [6].

Theorem 2. Let $X$ be a $\sigma$-discrete metric space of weight $k$ that is homogeneous with respect to $\sigma$. Then $X$ is homeomorphic to $Q(k)$.

Theorem 3. Let $X$ be a $u$-homogeneous space of first category. If $Y \in \mathcal{L}(X)$ and $w(Y) \leq w(X)$, then $Y$ is homeomorphic to a nowhere dense closed subset of $X$. In particular, $X$ contains a nowhere dense closed copy of itself.

Lemma 4. Let $F$ be a nowhere dense closed subset of a strongly zero-dimensional space $X$, $G$ be a nowhere dense closed subset of an $h$-homogeneous space $Y$, $F \approx G$, and every nonempty clopen subset of $X$ contains a nowhere dense closed copy of $F$. Then there exists a nowhere dense closed subset $F \ast X$ of $X$ such that $F \subseteq F \ast$ and $F \ast \approx Y$.

Proof. Fix a homeomorphism $h_0 : F \rightarrow G$, a retraction $r_X : X \rightarrow X$ with $r_X(F) = F$, and a retraction $r_Y : Y \rightarrow Y$ with $r_Y(Y) = G$.

For each $i \in \omega$ we construct inductively families $U_i = \{U_i(\gamma) : \gamma \in I_i\}$ of disjoint clopen subsets of $X$ and $V_i = \{V_i(\gamma) : \gamma \in I_i\}$ of disjoint clopen subsets of $Y$ such that

1. $U_0 = \{X\}$ and $V_0 = \{Y\}$;
2. $\bigcup U_i$ is clopen in $X$ and $\bigcup V_i$ is clopen in $Y$;
3. $\bigcap \{U_i : i \in \omega\} = F$ and $\bigcap \{V_i : i \in \omega\} = G$;
4. mesh$U_i \leq i^{-1}$ and mesh$V_i \leq i^{-1}$ for $i \geq 1$;
5. each $U \in U_{i+1}$ is contained in some $U \ast \in U_i$ and each $V \in V_{i+1}$ is contained in some $V \ast \in V_i$;
6. for each $U \in U_i$ the set $\triangle U = U \cup U_{i+1}$ is nonempty and clopen in $X$ and for each $V \in V_i$ the set $\triangle V = V \cup V_{i+1}$ is nonempty and clopen in $Y$;
7. $h_0(U_{i}(\gamma)) \cap F = V_i(\gamma) \cap G$ for each $\gamma \in I_i$.

Define $U_0$ and $V_0$ as in (1). Now suppose $U_j$, $V_j$, and $I_j$ have been defined for $j \leq i$. Fix $\gamma \in I_i$. For simplicity, let $U = U_i(\gamma)$, $V = V_i(\gamma)$, and $A = (3i + 3)^{-1}$. Take a disjoint clopen (in $U \cap F$) cover $W_1$ of $U \cap F$ with mesh$W_1 \leq A$ and a disjoint clopen (in $V \cap G$) cover $W_2$ of $V \cap G$ with mesh$W_2 \leq A$. The nonempty intersections $\{A \cap B : A \in W_1, \, B \in h_0^{-1}(W_2)\}$ form the disjoint clopen cover $W$ of $U \cap F$ with mesh$W \leq A$. Then $h_0(W)$ is a disjoint clopen cover of $V \cap G$ with mesh$h_0(W) \leq A$. Since $F$ is nowhere dense in $X$, there exists a nonempty clopen subset $D_G \subseteq U$ such that $D_G \cap F = \emptyset$. The $A$-ball about $U \cap F$ contains a clopen (in $X$) subset $B_U$ such that $U \cap F \subseteq B_U$. Put $\triangle U = D_U \cup (U \setminus D_U)$ and $W_U = (B_U \setminus D_U) \cap r_A^{-1}(W) : W \in W_1)$. It is clear that mesh$W_U \leq 3A$. Similarly, the $A$-ball about $V \cap G$ contains a clopen (in $Y$) subset $B_V$ such that $V \cap G \subseteq B_V$ and $\triangle V = V \setminus B_V \neq \emptyset$. Let $V_A = (B_V \cap r_A^{-1}(W) : W \in h_0(W))$.

Define $U_{i+1} = \{V_A : U \in U_i\}$ and $V_{i+1} = \{V_A : V \in V_i\}$. By construction, we can take a set $I_{i+1}$ of indexes such that $U_{i+1} = \{U_{i+1}(\gamma) : \gamma \in I_{i+1}\}$, $V_{i+1} = \{V_{i+1}(\gamma) : \gamma \in I_{i+1}\}$, and (7) is satisfied. Clearly, mesh$U_{i+1} \leq 3A = (i + 1)^{-1}$ and mesh$V_{i+1} \leq (i + 1)^{-1}$. This completes the induction.

By construction, we have $Y = G \cup \bigcup \{\triangle V_i(\gamma) : \gamma \in I_i, \, i \in \omega\}$. 

For each \( i \in \omega \) and \( y \in I_i \), choose a nowhere dense closed subset \( F_i(y) \subseteq \Delta U_i(y) \) such that \( F_i(y) \approx Y \). Since \( \Delta V_i(y) \approx Y \), there exists a homeomorphism \( h_{i,y} : F_i(y) \rightarrow \Delta V_i(y) \). The set \( F^* = F \cup \bigcup (F_i(y) : y \in I_i, i \in \omega) \) is nowhere dense and closed in \( X \). Define a mapping \( h : F^* \rightarrow Y \) as follows: \( h(x) = h_0(x) \) if \( x \in F \); \( h(x) = h_{i,y}(x) \) if \( x \in F_i(y) \). We claim that \( h \) is a homeomorphism. Since \( F_i(y) \cap F_j(\beta) = \emptyset \) if \( (i, y) \neq (j, \beta) \), \( h \) is well defined. Clearly, \( h \) is a bijection. Since each \( F_i(y) \) is clopen in \( F^* \) and each \( \Delta V_i(y) \) is clopen in \( Y \), the restriction \( h|F^* \setminus F \) is a local homeomorphism. Take a point \( x \in F \). Let \( V \) be a neighbourhood of the point \( y = h(x) \in G \). By (3) and (4) there exist \( i \in \omega \) and \( y \in I_i \) such that \( y \in V_i(y) \subseteq V \). Using (7), we obtain that the set \( U_i(y) \cap F^* = h^{-1}(V_i(y)) \) is open in \( F^* \); hence, \( h \) is continuous at \( x \). Similarly, \( h^{-1} \) is continuous at any point \( y \in G \). \( \square \)

**Theorem 4.** Suppose \( X \) and \( Y \) are \( h \)-homogeneous spaces of first category such that \( w(X) = w(Y) \), \( Y \in \sigma LF(X) \), and \( X \in \sigma LF(Y) \). Then \( X \) is homeomorphic to \( Y \).

**Proof.** Since \( X \) is of first category, we have \( X = \bigcup \{ X_i : i \in \omega \} \), where each \( X_i \) is a nowhere dense closed subset of \( X \). From Theorem 3 it follows that \( Y \) contains a nowhere dense closed copy of \( X \). Then each \( X_i \) is homeomorphic to a nowhere dense closed subset of \( Y \). Similarly, each nonempty clopen subset of \( X \) contains a nowhere dense closed copy of \( Y \). Hence, by Lemma 4, for each \( i \in \omega \) there exists a nowhere dense closed subset \( X^*_i \) of \( X \) such that \( X_i \subseteq X^*_i \) and \( X^*_i \approx Y \). Clearly, \( X = \bigcup \{ X^*_i : i \in \omega \} \). By virtue of the Ostrovsky theorem [10], \( X \approx Q \times Y \). Theorem 3 implies that every nonempty clopen subset of \( Y \) contains a nowhere dense closed copy of \( Y \). As above, we have \( Y \approx Q \times X \). Thus, \( X \approx Y \). \( \square \)

**Remark 1.** Note that Lemma 4 is true (see [7]) for any metric space \( X \) (if we will use canonical closed subsets of \( X \) instead of clopen subsets). In this paper we deal with strongly zero-dimensional metric spaces; therefore, our proof is modified for this case.

Theorem 1 was established by Ostrovsky [12], Theorem 3 was proved in [8]. Under the additional condition "\( Y \) is of first category" Theorem 3 was obtained in [7]. Theorem 4 was proved in [7]. In [5] we obtained that if \( X \) and \( Y \) are both \( h \)-homogeneous spaces of first category such that each space contains a nowhere dense closed copy of another space, then \( X \approx Y \). This theorem was the first variant of Theorem 4. Ostrovsky [12] showed that if \( X \) and \( Y \) are spaces of first category such that \( X \in u(Y) \) and \( Y \in u(X) \), then \( X \approx Y \). Theorem 5 is new.

**Theorem 5.** Let \( X \) be an \( h \)-homogeneous space of first category. Then for any strongly zero-dimensional space \( Y \) the product \( Z = X \times Y^{\omega} \) is an \( h \)-homogeneous space of first category.

Furthermore, if \( X \times X \approx X \), then \( F \times Z \approx Z \) for any \( F_{\sigma} \)-subset \( F \) of \( Z \).

**Proof.** Clearly, \( Z \) is of first category. Let \( Y^{\omega} = \prod \{ Y_i : i \in \omega \} \), where each \( Y_i = Y \). By the definition of the topology of the Cartesian product, for each open subset \( U \) of \( Z \) there exist clopen subset \( W \subseteq X \), \( n \in \omega \), and open subsets \( V_i \subseteq Y_i \) for \( i \leq n \) such that \( U \) contains the product \( W \times \prod \{ V_i : i \leq n \} \). Fix a point \( q \in \prod \{ V_i : i \leq n \} \). Then \( W \times \{ q \} \times \prod \{ Y_i : i \geq n \} \) is homeomorphic to \( Z \). Hence \( Z \) is \( h \)-homogeneous. From Theorem 1 it follows that \( Z \) is an \( h \)-homogeneous space.

Now assume that \( X \times X \approx X \). Take an \( F_{\sigma} \)-subset \( F \) of \( Z \). It follows from Theorem 3 that \( F \) is homeomorphic to a closed subset \( H \) of \( Z \). Clearly, \( H \times Z \) is of first category. Using \( Z \approx (X \times Y^{\omega}) \times (X \times Y^{\omega}) \), we see that \( Z \) contains a closed copy of \( H \times Z \). Each clopen subset \( U \) of \( H \times Z \) contains a canonical subset \( U_1 \times U_2 \), where \( U_1 \) is a clopen subset of \( H \) and \( U_2 \) is a clopen subset of \( Z \). Since \( Z \) is an \( h \)-homogeneous space, we have \( U_2 \approx Z \). Then \( U \) contains a closed copy of \( H \times Z \). Hence, \( H \times Z \) is \( h \)-homogeneous. Theorem 1 implies that \( H \times Z \) is an \( h \)-homogeneous space. Hence, by Theorem 4, \( H \times Z \approx Z \). So, \( F \times Z \approx Z \). \( \square \)

**Corollary 1.** Let \( \text{Ind} \ Y = 0 \). Then \( Q \times Y^{\omega} \) is an \( h \)-homogeneous space of first category.

**Corollary 2.** Let \( \text{Ind} \ Y = 0 \) and \( F \) be an \( F_{\sigma} \)-subset of \( Q \times Y^{\omega} \). Then \( Q \times Y^{\omega} \approx F \times Q \times Y^{\omega} \).

**Corollary 3.** Let \( \text{Ind} \ Y = 0, Y^{\omega} \) be a space of first category, and \( F \) be an \( F_{\sigma} \)-subset of \( Y^{\omega} \). Then \( F \times Y^{\omega} \approx Y^{\omega} \).

**Proof.** Clearly, the space \( X = Y^{\omega} \) is \( h \)-homogeneous. By Theorem 1 \( X \) is \( h \)-homogeneous. Since \( Y^{\omega} \approx Y^{\omega} \times Y^{\omega} \), we can apply Theorem 5. \( \square \)

**Remark 2.** Ostrovsky [11] proved that if \( X \) contains a nonempty open subset of first category, then \( X^{\omega} \) is of first category.

3. \( h \)-Homogeneous extensions

**Theorem 6.** Let \( X \) be a strongly zero-dimensional metric space of weight \( \leq k \). There exists a unique (up to homeomorphism) \( h \)-homogeneous space \( Z \) of first category such that \( w(Z) = k, Z \in \sigma LF(X) \), and \( Z \in u(X) \).
Proof. Suppose \( w(X) = \tau \). The Baire space \( B(\tau) \) is universal for metrizable spaces of weight \( \leq \tau \) (see [2]); therefore, we can assume that \( X \subset B(\tau) \). For \( t = (t_0, \ldots, t_{\omega - 1}) \in \tau^t, i \in N \), we define a translation mapping \( \psi_t : B(\tau) \to B(\tau) \) by the rule \( \psi_t(x) = (0, 0, 1_{0-1}, x_0, x_1, \ldots) \). When \( i = 0 \) and \( t \in \tau^t \), let \( \psi_t \) be an identity mapping. It is clear that \( X = \psi_t(X) \) for every \( t \in \tau^t \). Let \( X_t = \bigcup \{ \psi_t(X) : t \in \tau^t \} \), where \( i \in \omega \).

Let \( D = \{ (i, 0, 0, \ldots) \} \). For any \( i \in N \) take a metrically discrete set

\[
D_i = \{ d = (d_0, d_1, \ldots) \in k^\omega : d_{i-1} 0, d_i = d_{i+1} = \cdots = 0 \};
\]
then the set \( D = \bigcup \{ D_i : i \in \omega \} \) is dense in the Baire space \( B(k) \) and \( D_i \cap D_j = \emptyset \) whenever \( i \neq j \). For any \( i \in \omega \) take the subset \( Z_i = X_i \times D_i \) of the product \( B(\tau) \times B(k) \). Then \( Z_i \cap Z_j = \emptyset \) whenever \( i \neq j \), and the set \( Z = \bigcup \{ Z_i : i \in \omega \} \) is nowhere dense in \( D \).

For every \( i \in \omega \) and for every point \( d \in D_i \) the set

\[
Z_i \cap (B(\tau) \times \{ d \}) = \bigoplus \{ \psi_t(X) : t \in \tau^t \} \in LF(X).
\]

Each set \( Z_i \) is closed in \( Z \); therefore, \( Z \in \sigma LF(X) \). Since every \( D_i \) is a nowhere dense subset of \( D \), we see that every \( Z_i \) is nowhere dense in \( Z \); this implies that \( Z \) is a set of first category.

The set \( D \) is dense in \( B(k) \) and the union \( \bigcup \{ X_i : i \in \omega \} \) is dense in \( B(\tau) \), hence the set \( Z \) is dense in the product \( B(\times) \times B(k) \). Thus, \( Z \in \mathcal{X}_k \).

Suppose we are given a canonical clopen set \( U_{ta} = Z \cap (B(\tau) \times B(\times)) \), where \( t \in \tau^t, \alpha = (\alpha_0, \ldots, \alpha_{i-1}) \in \alpha^t \), \( i \in N \), and \( \alpha_{i-1} \neq 0 \). Define a one-to-one mapping \( g_{ta} : Z \to U_{ta} \) by the rule \( g_{ta}(x, d) = (\psi_t(x), \varphi_0(d)) \) whenever \( \varphi_0(d) = (\alpha_0, \ldots, \alpha_{i-1}, d_0, d_1, \ldots) \). One can readily verify that \( g_{ta}(Z_i) = Z_{j+i} \cap U_{ta} \) for \( j \in \omega \). Since \( Z_j \cap U_{ta} = \emptyset \) for \( j < i \), we see that

\[
g_{ta}(Z) = \left( \bigcup \{ Z_{j+i} : j \in \omega \} \right) \cap U_{ta} = \left( \bigcup \{ Z_j : j \in \omega \} \right) \cap U_{ta} = Z \cap U_{ta} = U_{ta}.
\]

The translation mapping \( g_{ta} \) is continuous and open; hence, \( U_{ta} \) is a homeomorphism and \( U_{ta} \approx Z \).

For any clopen subset \( V \subset Z \) there exist some \( i \in N, t \in \tau^t, \) and \( \alpha \in \alpha^t \) such that \( \alpha_{i-1} \neq 0 \) and \( U_{ta} \subset V \); therefore, \( V \) contains a closed copy of \( Z \). Thus, \( Z \in u(Z) \). By virtue Theorem 1 the space \( Z \) is \( h \)-homogeneous. Let \( Y \) be any \( h \)-homogeneous space of first category such that \( Y \in \sigma LF(X), Y \in u(X), \) and \( w(Y) = k \). By Lemma 3 \( Y \in \sigma LF(Z) \) and \( Z \in \sigma LF(Y) \). It follows from Theorem 4 that the spaces \( Y \) and \( Z \) are homeomorphic. \( \square \)

Remark 3. The set \( Z \) that is constructed in Theorem 6 is called the \( h \)-homogeneous extension of the space \( X \) of weight \( k \) with respect to spaces of first category and is denoted by \( h(X, k) \). According to Theorem 4, the topological characteristic of the space \( Z \) does not depend on a choice of an embedding of \( X \) into the Baire space \( B(\tau) \). So, our definition is well defined. It follows from Theorem 8 that this definition is equivalent to the definition of \( h(X, k) \) which is given in the Introduction.

Remark 4. The Cantor set \( C = 2^\omega \) is universal for separable zero-dimensional metric spaces. Therefore, in the case \( k = \omega \) we can modify the proof of Theorem 6 and construct the set \( Z \subset 2^\omega \times 2^\omega \).

Corollary 4. Let \( X \) be an \( h \)-homogeneous space of first category and of weight \( k \). Then \( X \) is homeomorphic to \( h(X, k) \).

Corollary 5. Let \( \text{Ind } X = 0 \) and \( w(X) \leq k \). Then \( h(X, k) \approx h(h(X, k), k) \).

Proof. Using Corollary 4 for a space \( Y = h(X, k) \), we obtain the corollary. \( \square \)

The following theorem observes the simplest case for constructing the \( h \)-homogeneous extension of a space.

Theorem 7. Let \( X \) be an \( h \)-homogeneous space of weight \( \leq k \). Then the space \( h(X, k) \) is homeomorphic to the product \( Q(k) \times X \).

Proof. Note that the space \( Q(k) \) is \( \sigma \)-discrete and \( Q(k) \in \mathcal{X}_k \); therefore, the product \( Q(k) \times X \) satisfies all the conditions of Theorem 6. This implies that \( h(X, k) \approx Q(k) \times X \). \( \square \)

Corollary 6. Let \( \text{Ind } X = 0 \) and \( w(X) \leq k_1 \leq k_2 \). Then \( h(X, k_2) \approx Q(k_2) \times h(X, k_1) \).

Corollary 7. Let \( \text{Ind } X = 0 \) and \( w(X) \leq k \). Then \( h(X, k) \approx Q \times h(X, k) \).

Theorem 8. Let \( X \) be a space of weight \( \leq k \), \( \text{Ind } X = 0 \), and \( Y \in \mathcal{H}_k(X) \). Then \( Y \) contains a nowhere dense closed copy of \( h(X, k) \).

Proof. Since \( h(X, k) \in \sigma LF(Y) \), the theorem follows from Theorem 3. \( \square \)
Theorem 9. Let $\text{Ind} X = 0$, $Y \in \sigma LF(X)$, and $k \geq \max\{w(X), w(Y)\}$. Then $h(Y, k)$ is homeomorphic to a nowhere dense closed subset of $h(X, k)$.

Proof. From Lemma 3 and Theorem 3 it follows that the space $h(X, k)$ contains a closed copy of $Y$. Then $h(X, k) \in \mathcal{H}_k(Y)$. Theorem 8 implies that $h(X, k)$ contains a nowhere dense closed copy of $h(Y, k)$. □

Theorem 10. Suppose $X$ and $Y$ are strongly zero-dimensional metric spaces of weight $\leq k$. Then $h(X, k)$ is homeomorphic to $h(Y, k)$ if and only if $X \in \sigma LF(Y)$ and $Y \in \sigma LF(X)$.

Proof. Since $h(X, k) \approx h(Y, k)$, we have $h(X, k) \in \sigma LF(Y)$. By definition, the space $h(X, k)$ contains a closed copy of $X$. Then $X \in \sigma LF(Y)$. Similarly, we obtain the second inclusion.

Conversely, Lemma 3 implies that if $X \in \sigma LF(Y)$, then $h(X, k) \in \sigma LF(h(Y, k))$. Likewise, $h(Y, k) \in \sigma LF(h(X, k))$. It remains to apply Theorem 4. □

Remark 5. Theorem 10 shows when spaces have homeomorphic $h$-homogeneous extensions.

Theorem 11. Suppose $\text{Ind} X = \text{Ind} Y = 0$ and $k \geq \max\{w(X), w(Y)\}$. Then the product $h(X, k) \times h(Y, k)$ is homeomorphic to $h(X \times Y, k)$.

Proof. Let $X$ and $Y$ be subsets of the Baire space $B(\tau)$, where $\tau = \max\{w(X), w(Y)\}$. As above, for $t \in \tau^I$, $i \in N$, define a translation mapping $\psi_t : B(\tau) \rightarrow B(\tau)$. For any $i \in N$ take sets $D_i = \{(d_0, d_1, \ldots) \in k^\omega : d_{i-1} \neq 0, d_i = \cdots = 0\}$, $X_i = \bigcup\{\psi_t(X) : t \in \tau^I\}$, $Y_i = \bigcup\{\psi_t(Y) : t \in \tau^I\}$, and $Z_i = X_i \times Y_i \times D_i$. For example,

$$Z_1 = \bigoplus\{\psi_t(X) \times \psi_t(Y) \times \{(d_0, 0, 0, \ldots)\} : t \in \tau, s \in \tau, d_0 \in k, d_0 \neq 0\}.$$ 

The set $Z = \bigcup\{Z_i : i \in N\}$ is dense in the product $B(\tau) \times B(\tau) \times B(k)$; hence, $Z \in \mathcal{E}_k$. Each $Z_i$ is a nowhere dense closed subset of $Z$; this implies that $Z$ is a set of first category and $Z \in \sigma LF(X \times Y)$. By construction, $Z \in u(X \times Y)$; therefore, from Theorem 6 it follows that $Z \approx h(X \times Y, k)$.

For any $i \in N$ take sets $X_i = X_i \times D_i$ and $Y_i = Y_i \times D_i$ in the product $B(\tau) \times B(k)$. As above, the set $\hat{X} = \bigcup\{\hat{X}_i : i \in N\}$ is homeomorphic to $h(X, k)$ and the set $\hat{Y} = \bigcup\{\hat{Y}_i : i \in N\}$ is homeomorphic to $h(Y, k)$. The product of two $h$-homogeneous spaces is $h$-homogeneous (see [7, Lemma 2.3]); hence, the space $\hat{X} \times \hat{Y}$ is $h$-homogeneous.

Since the set $\hat{X} \times \hat{Y}$ contains a closed copy of $X \times Y$, we see that each $Z_i \in LF(\hat{X} \times \hat{Y})$; hence, $Z \in \sigma LF(\hat{X} \times \hat{Y})$. Similarly, $\hat{X} \times \hat{Y} \in \sigma LF(Z)$. The spaces $Z$ and $\hat{X} \times \hat{Y}$ are sets of first category and $w(Z) = w(\hat{X} \times \hat{Y})$. Now by Theorem 4 the spaces $Z$ and $\hat{X} \times \hat{Y}$ are homeomorphic. □

Corollary 8. Suppose $\text{Ind} X = \text{Ind} Y = 0$, $w(X) \leq k_1$, $w(Y) \leq k_2$, and $k = \max\{k_1, k_2\}$. Then $h(X, k_1) \times h(Y, k_2)$ is homeomorphic to $h(X \times Y, k)$.

Proof. Let $k_1 \leq k_2$. From Corollary 6 and Theorem 11 it follows that

$$h(X, k_1) \times h(Y, k_2) \approx h(X, k_1) \times (Q(k) \times h(Y, k_2)) \approx (Q(k) \times h(X, k_1)) \times h(Y, k_2) \approx h(X, k) \times h(Y, k) \approx h(X \times Y, k).$$ □

Example 1. Let $X$ be a one-point set. Then $h(X, k) \approx Q(k)$.

Example 2. Let $C$ be the Cantor set. Then $h(C \oplus \{\text{point}\}, k) \approx Q(k) \times C$.

4. Canonical elements

Recall the definition of a canonical element. Let $\mathcal{U}$ be a family of subsets of the Baire space $B(k)$. A set $U \in \mathcal{U}$ is called universal for $\mathcal{U}$ if for each $X \in \mathcal{U}$ there exists a perfect set $P \subset B(k)$ such that $P \cap U \approx X$. A set $E \subset B(k)$ of first category is called a canonical element for $\mathcal{U}$ if it is everywhere universal for $\mathcal{U}$.

Lemma 5. For every subset $A \subset B(k)$ of first category there exists a closed subset $H \subset B(k)$ such that $H \cap A = \emptyset$ and $H \approx B(k)$.

Proof. Let $A = \bigcup\{A_i : i \in \omega\}$, where each $A_i$ is nowhere dense in $B(k)$. Then the closure $\overline{A_i}$ is nowhere dense in $B(k)$. Clearly, $A \subset \bigcup\{\overline{A_i} : i \in \omega\}$. 
For each $i \in \omega$ we define inductively a family $\mathcal{V}_i = \{V_i(t): t \in k^i\}$ of disjoint clopen subsets of $B(k)$ such that

1. $\bigcup \mathcal{V}_i$ is clopen in $B(k)$ and mesh $\mathcal{V}_i \leq (i + 1)^{-1}$;
2. $\bigcup \{V_{i+1}(t^n t_i): t_i \in k \}$ for each $t \in k^i$ and $\bigcup \mathcal{V}_{i+1} \subseteq \bigcup \mathcal{V}_i$;
3. $(\bigcup \mathcal{V}_i) \cap (\bigcup (\overline{\mathcal{A}_j}: j \leq i)) = \emptyset$.

Since $\mathcal{A}_0^k$ is nowhere dense in $B(k)$, there exists a clopen subset $V_0(0)$ such that $V_0(0) \cap \mathcal{A}_0^k = \emptyset$. Let $\mathcal{V}_0 = \{V_0(0)\}$. Assume that the families $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_i$ are already defined. Fix a $t \in k^i$. The set $\mathcal{A}_{i+1}^* = \bigcup \{A_j: j \leq i + 1\}$ is nowhere dense in $B(k)$. Then the set $V_i(t) \setminus \mathcal{A}_{i+1}^*$ contains a Baire interval $\mathcal{B}(s,k^m)$ for a tuple $s \in k^m$ with $m \geq i + 1$. Let $V_{i+1}(t^n t_i) = \mathcal{B}(s,k^m)$ for each $t_i \in k$. Define $\mathcal{V}_{i+1} = \{V_{i+1}(t^n t_i): t \in k^i, t_i \in k\}$. Clearly, the conditions (1)–(3) are satisfied. This completes the induction.

The set $H = \bigcap \{V_i: i \in \omega\}$ is closed in $B(k)$. For each $\xi \in k^\omega$, by the Cantor theorem (see [2]), the intersection $\bigcap \{V_i(\xi | i): i \in \omega\}$ is a nonempty one-point set. By construction, $H \approx B(k)$. It follows from (3) that $H \cap (\bigcup (\overline{\mathcal{A}_j}: i \in \omega)) = \emptyset$. Thus, $H \cap \mathcal{A} = \emptyset$. □

**Theorem 12.** Let $E$ be an $h$-homogeneous subset of first category of $B(k)$, $w(E) = k$, and $\mathcal{U}$ be a family of subsets of $B(k)$. Then $E$ is a canonical element for $\mathcal{U}$ if and only if $\mathcal{U} \subseteq \{T \subseteq B(k): T \in \sigma LF(Y^\omega)\}$ and $E \in \mathcal{U}$.

**Proof.** Let $E$ be a canonical element for $\mathcal{U}$. By definition, $E \in \mathcal{U}$ and for any $X \in \mathcal{U}$ there exists a perfect set $P \subseteq B(k)$ such that $P \cap E \approx X$. Then the set $P \cap E$ is closed in $E$. So, $\mathcal{U} \subseteq \{T \subseteq B(k): T \in \sigma LF(E)\}$.

Conversely, since $E \in \mathcal{U}$, we have $E \in \mathcal{U}(E)$. Theorem 1 implies that $E$ is $h$-homogeneous. Take a clopen (in $E$) subset $V \subseteq E$ and any $T \in \mathcal{U}$. Then $T \in \sigma LF(E)$. By Theorem 3 $E$ contains a nowhere dense closed copy of $T$. Since $E$ is $h$-homogeneous, $T \approx F$ for a nowhere dense closed subset $F \subseteq V$. Then the closure $\overline{F}$ is nowhere dense in $B(k)$. By Lemma 4 with $Y = B(k)$, there exists a nowhere dense closed subset $F^* \subseteq B(k)$ such that $F \subseteq F^*$ and $F^* \approx B(k)$. Clearly, $F^*$ is a perfect subset of $B(k)$. From Lemma 5 it follows that we can choose the closed sets $F_\gamma(\gamma) \subseteq \Delta_{U_\gamma}(\gamma)$, which were constructed in the proof of Lemma 4, such that $F_\gamma(\gamma) \cap E \approx \emptyset$. Since $F^* \approx F \cup \bigcup \{F_\gamma(\gamma): \gamma \in I_\gamma, i \in \omega\}$, we have $F^* \cap F = \approx \approx T$. Thus, $V$ is universal for $\mathcal{U}$. □

**Theorem 13.** Let Ind $Y = 0$, $w(Y) \leq k$, and $E \subset B(k)$. Then $E$ is a canonical element for the family $\mathcal{U} = \{T \subseteq B(k): T \in \sigma LF(Y^\omega)\}$ if and only if $E \approx Q(k) \times Y^\omega$.

**Proof.** According to Theorem 5, $X = Q(k) \times Y^\omega$ is an $h$-homogeneous space of first category. Clearly, $w(X) = k$. Since the space $Q(k)$ is $\sigma$-discrete, we have $X \in \sigma LF(Y^\omega)$. From Lemma 3 it follows that $\sigma LF(X) = \sigma LF(Y^\omega)$.

Suppose $E$ is a canonical element for the family $\mathcal{U}$. By definition, $E$ is of first category and $E \in \mathcal{U}(E)$. Theorem 1 implies that $E$ is $h$-homogeneous. The set $E$ contains a closed copy of $X$. Then $w(E) = k$. Since $E \in \sigma LF(X)$, Theorem 4 implies that $E \approx X \approx Q(k) \times Y^\omega$.

Now, let $E \approx Q(k) \times Y^\omega$. Since $\sigma LF(X) = \sigma LF(Y^\omega)$, Theorem 12 implies that $E$ is a canonical element for the family $\{T \subseteq B(k): T \in \sigma LF(Y^\omega)\}$. □

5. Applications to extended Borel sets

Below we obtain a scheme of an employment Theorems 6 and 12 respect to the family of extended Borel sets. The following theorem was proved in [7].

**Theorem 14.** Let $X$ be an absolutely $\mathcal{F}_\omega^\text{co}$-set, $X \in \mathcal{E}_k$, and $X$ be nowhere $\sigma$-discrete and nowhere compact. Then the space $X$ is homeomorphic to the product $Q(k) \times \mathcal{C}$, where $\mathcal{C}$ is the Cantor set.

**Proof.** By virtue of the Stone theorem [15] $X$ is a $\sigma$-locally compact space, so $X \in \sigma LF(C)$. Since $X$ is a nowhere compact space, we see that $X$ is a set of first category. Under the conditions of the theorem, $X$ is a nowhere $\sigma$-discrete space; hence by the El’kin theorem [1] we have $X \in u(C)$. From Theorem 6 it follows that $X \approx Q(k) \times C$. □

Hansell [3] defined the family of extended Borel sets of a space $X$ to be the smallest $\sigma$-algebra of subsets of $X$, containing the opens sets, and closed with respect to unions of discrete subfamilies. The extended Borel sets of a separable space coincide with the classical Borel sets. For an ordinal $\alpha$ let $\Sigma^0_\alpha(k)$ and $\Pi^0_\alpha(k)$ denote, respectively, the extended Borel subsets of the Baire space $B(k)$ of additive and multiplicative class $\alpha$, starting with $\Sigma^0_1(k) = \text{Open}$ and $\Pi^0_1(k) = \text{Closed}$; in particular, $\Pi^0_0 = \mathcal{F}_\omega$. The successor of a cardinal $k$ is denoted by $k^*$. From the classification theorem (see [3, Theorem 2.7]) it follows that for every extended Borel set $X \subseteq B(k)$ there exists an ordinal $\alpha$ such that $X \in \Sigma^0_\alpha(k) \cup \Pi^0_\alpha(k)$ and $\alpha < k^*$. On the other hand, for each ordinal $\alpha$, where $\alpha < k^*$, Hansell [3] constructed subsets $M_\alpha(k)$ and $A_\alpha(k)$ of $B(k)$ of multiplicative and additive class $\alpha$, respectively, and which are not of any lower class. Hansell used the Hausdorff classification for Borel
sets (families $G_0$ and $F_0$ instead of $\Sigma^0_4(k)$ and $\Pi^0_4(k)$); therefore, we slightly change the definitions of $M_\alpha(k)$ and $A_\alpha(k)$. Fix homeomorphisms $g : B(k)^\omega \to B(k)$ and $h : (kB(k))^\omega \to B(k)$. Let $M_1(k)$ be any (fixed) one-point subset of $B(k)$ and $A_1(k) = B(k) \setminus M_1(k)$. If $\alpha = \beta + 1$, we define $M_\alpha(k) = g((A_\beta(k))^\omega)$ and $A_\alpha(k) = B(k) \setminus M_\alpha(k)$. If $\alpha$ is a limit ordinal, we first define $D_\alpha(k) = \bigoplus \{ A_\beta : \beta < \alpha \} \subset B(k)$, and put $C_\alpha(k) = kB(k) \setminus D_\alpha(k)$. Then $M_\alpha(k) = h((C_\alpha(k))^\omega)$ and $A_\alpha(k) = B(k) \setminus M_\alpha(k)$. Thus the sets $M_\alpha(k)$ and $A_\alpha(k)$ are defined for all $\alpha < k^+$. Note that $w(M_\alpha(k)) = w(A_\alpha(k)) = k$ for all $1 \leq \alpha < k^+$ (except $M_1(k)$).

We say [see [16]] that a space $X$ is $\sigma$-locally of weight $< \tau$ (in symbols, $X \in \sigma LW(< \tau)$) if $X = \bigcup \{ X_i : i \in \omega \}$, where each $X_i$ (in its relative topology) is locally of weight $< \tau$. In particular, a space $X \in \sigma LW(< \omega^\omega)$ if it is a countable union of locally separable sets. Stone [16] proved that the Baire space $B(\tau)$ is nowhere $\sigma LW(< \tau)$.

Let $\Pi^0_\alpha(\tau,k) = \Pi^0_\alpha(\tau) \cap \sigma LW(< \tau)$ and $\Sigma^0_\alpha(\tau,k) = \Sigma^0_\alpha(\tau) \cap \sigma LW(< \tau)$. For a cardinal $k$ take a cardinal $\tau \leq k$ and an ordinal $\alpha < \tau^+$. A set $E \subset B(k)$ of first category is called a canonical element of multiplicative class $\alpha$ and of type $(k, \tau)$ if every nonempty clopen in $E$ subset $V \subset E$ is universal for the family $\Pi^0_\alpha(\tau,k)$. Clearly, $E \in \mathcal{E}_k$. Similarly, we define a canonical element of additive class $\alpha$ and of type $(k, \tau)$.

**Lemma 6.** Let $\alpha < \tau^+$. Then every $\Pi^0_\alpha(\tau,k)$-set is homeomorphic to a closed subset of $B(\tau) \times M_\alpha(\tau)$ and every $\Sigma^0_\alpha(\tau,k)$-set is homeomorphic to a closed subset of $B(\tau) \times A_\alpha(\tau)$.

**Proof.** Let $X \in \Pi^0_\alpha(\tau,k)$. Hansel proved [see [3, Lemma 2.18]] that there exists a continuous mapping $f : B(\tau) \to B(\tau)$ such that $X = f^{-1}(M_\alpha(\tau))$. The graph $G(f)$ is closed in $B(\tau) \times B(\tau)$. Hence, the intersection $Y = G(f) \cap (B(\tau) \times M_\alpha(\tau))$ is a closed subset of $B(\tau) \times M_\alpha(\tau)$. It is clear that $X \approx Y$. The proof for the second part of the lemma is similar. □

**Theorem 15.** Let $\omega \leq \tau \leq k$ and $3 \leq \alpha < \tau^+$. A subset $E \subset B(k)$ is a canonical element of multiplicative class $\alpha$ and of type $(k, \tau)$ if and only if $E$ is homeomorphic to the space $Q(k) \times M_\alpha(\tau)$.

**Proof.** Define $Y$ as follows: $Y = A_\beta(\tau)$ if $\alpha = \beta + 1$; or $Y = C_\alpha(\tau)$ if $\alpha$ is a limit ordinal. By construction, $Q \times M_\alpha(\tau) \approx Q \times Y^\omega$.

Let us show that the family $\mathcal{U} = \{ T \subset B(k) : T \in \sigma LF(Y^\omega) \}$ coincides with $\Pi^0_\alpha(\tau,k)$. Take any $T \in \mathcal{U}$. Then $T = \bigcup \{ T_{i,y} : i \in \omega, y \in \Gamma_i \}$, where each $T_{i,y}$ is homeomorphic to a closed subset of $Y^\omega$, $T_{i,y} \cap T_{i,\delta} = \emptyset$ whenever $\gamma \neq \delta$, the set $T_i = \bigcup \{ T_{i,y} : y \in \Gamma_i \}$ is closed in $T$ for each $i$, and each $T_{i,y}$ is clopen in $T_i$. Since $w(Y^\omega) = \tau$, we have $T \in \sigma LW(< \tau)$. Since $\alpha \geq 3$, we see [3] that $Y^\omega$ is absolutely of multiplicative class $\alpha$. Hence, each $T_i$ is locally of multiplicative class $\alpha$ in $B(k)$. This implies that each $T_i$ is of multiplicative class $\alpha$ (see [3, Theorem 2.2]). Now, $B(k) \setminus T = \left( B(k) \setminus \bigcup \{ T_i : i \in \omega \} \cup \left( \bigcup \{ T_i \setminus T_i : i \in \omega \} \right) \right) \in \Sigma^0_\alpha(k)$.

Whence, $T \in \Pi^0_\alpha(\tau,k)$. Thus, $\mathcal{U} \subseteq \Pi^0_\alpha(\tau,k)$.

Conversely, take any $T \in \Pi^0_\alpha(\tau,k)$. Since $T$ is $\sigma$-locally of weight $< \tau^+$, we have $T = \bigcup \{ A_i : i \in \omega \}$, where each $A_i$ is a closed subset of $T$ and each $A_i$ is locally of weight $\leq \tau$. Then each $A_i \approx \bigoplus \{ A_{i,y} : y \in \Gamma_i \}$, where $w(A_{i,y}) \leq \tau$ for every $\gamma \in \Gamma_i$ (see [15]). Since $A_{i,y}$ is absolutely of multiplicative class $\alpha$, it follows from Lemma 6 that $A_{i,y}$ is homeomorphic to a closed subset of $B(\tau) \times M_\alpha(\tau)$. On the other hand, $w(Y) = \tau$ and $Y$ is nowhere compact. Then the space $Y$ contains a closed discrete subset of cardinality $\tau$. Hence, $Y^\omega$ contains a closed copy of $B(\tau) = \tau^\omega$. Corollary 2 implies that $B(\tau) \times Q \approx Q \times Y^\omega$. Then each $A_{i,y}$ is homeomorphic to a closed subset of $Q \times Y^\omega$. Since $Q \times Y^\omega \in \sigma LF(Y^\omega)$, by Lemma 3 we obtain $T \in \sigma LF(Y^\omega)$. Clearly, $T \subset B(k)$. Thus, $\mathcal{U} \supseteq \Pi^0_\alpha(\tau,k)$.

So, we have $\mathcal{U} = \Pi^0_\alpha(\tau,k)$. Using Theorem 13, we conclude the proof of the theorem. □

**Corollary 9.** Let $\omega \leq \tau \leq k$, $3 \leq \alpha < \tau^+$, and $X \in \Pi^0_\alpha(k)$. Then $X \times Q(k) \times M_\alpha(\tau) \approx Q(k) \times M_\alpha(\tau)$.

**Proof.** It follows from Theorem 15 that $X$ is homeomorphic to a closed subset $F \subset Q(k) \times M_\alpha(\tau)$. By construction, $(M_\alpha(\tau))^\omega \approx M_\alpha(\tau)$. It remains to apply Theorem 5 with $Z = Q(k) \times M_\alpha(\tau)$. □

**Remark 6.** In reality, in the proof of Theorem 15 we establish that if $\omega \leq \tau \leq k$ and $2 \leq \alpha < \tau^+$, then $h(B(\tau) \times M_\alpha(\tau), k) \approx Q(k) \times M_\alpha(\tau)$. Note that the space $Q(k) \times M_2(\tau) \approx Q(k) \times B(\tau)$ is not absolutely of multiplicative class $\alpha = 2$; therefore, in Theorem 15 we deal with $\alpha \geq 3$.

In the separable case, i.e., $k = \omega$, Lemma 6 and Theorem 16 were obtained by Ostrovsky [12].

**Theorem 16.** Let $3 \leq \alpha < \omega^\omega$. Suppose $X \subset B(k)$ is of first category, $X \in \mathcal{E}_k$, $X \in \Pi^0_\alpha(k,\omega)$, and $X$ is nowhere of additive class $\alpha$. Then $X$ is homeomorphic to the space $Q(k) \times M_\alpha(\omega)$.

**Proof.** Since $X$ is a $\sigma$-locally separable space, we have $X = \bigcup \{ X_i : i \in \omega \}$, where each $X_i$ is a locally separable closed subset of $X$. Then each $X_i$ is of multiplicative class $\alpha$. 

Take a nonempty clopen subset $U \subset X$. Since $X$ is nowhere of additive class $\alpha$, there exists $j$ such that $U \cap X_j$ is not of additive class $\alpha$. It follows from local separability of $X_j$ that there exists a separable clopen (in $U \cap X_j$) subset $V$ of $U \cap X_j$ such that $V$ is of multiplicative class $\alpha$ but not of additive class $\alpha$. Now, by the Harrington theorem (see [14, Lemma 3]) the set $V$ contains a closed copy of $M_{\alpha}(\omega)$; hence, $X \in u(M_{\alpha}(\omega))$. As above, $X \in \sigma LF(M_{\alpha}(\omega))$. By Theorem 6, we have $X \approx h(M_{\alpha}(\omega), k) \approx Q(k) \times M_{\alpha}(\omega)$. □

**Theorem 17.** Let $\omega \leq \tau \leq k$ and $3 \leq \alpha < \tau^\dagger$. A subset $E \subset B(k)$ is a canonical element of additive class $\alpha$ and of type $(k, \tau)$ if and only if $E$ is homeomorphic to $h(B(\tau) \times A_\alpha(\tau), k)$.

**Proof.** By $G$ we denote the product $B(\tau) \times A_\alpha(\tau)$. We shall verify that the family $\mathcal{U} = \{ T \subset B(k) : T \in \sigma LF(G) \}$ coincides with $\Sigma_0^G(k, \tau)$. Let any $T \in \mathcal{U}$. Then $T = \bigcup \{ T_{i,y} : i \in \omega, y \in \Gamma \}$, where each $T_{i,y}$ is homeomorphic to a closed subset of $G$, $T_{i,\gamma} \cap T_{i,\delta} = \emptyset$ whenever $\gamma \neq \delta$, the set $T_i = \bigcup \{ T_{i,y} : y \in \Gamma \}$ is closed in $T$ for each $i$, and each $T_{i,y}$ is clopen in $T_i$. Since $w(G) = \tau$, we have $T \in \sigma LW(\tau^\dagger)$. Since $G$ is absolutely of additive class $\alpha$, we have that each $T_i$ is locally of additive class $\alpha$ in $B(k)$. Then each $T_i$ is of additive class $\alpha$. Whence, $T \in \Sigma_0^G(k, \tau)$. Thus, $\mathcal{U} \subseteq \Sigma_0^G(k, \tau)$.

Conversely, for any $T \in \Sigma_0^G(k, \tau)$ we have $T = \bigcup \{ A_i : i \in \omega \}$, where each $A_i$ is a closed subset of $T$ and each $A_i$ is locally of weight $\leq \tau$. Then each $A_i \approx \bigoplus \{ T_{i,y} : y \in \Gamma \}$, where $w(A_i) \leq \tau$ for every $y \in \Gamma$ (see [15]). As a closed subset of $T$, each $A_i$ is absolutely of additive class $\alpha$. It follows from Lemma 6 that each $A_i, y$ is homeomorphic to a closed subset of $G$. Then $T \in \sigma LF(G)$. Thus, $\mathcal{U} \subseteq \Sigma_0^G(k, \tau)$. So, we have $\mathcal{U} = \Sigma_0^G(k, \tau)$.

Let $E$ be a canonical element for $\Sigma_0^G(k, \tau)$. Then $E$ is a canonical element for $\mathcal{U}$. By definition, we have $E \in u(G)$, $E \in \sigma LF(G)$, and $E$ is of first category. Clearly, $E$ contains a closed copy of $Q(k) \in \Sigma_0^G(k, \omega)$. Then $w(E) = k$. From Theorem 6 it follows that $E \approx h(G, k)$.

Conversely, if $E \approx h(G, k)$, then $E \in \mathcal{U}$ and $\mathcal{U} \subseteq \sigma LF(E)$. Theorem 12 implies that $E$ is a canonical element for $\Sigma_0^G(k, \tau)$. □

**Theorem 18.** Let $3 \leq \alpha < \omega^\dagger$. Suppose $X \subset B(k)$ is of first category, $X \in E_k$, $X \in \Sigma_0^G(k, \omega)$, and $X$ is nowhere of multiplicative class $\alpha$. Then $X$ is homeomorphic to $h(B(\omega) \times A_\alpha(\omega), k)$.

**Proof.** Since $X$ is a $\sigma$-locally separable space, we have $X = \bigcup \{ X_i : i \in \omega \}$, where each $X_i$ is a locally separable closed subset of $X$. Then $X_i \in \Sigma_0^G(k)$. From Lemma 6 it follows that $X \in \sigma LF(B(\omega) \times A_\alpha(\omega))$.

Take a clopen subset $U \subset B(k)$ with $U \cap X \neq \emptyset$. Under the conditions of the theorem, $U \cap X$ is not of multiplicative class $\alpha$; hence, $(U \setminus X) \not\in \Sigma_0^G(k)$. Assume that each $U \cap (X_i \setminus X) \in \Sigma_0^G(k)$. Since $\alpha \geq 3$, we have

$$U \setminus X = \left( U \setminus \bigcup \{ X_i : i \in \omega \} \right) \cup \left( \bigcup \{ X_i \setminus X : i \in \omega \} \right) \in \Sigma_0^G(k),$$

a contradiction. Hence, there exists $j$ such that $U \cap (X_j \setminus X_j)$ is of multiplicative class $\alpha$ but not of additive class $\alpha$. Then $U \cap X_j \in \Sigma_0^G(k) \setminus \Pi_0^G(k)$.

It follows from local separability of $X_j$ that there exists a separable clopen (in $U \cap X_j$) subset $V$ of $U \cap X_j$ such that $V \in \Sigma_0^G(k) \setminus \Pi_0^G(k)$. By the Harrington theorem the set $V$ contains a closed copy of $B(\omega) \times A_\alpha(\omega)$; hence, $X \in u(B(\omega) \times A_\alpha(\omega))$. By Theorem 6 we have $X \approx h(B(\omega) \times A_\alpha(\omega), k)$. □

**Remark 7.** In Theorem 18 we can take any set $Y \in \Sigma_0^G(\omega) \setminus \Pi_0^G(\omega)$ instead of $B(\omega) \times A_\alpha(\omega)$. By the theorem of Harrington (see [14]) $Y$ contains a closed copy of $B(\omega) \times A_\alpha(\omega)$; hence, $X \in u(B(\omega) \times A_\alpha(\omega))$. Lemma 6 implies that $B(\omega) \times A_\alpha(\omega)$ contains a closed copy of $Y$. From Theorem 10 it follows that $h(B(\omega) \times A_\alpha(\omega), k) \approx h(Y, k)$.

If we had embedding theorems for the sets $M_\alpha(\tau)$ and $A_\alpha(\tau)$ with $\tau > \omega$, we would prove theorems (similar to Theorems 16 and 18) about extended Borel sets for nonseparable ordinals $\alpha$.

**Acknowledgement**

I thank the referee for his helpful comments.

**References**


