Generalized bi-quasi-variational inequalities for quasi-pseudo-monotone type I operators on non-compact sets

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ABSTRACT
In this paper, the authors prove some existence results for solutions for a new class of generalized bi-quasi-variational inequalities (GBQVI) for quasi-pseudo-monotone type I and strongly quasi-pseudo-monotone type I operators defined on non-compact sets in locally convex Hausdorff topological vector spaces. In obtaining these results on GBQVI for quasi-pseudo-monotone type I and strongly quasi-pseudo-monotone type I operators, we shall use Chowdhury and Tan’s generalized version of Ky Fan’s minimax inequality as the main tool.

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1. Introduction

The generalized bi-quasi-variational inequality problem was first introduced by Shih and Tan [1] in 1989. Since Shih and Tan’s work, some authors have obtained many results on generalized (quasi-)variational inequalities, generalized (quasi-) variational-like inequalities and generalized bi-quasi-variational inequalities (see [2–17] and [18]).

The following is the definition due to Shih and Tan [1]:

Let $E$ and $F$ be vector spaces over $\Phi$, $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional and $X$ be a non-empty subset of $E$. If $S : X \rightarrow 2^X$ and $M, T : X \rightarrow 2^F$, the **generalized bi-quasi-variational inequality problem** (GBQVI) for the triple $(S, M, T)$ is to find $\hat{y} \in X$ satisfying the following properties:

1. $\hat{y} \in S(\hat{y})$,
2. $\inf_{w \in T(\hat{y})} \Re \langle f - w, \hat{y} - x \rangle \leq 0$ for any $x \in S(\hat{y})$ and $f \in M(\hat{y})$.

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We shall now give some very basic notation, definitions and concepts which will be used throughout this paper.

Let $X$ be a non-empty set, $2^X$ be the family of all non-empty subsets of $X$, and $\mathcal{F}(X)$ denote the family of all non-empty finite subsets of $X$. If $X$ and $Y$ are topological spaces and $T : X \rightarrow 2^Y$, then the graph of $T$ is the set $G(T) := \{(x, y) \in X \times Y : y \in T(x)\}$. Throughout this paper, $\Phi$ will denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

Let $E$ be a topological vector space over $\Phi$, $F$ be a vector space over $\Phi$ and $\langle , \rangle : F \times E \rightarrow \Phi$ be a bilinear functional.

For each $x_0 \in E$, each non-empty subset $A$ of $E$ and each $\epsilon > 0$, let $W(x_0; \epsilon) := \{y \in F : \|y, x_0\| < \epsilon\}$ and $U(A; \epsilon) := \{y \in F : \sup_{x \in A} |\langle y, x \rangle| < \epsilon\}$. Let $\sigma(F, E)$ be the (weak) topology on $F$ generated by the family $\{W(x; \epsilon) : x \in E$ and $\epsilon > 0\}$ as a subbase for the neighbourhood system at $0$ and $\delta(F, E)$ be the (strong) topology on $F$ generated by the family $\{U(A; \epsilon) : A$ is a non-empty bounded subset of $E$ and $\epsilon > 0\}$ as a base for the neighbourhood system at $0$. We note then that $F$, when equipped with the (weak) topology $\sigma(F, E)$ or the (strong) topology $\delta(F, E)$, becomes a locally convex topological vector space which is not necessarily Hausdorff. But if the bilinear functional $\langle , \rangle : F \times E \rightarrow \Phi$ separates points in $F$, i.e., for each $y \in F$ with $y \neq 0$, there exists $x \in E$ such that $\langle y, x \rangle \neq 0$, then $F$ also becomes Hausdorff.

Then, we have the following three maps:

(i) $h : X \rightarrow \mathbb{R}$
(ii) $M : X \rightarrow 2^E$ and
(iii) $T : X \rightarrow 2^F$.

Then $T$ is said to be an $h$-quasi-pseudo-monotone (respectively, strongly $h$-quasi-pseudo-monotone) type I operator if for each $y \in X$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in $X$ converging to $y$ (respectively, weakly to $y$) with

$$\limsup_{\alpha} \left[ \inf_{f \in M(y)} \inf_{u \in T(y_\alpha)} \Re\langle f - u, y_\alpha - y \rangle + h(y_\alpha) - h(y) \right] \leq 0$$

we have

$$\limsup_{\alpha} \left[ \inf_{f \in M(x)} \inf_{u \in T(y_\alpha)} \Re\langle f - u, y_\alpha - x \rangle + h(y_\alpha) - h(x) \right]$$

$$\geq \inf_{f \in M(x)} \inf_{x \in T(y_\alpha)} \Re\langle f - u, y \rangle + h(y) - h(x) \quad \text{for all } x \in X.$$

$T$ is said to be a quasi-pseudo-monotone (respectively, strongly quasi-pseudo-monotone) type I operator if $T$ is an $h$-quasi-pseudo-monotone (respectively, strongly $h$-quasi-pseudo-monotone) type I operator with $h \equiv 0$.

Note that when $M \equiv 0$, and $T$ is replaced by $-T$, an $h$-quasi-pseudo-monotone type I operator is reduced to an $h$-pseudo-monotone (or an $h$-demi-monotone) operator defined in [5]. The $h$-pseudo-monotone (or $h$-demi-monotone) operators defined in [5] are slightly more general than the $h$-pseudo-monotone operators with the definition given in [6].

Later, in the year 2000, the first author (M.S.R. Chowdhury) renamed the above $h$-pseudo-monotone (or $h$-demi-monotone) operators as pseudo-monotone type I operators [19]. The pseudo-monotone type I operators are a set-valued generalization of the classical (single-valued) pseudo-monotone operators with slight variations. The classical definition of a single-valued pseudo-monotone operator was introduced by Brézis et al. in [20]. We first introduced quasi-pseudo-monotone type I operators in [9, Definition 1.1], as a generalization of pseudo-monotone type I operators.

We shall establish the following result:

**Proposition 1.1.** Let $X$ be a non-empty subset of a topological vector space $E$. Let $T : X \rightarrow E^*$ and $M : X \rightarrow E^*$ be two single-valued maps. Suppose that the operator $T$ is monotone, and both $M$ and $T$ are continuous maps from the relatively weak topology on $X$ to the weak* topology on $E^*$. Then $T$ is both quasi-pseudo-monotone type I and strongly quasi-pseudo-monotone type I operator.

**Proof.** Suppose that $\{y_\alpha\}_{\alpha \in \Gamma}$ is a net in $X$ and $y \in X$ with $y_\alpha \rightarrow y$ (respectively, $y_\alpha \rightarrow y$ weakly) and that $\limsup_{\alpha} \Re\langle M(y) - T(y_\alpha), y_\alpha - y \rangle \leq 0$.

Now, for any $x \in X$,

$$\Re\langle M(x) - T(y_\alpha), y_\alpha - x \rangle = \Re\langle M(x) - T(y_\alpha), y_\alpha - y \rangle + \Re\langle M(x) - T(y_\alpha), y - x \rangle$$

$$\geq \Re\langle M(x) - T(y), y_\alpha - y \rangle + \Re\langle M(x) - T(y), y - x \rangle$$

$$= \Re\langle M(x) - T(y), y_\alpha - y \rangle + \Re\langle M(x) - T(y), y - x \rangle - (M(x) - T(y), y - x) + \Re\langle M(x) - T(y), y - x \rangle.$$

\(\ast\)
Then given $\epsilon > 0$, there exists $\beta_1 \in \Gamma$ such that
\[
|\Re\langle M(x) - T(y), y_\alpha - y \rangle| < \epsilon/2 \quad \text{for all } \alpha \geq \beta_1;
\]
i.e., $-\epsilon/2 < \Re\langle M(x) - T(y), y_\alpha - y \rangle < \epsilon/2$ for all $\alpha \geq \beta_1$.
Again, for the same $\epsilon > 0$, there exists $\beta_2 \in \Gamma$ such that
\[
|\Re\langle M(x) - T(y_\alpha) - \langle M(x) - T(y)\rangle, y - x \rangle| < \epsilon/2 \quad \text{for all } \alpha \geq \beta_2.
\]
Let us choose $\beta_0 \in \Gamma$ with $\beta_0 \geq \beta_1$ and $\beta_0 \geq \beta_2$. Thus (\#) becomes
\[
\Re\langle M(x) - T(y_\alpha), y_\alpha - x \rangle > -\epsilon/2 - \epsilon/2 + \Re\langle M(x) - T(y)\rangle, y - x \rangle
\]
for all $\alpha \geq \beta$. Hence,
\[
\limsup_{\alpha \in \Gamma} \Re\langle M(x) - T(y_\alpha), y_\alpha - x \rangle \geq -\epsilon + \Re\langle M(x) - T(y)\rangle, y - x \rangle.
\]
As $\epsilon > 0$ is arbitrary, we have
\[
\limsup_{\alpha \in \Gamma} \Re\langle M(x) - T(y_\alpha), y_\alpha - x \rangle \geq \Re\langle M(x) - T(y)\rangle, y - x \rangle \quad \forall x \in X.
\]
Consequently, $T$ is both a quasi-pseudo-monotone type I and strongly quasi-pseudo-monotone type I operator. 

Note that the above Proposition 1.1 is a slight modification and or extension of Proposition 1.1 in [9]. Moreover, with our modified Definition 1.1 above, the operator $T$ in Proposition 1.1 is now both a quasi-pseudo-monotone type I and strongly quasi-pseudo-monotone type I operator.

In this paper we shall obtain some general theorems on solutions for a new class of generalized bi-quasi-variational inequalities for quasi-pseudo-monotone type I and strongly quasi-pseudo-monotone type I operators defined on non-compact sets in topological vector spaces. In obtaining these results we shall mainly use the following generalized version of Ky Fan’s minimax inequality [21] due to Chowdhury and Tan [5].

**Theorem 1.1.** Let $E$ be a topological vector space, $X$ be a non-empty convex subset of $E$, $\mathcal{F}(X)$ denote the family of all non-empty finite subsets of $X$ and $f : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be such that

(a) for each $A \in \mathcal{F}(X)$ and each fixed $x \in \text{co}(A)$, $y \mapsto f(x, y)$ is lower semi-continuous on $\text{co}(A)$;

(b) for each $A \in \mathcal{F}(X)$ and each $y \in \text{co}(A)$, $\min_{x \in A} f(x, y) \leq 0$;

(c) for each $A \in \mathcal{F}(X)$ and each $x, y \in \text{co}(A)$, every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in $X$ converging to $y$ with $f(tx + (1-t)y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$ and all $t \in [0, 1]$, we have $f(x, y) \leq 0$;

(d) there exist a non-empty closed and compact subset $K$ of $X$ and $x_0 \in K$ such that $f(x_0, y) > 0$ for all $y \in X \setminus K$.

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$.

2. Preliminaries

We shall first state the following result which is Lemma 1 of Shih and Tan in [22, pp. 334–335]:

**Lemma 2.1.** Let $X$ be a non-empty subset of a Hausdorff topological vector space $E$ and $S : X \to 2^E$ be an upper semi-continuous map such that $S(x)$ is a bounded subset of $E$ for each $x \in X$. Then for each continuous linear functional $p$ on $E$, the functional $f_p : X \to \mathbb{R}$ defined by $f_p(y) = \sup_{x \in S(y)} \Re(p, x)$ is upper semi-continuous; i.e., for each $\lambda \in \mathbb{R}$, the set $\{y \in X : f_p(y) = \sup_{x \in S(y)} \Re(p, x) < \lambda\}$ is open in $X$.

The following result is Lemma 3 of Takahashi in [23, p. 177] (see also Lemma 3 in [1, pp. 71–72]):

**Lemma 2.2.** Let $X$ and $Y$ be topological spaces, $f : X \to \mathbb{R}$ be non-negative and continuous and $g : Y \to \mathbb{R}$ be lower semi-continuous. Then the functional $F : X \times Y \to \mathbb{R}$, defined by $F(x, y) = f(x)g(y)$ for all $(x, y) \in X \times Y$, is lower semi-continuous.

The following result is Lemma 1 in [7]:

**Lemma 2.3.** Let $E$ be a topological vector space over $\Phi$, $X$ be a non-empty compact subset of $E$ and $F$ be a Hausdorff topological vector space over $\Phi$. Let $(\cdot, \cdot) : F \times E \to \Phi$ be a bilinear functional and $T : X \to 2^X$ be an upper semi-continuous map such that each $T(x)$ is compact. Let $M$ be a non-empty compact subset of $F$, $x_0 \in X$ and $h : X \to \mathbb{R}$ be continuous. Define $g : X \to \mathbb{R}$ by $g(y) = \inf_{f \in M} \inf_{w \in T(y)} \Re(f - w, y - x_0) + h(y)$ for each $y \in X$. Suppose that $(\cdot, \cdot)$ is continuous on the (compact) subset $[M - \cup_{y \in X} T(y)] \times X$ of $F \times E$. Then $g$ is lower semi-continuous on $X$.

When $h \equiv 0$ and $M = \{0\}$, replacing $T$ by $-T$, Lemma 2.3 reduces to Lemma 2 of Shih and Tan from [1, pp. 70–71].

The following result is a slight modification of Lemma 4 in [1]:

[Note: The rest of the text has been truncated for brevity. For the full context, please refer to the original document.]
Lemma 2.4. Let $E$ be a topological vector space over $\Phi$, $F$ be a vector space over $\Phi$ and $X$ be a non-empty convex subset of $E$. Let $\langle , \rangle : F \times E \to \Phi$ be a bilinear functional. Equip $F$ with the $\sigma (F, E)$-topology.

Let $h : X \to \mathbb{R}$ be convex and $M : X \to 2^F$ be lower semi-continuous along line segments in $X$ to the $\sigma (F, E)$-topology on $F$. Let $S : X \to 2^\mathbb{R}$ and $T : X \to 2^F$ be two maps. Suppose that there exists $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$, $S(\hat{y})$ is convex and $\inf_{f \in M(x, y) \in T(\hat{y})} \inf_{w \in T(y)} \Re\langle f - w, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$. Then

$$\inf_{f \in M(x, y) \in T(\hat{y})} \inf_{w \in T(y)} \Re\langle f - w, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$$

for all $x \in S(\hat{y})$.

We shall need the following Kneser’s minimax theorem from [24, pp. 2418–2420] (see also Aubin [25, pp. 40–41]):

Theorem 2.1. Let $X$ be a non-empty compact convex subset of a vector space and $Y$ be a non-empty compact convex subset of a Hausdorff topological vector space. Suppose that $f$ is a real-valued function on $X \times Y$ such that for each fixed $x \in X$, the map $y \mapsto f(x, y)$, i.e., $f(x, \cdot)$ is lower semi-continuous and convex on $Y$ and for each fixed $y \in Y$, the map $x \mapsto f(x, y)$, i.e., $f(\cdot, y)$ is concave on $X$. Then

$$\min_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \min_{y \in Y} f(x, y).$$

3. Existence theorems for generalized bi-quasi-variational inequalities for quasi-pseudo-monotone type I operators

In this section, we shall obtain and prove some existence theorems for the solutions to the generalized bi-quasi-variational inequalities for quasi-pseudo-monotone type I and strongly quasi-pseudo-monotone type I operators $T$ with non-compact domain in locally convex Hausdorff topological vector spaces. Our results extend and/or generalize the corresponding results in [1].

We shall first establish the following result:

Theorem 3.1. Let $E$ be a locally convex Hausdorff topological vector space over $\Phi$, $X$ be a non-empty para-compact convex and bounded subset of $E$ and $F$ be a Hausdorff topological vector space over $\Phi$. Let $\langle , \rangle : F \times E \to \Phi$ be a bilinear functional which is continuous on compact subsets of $F \times X$. Suppose that:

(a) $S : X \to 2^X$ is upper semi-continuous such that each $S(x)$ is compact and convex;
(b) $h : E \to (0, \infty)$ is convex and $h(X)$ is bounded;
(c) $T : X \to 2^F$ is an $h$-quasi-pseudo-monotone type I (respectively, strongly $h$-quasi-pseudo-monotone type I) operator and is upper semi-continuous such that each $T(x)$ is compact (respectively, weakly compact) and convex and $T(X)$ is strongly bounded;
(d) $M : X \to 2^F$ is a linear map in $X$ (and is therefore single-valued for each $x \in X$);
(e) the set $\Sigma = \{ y \in X : \sup_{x \in S(y)} (\inf_{w \in T(y)} \Re\langle M(x) - w, y - x \rangle + h(y) - h(x)) > 0 \}$ is open in $X$.

Suppose further that there exist a non-empty open and compact (respectively, weakly closed and weakly compact) subset $K$ of $X$ and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w \in T(y)} \Re\langle M(x_0) - w, y - x_0 \rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$.

Then there exists a point $\hat{y}$ such that

(i) $\hat{y} \in S(\hat{y})$ and
(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $\Re\langle M(\hat{y}) - \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Moreover, if $S(x) = X$ for all $x \in X$, $E$ is not required to be locally convex and if $T \equiv 0$, the continuity assumption on $\langle , \rangle$ can be weakened to the assumption that for each $f \in F$, the map $x \mapsto f(x, y)$ is continuous (respectively, weakly continuous) on $X$.

Proof. We divide the proof into three steps:

Step 1. There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} \left[ \inf_{\hat{w} \in T(\hat{y})} \Re\langle M(x) - w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \right] \leq 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(\hat{y})$ or there exists $x \in S(\hat{y})$ such that $\inf_{w \in T(y)} \Re\langle M(x) - w, y - x \rangle + h(y) - h(x) > 0$; that is, for each $y \in X$, either $y \notin S(\hat{y})$ or $y \in \Sigma$. If $y \notin S(\hat{y})$, then by a separation theorem for convex sets in locally convex Hausdorff topological vector spaces, there exists $p \in E^*$ such that $\Re\langle p, y \rangle - \sup_{x \in S(\hat{y})} \Re\langle p, x \rangle > 0$. Let $\gamma(y) = \sup_{x \in S(\hat{y})} \inf_{w \in T(y)} \Re\langle M(x) - w, y - x \rangle + h(y) - h(x)$ and let

$$V_0 := \{ y \in X : \gamma(y) > 0 \} = \Sigma$$

and for each $p \in E^*$, set

$$V_p := \left\{ y \in X : \Re\langle p, y \rangle - \sup_{x \in S(\hat{y})} \Re\langle p, x \rangle > 0 \right\}.$$
Then \( X = V_0 \cup \bigcup_{p \in E^*} V_p \). Since each \( V_p \) is open in \( X \) by Lemma 2.1 and \( V_0 \) is open in \( X \) by hypothesis, \( \{V_0, V_p : p \in E^*\} \) is an open covering for \( X \). Since \( X \) is para-compact, there is a continuous partition of unity \( \{\beta_0, \beta_p : p \in E^*\} \) for \( X \) subordinated to the open cover \( \{V_0, V_p : p \in E^*\} \) (see, e.g., Theorem VIII.4.2 of Dugundji in [26]); that is for each \( p \in E^* \), \( \beta_0 : X \to [0, 1] \) and \( \beta_0 : X \to [0, 1] \) are continuous functions such that for each \( p \in E^*, \beta_p(y) = 0 \) for all \( y \in X \setminus V_p \) and \( \beta_0(y) = 0 \) for all \( y \in X \setminus V_0 \) and \{support \( \beta_0 \), support \( \beta_p : p \in E^* \} \) is locally finite and \( \beta_0(y) + \sum_{p \in E^*} \beta_p(y) = 1 \) for each \( y \in X \). Note that for each \( A \in \mathcal{F}(X), h \) is continuous on \( co(A) \) (see e.g. [27], Corollary 10.1.1, p. 83).

Define \( \phi : X \times X \to \mathbb{R} \) by

\[
\phi(x, y) = \beta_0(y) \left[ \inf_{w \in T(y)} \Re(M(x) - w, y - x) + h(y) - h(x) \right] + \sum_{p \in E^*} \beta_p(y) \Re(p, y - x)
\]

for each \( x, y \in X \). Then we have the following:

1. Since \( E \) is Hausdorff, for each \( A \in \mathcal{F}(X) \) and each fixed \( x \in co(A) \), the map

\[
y \mapsto \inf_{w \in T(y)} \Re(M(x) - w, y - x) + h(y) - h(x)
\]

is lower semi-continuous (respectively, weakly lower semi-continuous) on \( co(A) \) by Lemma 2.3 and the fact that \( h \) is continuous on \( co(A) \) and therefore the map

\[
y \mapsto \sum_{p \in E^*} \beta_p(y) \Re(p, y - x)
\]

is continuous on \( X \). Hence, for each \( A \in \mathcal{F}(X) \) and each fixed \( x \in co(A) \), the map \( y \mapsto \phi(x, y) \) is lower semi-continuous (respectively, weakly lower semi-continuous) on \( co(A) \).

2. For each \( A \in \mathcal{F}(X) \) and for each \( y \in co(A) \), \( \min_{x \in A} \phi(x, y) \leq 0 \). Indeed, if this were false, then for some \( A = \{x_1, x_2, \ldots, x_n\} \in \mathcal{F}(X) \) and some \( y \in co(A) \) (say \( y = \sum_{i=1}^n \lambda_i x_i \) where \( \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0 \) with \( \sum_{i=1}^n \lambda_i = 1 \)), we have \( \min_{1 \leq i \leq n} \phi(x_i, y) > 0 \). Then for each \( i = 1, 2, \ldots, n \),

\[
\beta_0(y) \left[ \inf_{w \in T(y)} \Re(M(x_i) - w, y - x_i) + h(y) - h(x_i) \right] + \sum_{p \in E^*} \beta_p(y) \Re(p, y - x_i) > 0
\]

and so

\[
0 = \phi(y, y) = \beta_0(y) \left[ \inf_{w \in T(y)} \Re(M \left( \sum_{i=1}^n \lambda_i x_i \right) - w, y - \sum_{i=1}^n \lambda_i x_i) + h(y) - h \left( \sum_{i=1}^n \lambda_i x_i \right) \right]
\]

\[
+ \sum_{p \in E^*} \beta_p(y) \Re(p, y - \sum_{i=1}^n \lambda_i x_i)
\]

\[
= \beta_0(y) \left[ \inf_{w \in T(y)} \Re \left( \sum_{i=1}^n \lambda_i (M(x_i) - w, y - \sum_{i=1}^n \lambda_i x_i) + h(y) - h \left( \sum_{i=1}^n \lambda_i x_i \right) \right) \right] + \sum_{p \in E^*} \beta_p(y) \Re(p, y - \sum_{i=1}^n \lambda_i x_i)
\]

\[
\geq \sum_{i=1}^n \lambda_i \left( \beta_0(y) \left[ \inf_{w \in T(y)} \Re(M(x_i) - w, y - x_i) + h(y) - h(x_i) \right] + \sum_{p \in E^*} \beta_p(y) \Re(p, y - x_i) \right) > 0,
\]

which is a contradiction.

3. Suppose that \( A \in \mathcal{F}(X), x, y \in co(A) \) and \( \{y_\alpha\}_{\alpha \in \Gamma} \) is a net in \( X \) converging to \( y \) (respectively, weakly to \( y \)) with \( \phi(tx + (1 - t)y, y_\alpha) \leq 0 \) for all \( \alpha \in \Gamma \) and all \( t \in [0, 1] \).

Case 1: \( \beta_0(y_\alpha) = 0 \).

Note that \( \beta_0(y_\alpha) \geq 0 \) for each \( \alpha \in \Gamma \) and \( \beta_0(y_\alpha) \to 0 \). Since \( T(X) \) is strongly bounded and \( \{y_\alpha\}_{\alpha \in \Gamma} \) is a bounded net, it follows that

\[
\limsup_{\alpha} \left[ \beta_0(y_\alpha) \left( \min_{w \in T(y_\alpha)} \Re(M(x) - w, y_\alpha - x) + h(y_\alpha) - h(x) \right) \right] = 0.
\]

Also

\[
\beta_0(y) \left[ \min_{w \in T(y)} \Re(M(x) - w, y - x) + h(y) - h(x) \right] = 0.
\]
Thus
\[
\limsup_{\alpha} \left[ \beta_0(y_\alpha) \left( \min_{w \in T(y_\alpha)} \text{Re}(M(x) - w, y_\alpha - x) + h(y_\alpha) - h(x) \right) \right] + \sum_{p \in E^*} \beta_p(y) \text{Re}(p, y - x) \\
= \sum_{p \in E^*} \beta_p(y) \text{Re}(p, y - x) \quad \text{by (3.1)}
\]
\[
= \beta_0(y) \left[ \min_{w \in T(y)} \text{Re}(M(x) - w, y - x) + h(y) - h(x) \right] + \sum_{p \in E^*} \beta_p(y) \text{Re}(p, y - x).
\]
(3.2)

When \( t = 1 \) we have \( \phi(x, y_\alpha) \leq 0 \) for all \( \alpha \in \Gamma' \), i.e.,
\[
\beta_0(y_\alpha) \left[ \min_{w \in T(y_\alpha)} \text{Re}(M(x) - w, y_\alpha - x) + h(y_\alpha) - h(x) \right] + \sum_{p \in E^*} \beta_p(y_\alpha) \text{Re}(p, y_\alpha - x) \leq 0 \quad \forall \alpha \in \Gamma'.
\]
(3.3)

Therefore
\[
\limsup_{\alpha} \left[ \beta_0(y_\alpha) \min_{w \in T(y_\alpha)} \text{Re}(M(x) - w, y_\alpha - x) + h(y_\alpha) - h(x) \right] + \liminf_{\alpha} \left[ \sum_{p \in E^*} \beta_p(y_\alpha) \text{Re}(p, y_\alpha - x) \right]
\]
\[
\leq \limsup_{\alpha} \left[ \beta_0(y_\alpha) \min_{w \in T(y_\alpha)} \text{Re}(M(x) - w, y_\alpha - x) + h(y_\alpha) - h(x) + \sum_{p \in E^*} \beta_p(y_\alpha) \text{Re}(p, y_\alpha - x) \right]
\]
\[
\leq 0 \quad \text{(by (3.3)).}
\]

Thus
\[
\limsup_{\alpha} \left[ \beta_0(y_\alpha) \min_{w \in T(y_\alpha)} \text{Re}(M(x) - w, y_\alpha - x) + h(y_\alpha) - h(x) \right] + \sum_{p \in E^*} \beta_p(y) \text{Re}(p, y - x) \leq 0.
\]
(3.4)

Hence by (3.2) and (3.4), we have \( \phi(x, y) \leq 0 \).

Case 2: \( \beta_0(y) > 0 \).

Since \( \beta_0(y_\alpha) \to \beta_0(y) \), there exists \( \lambda \in \Gamma' \) such that \( \beta_0(y_\alpha) > 0 \) for all \( \alpha \geq \lambda \). When \( t = 0 \), we have \( \phi(y, y_\alpha) \leq 0 \forall \alpha \in \Gamma' \), i.e.,
\[
\beta_0(y_\alpha) \left[ \inf_{w \in T(y_\alpha)} \text{Re}(M(y) - w, y_\alpha - y) + h(y_\alpha) - h(y) \right] + \sum_{p \in E^*} \beta_p(y_\alpha) \text{Re}(p, y_\alpha - y) \leq 0 \quad \forall \alpha \in \Gamma'.
\]

Thus
\[
\limsup_{\alpha} \left[ \beta_0(y_\alpha) \left( \inf_{w \in T(y_\alpha)} \text{Re}(M(y) - w, y_\alpha - y) + h(y_\alpha) - h(y) \right) + \sum_{p \in E^*} \beta_p(y_\alpha) \text{Re}(p, y_\alpha - y) \right] \leq 0.
\]
(3.5)

Hence
\[
\limsup_{\alpha} \left[ \beta_0(y_\alpha) \left( \inf_{w \in T(y_\alpha)} \text{Re}(M(y) - w, y_\alpha - y) + h(y_\alpha) - h(y) \right) \right] + \liminf_{\alpha} \left[ \sum_{p \in E^*} \beta_p(y_\alpha) \text{Re}(p, y_\alpha - y) \right]
\]
\[
\leq \limsup_{\alpha} \left[ \beta_0(y_\alpha) \left( \inf_{w \in T(y_\alpha)} \text{Re}(M(y) - w, y_\alpha - y) + h(y_\alpha) - h(y) \right) + \sum_{p \in E^*} \beta_p(y_\alpha) \text{Re}(p, y_\alpha - y) \right]
\]
\[
\leq 0 \quad \text{(by (3.5)).}
\]

Since \( \liminf_{\alpha} \left[ \sum_{p \in E^*} \beta_p(y_\alpha) \text{Re}(p, y_\alpha - y) \right] = 0 \), we have
\[
\limsup_{\alpha} \left[ \beta_0(y_\alpha) \left( \min_{w \in T(y_\alpha)} \text{Re}(M(y) - w, y_\alpha - y) + h(y_\alpha) - h(y) \right) \right] \leq 0.
\]
(3.6)

Since \( \beta_0(y_\alpha) > 0 \) for all \( \alpha \geq \lambda \), it follows that
\[
\beta_0(y) \limsup_{\alpha} \left[ \left( \min_{w \in T(y_\alpha)} \text{Re}(M(y) - w, y_\alpha - y) + h(y_\alpha) - h(y) \right) \right]
\]
\[
= \limsup_{\alpha} \left[ \beta_0(y_\alpha) \left( \min_{w \in T(y_\alpha)} \text{Re}(M(y) - w, y_\alpha - y) + h(y_\alpha) - h(y) \right) \right].
\]
(3.7)
Since $\beta_0(y) > 0$, by (3.6) and (3.7) we have

$$\limsup_{\alpha} \left[ \min_{w \in T(y)} \operatorname{Re}(M(y) - w, y_a - y) + h(y_a) - h(y) \right] \leq 0.$$ 

Since $T$ is an $h$-quasi-pseudo-monotone type I (respectively, strongly $h$-quasi-pseudo-monotone type I) operator, we have

$$\limsup_{\alpha} \left[ \min_{w \in T(y)} \operatorname{Re}(M(x) - w, y_a - x) + h(y_a) - h(x) \right] \geq \min_{w \in T(y)} \operatorname{Re}(M(x) - w, y - x) + h(y) - h(x)$$

for all $x \in X$. Since $\beta_0(y) > 0$, we have

$$\beta_0(y) \left[ \limsup_{\alpha} \left( \min_{w \in T(y)} \operatorname{Re}(M(x) - w, y_a - x) + h(y_a) - h(x) \right) \right] \geq \beta_0(y) \left[ \min_{w \in T(y)} \operatorname{Re}(M(x) - w, y - x) + h(y) - h(x) \right].$$

Thus

$$\beta_0(y) \left[ \limsup_{\alpha} \left( \min_{w \in T(y)} \operatorname{Re}(M(x) - w, y_a - x) + h(y_a) - h(x) \right) \right] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}(p, y - x) \geq \beta_0(y) \left[ \min_{w \in T(y)} \operatorname{Re}(M(x) - w, y - x) + h(y) - h(x) \right] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}(p, y - x).$$

(4) By hypothesis, there exist a non-empty compact and therefore closed (respectively, weakly closed and weakly compact) subset $K$ of $X$ and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w \in T(y)}[\operatorname{Re}(M(x_0) - w, y - x_0) + h(y) - h(x_0)] > 0$ for all $y \in X \setminus K$. Thus, for all $y \in X \setminus K$, $\beta_0(y)[\inf_{w \in T(y)} \operatorname{Re}(M(x_0) - w, y - x_0) + h(y) - h(x_0)] > 0$ whenever $\beta_0(y) > 0$, and $\operatorname{Re}(p, y - x_0) > 0$ whenever $\beta_p(y) > 0$ for $p \in E^*$. Consequently,

$$\phi(x_0, y) = \beta_0(y) \left[ \inf_{w \in T(y)} \operatorname{Re}(M(x_0) - w, y - x_0) + h(y) - h(x_0) \right] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}(p, y - x_0) > 0 \quad \forall y \in X \setminus K.$$ 

(If $T$ is a strongly $h$-quasi-pseudo-monotone type I operator, we equip $E$ with the weak topology.) Thus $\phi$ satisfies all the hypotheses of Theorem 1.1. Hence by Theorem 1.1, there exists a point $\hat{y} \in K$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$; i.e.,

$$\beta_0(\hat{y}) \left[ \inf_{w \in T(\hat{y})} \operatorname{Re}(M(x) - w, \hat{y} - x) + h(\hat{y}) - h(x) \right] + \sum_{p \in E^*} \beta_p(\hat{y}) \operatorname{Re}(p, \hat{y} - x) \leq 0 \quad \forall x \in X.$$ 

(3.10)

Now the rest of the proof of Step 1 is similar to the proof in Step 1 of Theorem 1 in [7]. Hence Step 1 is proved.

Step 2. $\inf_{w \in T(\hat{y})} \operatorname{Re}(M(\hat{y}) - w, \hat{y} - x) \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

From Step 1 we have $\hat{y} \in S(\hat{y})$ and
\[ \inf_{w \in T(y)} \text{Re}(M(x) - w, \hat{y} - x) \leq h(x) - h(\hat{y}) \]

for all \( x \in S(\hat{y}) \). Since \( S(\hat{y}) \) is a convex subset of \( X \), and \( M \) is linear and therefore continuous along line segments in \( X \), by Lemma 2.4 we have
\[ \inf_{w \in T(y)} \text{Re}(M(\hat{y}) - w, \hat{y} - x) \leq h(x) - h(\hat{y}) \]

for all \( x \in S(\hat{y}) \).

Step 3. There exists a point \( \hat{w} \in T(\hat{y}) \) with \( \text{Re}(M(\hat{y}) - \hat{w}, \hat{y} - x) \leq h(x) - h(\hat{y}) \) for all \( x \in S(\hat{y}) \).

By Step 2 above and applying Theorem 2.1, as we proved in Step 3 of Theorem 1 in [7], we can show that there exists a point \( \hat{w} \in T(\hat{y}) \) such that \( \text{Re}(M(\hat{y}) - \hat{w}, \hat{y} - x) \leq h(x) - h(\hat{y}) \) for all \( x \in S(\hat{y}) \).

We observe from the above proof that the requirement that \( E \) needs to be locally convex is needed when and only when the separation theorem is applied to the case \( y \not\in S(y) \). Thus if \( S : X \rightarrow 2^X \) is the constant map \( S(x) = X \) for all \( x \in E \), \( E \) is not required to be locally convex.

Finally, if \( T \equiv 0 \), in order to show that for each \( x \in X \), \( y \mapsto \phi(x, y) \) is lower semi-continuous (respectively, weakly lower semi-continuous), Lemma 2.3 is no longer needed and the weaker continuity assumption on \( \langle \cdot, \cdot \rangle \) that for each \( f \in F \), the map \( x \mapsto \langle f, x \rangle \) is continuous (respectively, weakly continuous) on \( X \) is sufficient. This completes the proof. \( \square \)

We shall now establish our last result of this section:

**Theorem 3.2.** Let \( E \) be a locally convex Hausdorff topological vector space over \( \Phi \), \( X \) be a non-empty para-compact convex and bounded subset of \( E \) and \( F \) be a vector space over \( \Phi \). Let \( \langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi \) be a bilinear functional such that \( \langle \cdot, \cdot \rangle \) separates points in \( F \) and for each \( f \in F \), the map \( x \mapsto \langle f, x \rangle \) is continuous on \( X \). Equip \( F \) with the strong topology \( \delta(F, E) \) . Suppose that

(a) \( S : X \rightarrow 2^X \) is a continuous map such that each \( S(x) \) is compact and convex;

(b) \( h : E \mapsto \mathbb{R} \) is convex and \( h(X) \) is bounded;

(c) \( T : X \rightarrow 2^X \) is a quasi-pseudo-monotone type I (respectively, strongly \( h \)-quasi-pseudo-monotone type \( I \))-operator and is upper semi-continuous such that each \( T(x) \) is strongly, i.e., \( \delta(F, E) \)-compact and convex (respectively, weakly, i.e., \( \sigma(F, E) \)-compact and convex);

(d) \( M : X \rightarrow 2^F \) is a continuous linear map in \( X \) and for each \( y \in \Sigma = \{ y \in X : \sup_{x \in S(\hat{y})} \text{Re}(M(x) - w, y - x) + h(y) - h(x) > 0 \} \), \( \inf_{w \in T(y)} \text{Re}(M(x) - w, y - x) + h(y) - h(x) > 0 \) for some point \( x \in S(\hat{y}) \).

Suppose further that there exist a non-empty closed and compact (respectively, weakly closed and weakly compact) subset \( K \) of \( X \) and a point \( x_0 \in X \) such that \( x_0 \in K \cap S(\hat{y}) \) and \( \inf_{w \in T(y)} \text{Re}(M(x_0) - w, y - x_0) + h(y) - h(x_0) > 0 \) for all \( y \in X \setminus K \).

Then there exists a point \( \hat{y} \in X \) such that

(i) \( \hat{y} \in S(\hat{y}) \) and

(ii) there exists a point \( \hat{w} \in T(\hat{y}) \) with \( \text{Re}(M(\hat{y}) - \hat{w}, \hat{y} - x) \leq h(x) - h(\hat{y}) \) for all \( x \in S(\hat{y}) \).

Moreover, if \( S(x) = X \) for all \( x \in X \), \( E \) is not required to be locally convex.

**Proof.** The proof is similar to the proof of Theorem 2 in [7]. Hence the proof is omitted here. \( \square \)

**Remark 3.1.** (1) Theorems 3.1 and 3.2 of this paper are generalizations of Theorems 3.2 and 3.3 in [9], respectively, on non-compact sets. In Theorems 3.1 and 3.2 of this paper, \( X \) is considered to be a para-compact convex and bounded subset of locally convex Hausdorff topological vector space \( E \) whereas, in [9], \( X \) is just a compact and convex subset of \( E \). Hence our results generalize the corresponding results from [9].

(2) The first paper on generalized bi-quasi-variational inequalities was written by Shih and Tan in 1989 in [1] and the results were obtained on compact sets where the set-valued mappings were either lower semi-continuous or upper semi-continuous. Our present paper is another extension of the original work from [1] using quasi-pseudo-monotone type I operators on non-compact sets. The quasi-pseudo-monotone type I operators are generalizations of pseudo-monotone type I operators introduced first in [5].

(3) In all of our results on generalized bi-quasi-variational inequalities, if the operators \( M \equiv 0 \) and the operator \( T \) is replaced by \(-T\), then we obtain results on generalized quasi-variational inequalities which generalize the corresponding results given in the literature (see [28]).

(4) The results on generalized bi-quasi-variational inequalities given in [2] were obtained for set-valued quasi-semi-monotone and bi-quasi-semi-monotone operators and the corresponding results in [17] were obtained for set-valued upper hemi-continuous operators introduced in [14]. Our results in this paper are also further extensions of the corresponding results given in [2,17] using set-valued quasi-pseudo-monotone type I operators on non-compact sets.

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