Positive solutions of nonlinear singular third-order two-point boundary value problem

Shuhong Li

Department of Mathematics and Department of Adult Education, Lishui University, Lishui 323000, Zhejiang, PR China

Received 8 September 2005
Available online 28 November 2005
Submitted by S. Heikkilä

Abstract

In this paper, we are concerned with the existence of single and multiple positive solutions to the nonlinear singular third-order two-point boundary value problem

\[ u'''(t) + \lambda a(t)f(u(t)) = 0, \quad 0 < t < 1, \]
\[ u(0) = u'(0) = u''(1) = 0, \]

where \( \lambda \) is a positive parameter. Under various assumptions on \( a \) and \( f \) we establish intervals of the parameter \( \lambda \) which yield the existence of at least one, at least two, and infinitely many positive solutions of the boundary value problem by using Krasnoselskii’s fixed point theorem of cone expansion–compression type.

© 2005 Elsevier Inc. All rights reserved.

Keywords: Positive solution; Nonlinear singular boundary value problem; Fixed point theorem

1. Introduction

In this paper, we are concerned with the existence of single and multiple positive solutions to nonlinear singular third-order two-point boundary value problem:

\[
\begin{cases}
  u'''(t) + \lambda a(t)f(u(t)) = 0, & 0 < t < 1, \\
  u(0) = u'(0) = u''(1) = 0,
\end{cases}
\]  

(1.1)
where $\lambda > 0$ is a positive parameter and $a : (0, 1) \to [0, \infty)$ is continuous and may be singular at $t = 0$ and/or $1$, $f : [0, \infty) \to [0, \infty)$ is continuous. Here, by a positive solution $u^*$ of BVP (1.1), we mean a function $u^*$ satisfies BVP (1.1) and $u^*(t) > 0$, $0 < t < 1$. Therefore, our positive solutions are nontrivial ones.

Third order equations arise in a variety of different areas of applied mathematics and physics, as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves or gravity driven flows and so on [13]. Different type of techniques have been used to study such problems: reduce them to first and/or second order equations [8]; use Green’s functions and comparison principles [5,6,19] (for periodic boundary value conditions), [20–25] (three point boundary conditions), and [7,9,10,15,27] (two point ones).

A large part of the literature on multiple solutions to boundary value problems seems to be traced back to Krasnoselskii’s work on nonlinear operator equations [1], especially the part dealing with the theory of cones in Banach spaces. In 1994, Erbe and Wang [12] applied Krasnoselskii’s work to eigenvalue problems to establish intervals of the parameter $\lambda$ for which there is at least one positive solution. The upshot of their technique was assuming that the nonlinearity grew either superlinearly or sublinearly and $a(t)$ did not vanish identically on any subinterval of the domain. The growth assumptions and calculations involving the Greens function followed by an application of Krasnoselskii’s theorem yield the result. Many authors have used this approach or a variation to obtain eigenvalue intervals; see [1,2,14]. In 1998, Anderson [3], and Davis and Henderson [11] each applied seldom used fixed-point theorems due to Leggett and Williams [18] to obtain the existence of at least three solutions for certain types of boundary value problems. Anderson and Avery [4] applied a generalization of the Leggett–Williams fixed-point theorem to obtain the existence of at least three solutions for to a third-order discrete focal boundary value problem. Motivated by the papers mentioned above, the aim of this paper is to establish some simple criteria for the existence of single and multiple positive solutions for BVP (1.1) in an explicit interval of $\lambda$. Also, we allow our nonlinearity $a(t)$ to have suitable singularities (such as $t = 0$ and/or $t = 1$). This paper is organized as follows. In Section 2, we present some lemmas that will be used to prove our main results. In Section 3, we discuss the existence of single positive solution of BVP (1.1). In Section 4, we discuss the existence conditions of multiple positive solutions of BVP (1.1). It must be pointed out that in this section, we established the existence results on infinitely many positive solutions of BVP (1.1) which less attention has been given in all existing literature.

2. Preliminaries

We shall consider the Banach space $C[0, 1]$ equipped with sup norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$. $C^+[0, 1]$ is the cone of nonnegative functions in $C[0, 1]$. In arriving our results, we need the following three preliminary lemmas. The first is well known.

Lemma 2.1. Let $y(t) \in C[0, 1]$, then the BVP

$$
\begin{align*}
  u'''(t) + y(t) &= 0, \quad 0 \leq t \leq 1, \\
  u(0) = u'(0) &= u''(1) = 0,
\end{align*}
$$

(2.1)

has a unique solution

$$
  u(t) = \int_0^1 G(t, s)y(s) \, ds,
$$

where

$$
  G(t, s) = \frac{1}{2}\left(\min\{t, s\} - |t - s|\right),
$$

for $0 \leq t, s \leq 1$. Here $G(t, s)$ is the Green’s function for BVP (2.1).
where
\[ G(t, s) = \begin{cases} \frac{1}{2} t^2, & 0 \leq t \leq s \leq 1, \\ \frac{1}{2} t^2 - \frac{1}{2} (t - s)^2, & 0 \leq s \leq t \leq 1. \end{cases} \]

It is obvious that \( 0 \leq G(t, s) \leq G(1, s) = s - \frac{1}{2} s^2, \ 0 \leq t, s \leq 1. \)

**Lemma 2.2.** Let \( 0 < \theta < 1. \) Then for \( y(t) \in C^+[0, 1], \) the unique solution \( u(t) \) of BVP (2.1) is nonnegative and satisfies

\[ \min_{t \in [\theta, 1]} u(t) \geq \frac{\theta^2}{2} \|u\|. \]

**Proof.** Let \( y \in C^+[0, 1], \) then from \( G(t, s) \geq 0 \) we know \( u \in C^+[0, 1]. \) Set \( u(t_0) = \|u\|, t_0 \in (0, 1]. \) We first prove that

\[ \frac{G(t, s)}{G(t_0, s)} \geq \frac{t^2}{2}, \quad t, t_0, s \in (0, 1]. \]

In fact, we can consider four cases:

1. If \( 0 < t, t_0 \leq s \leq 1, \) then
   \[ \frac{G(t, s)}{G(t_0, s)} = \frac{t^2}{t_0^2} \geq \frac{t^2}{2}; \]

2. If \( 0 < t \leq s \leq t_0 \leq 1, \) then
   \[ \frac{G(t, s)}{G(t_0, s)} = \frac{t^2}{t_0^2 - (t_0 - s)^2} \geq \frac{t^2}{2}; \]

3. If \( 0 < s \leq t, \) \( t_0 \leq 1, \) then
   \[ \frac{G(t, s)}{G(t_0, s)} = \frac{t^2 - (t - s)^2}{t_0^2 - (t_0 - s)^2} = \frac{2t - s}{2t_0 - s} \geq \frac{t + (t - s)}{2} \geq \frac{t}{2} \geq \frac{t^2}{2}; \]

4. If \( 0 < t_0 \leq s \leq t \leq 1, \) then
   \[ \frac{G(t, s)}{G(t_0, s)} = \frac{t^2 - (t - s)^2}{t_0^2} \geq \frac{t^2 - (t - s)^2}{s^2} = \frac{2t - s}{s} \geq 2t - s \geq t \geq \frac{t^2}{2}. \]

Therefore, for \( t \in [\theta, 1], \) we have

\[ u(t) = \int_0^1 G(t, s)y(s) \, ds = \int_0^1 \frac{G(t, s)}{G(t_0, s)} G(t_0, s)y(s) \, ds \geq \frac{t^2}{2} u(t_0) \geq \frac{\theta^2}{2} \|u\|. \]

The proof is complete. \( \square \)

Define a cone \( K \) by

\[ K = \left\{ u \in C^+[0, 1]: \min_{t \in [\theta, 1]} u(t) \geq \frac{\theta^2}{2} \|u\| \right\}. \]
Define an integral operator $T_\lambda : K \to C^+[0, 1]$ by
\[
T_\lambda u(t) = \lambda \int_0^1 G(t, s)a(s)f(u(s)) \, ds.
\] (2.2)

By Lemma 2.1, BVP (1.1) has a solution $u = u(t)$ if and only if $u$ is a fixed point of $T_\lambda$.

We adopt the following assumptions:

\begin{enumerate}
\item[(H1)] $a \in C((0, 1), [0, \infty))$ and $0 < \int_0^1 G(1, s)a(s) \, ds < \infty$.
\item[(H2)] $f \in C([0, 1], [0, \infty))$.
\end{enumerate}

By (H1), there exists $t_0 \in (0, 1)$ such that $a(t_0) > 0$.

**Lemma 2.3.** Assume that (H1), (H2) hold. Then $T_\lambda : K \to K$ is completely continuous.

**Proof.** By Lemma 2.2, we know that $T_\lambda (K) \subset K$. Now, we shall prove that the operator $T_\lambda$ is completely continuous. Defined the function $a_n$ for $n \geq 2$, by
\[
a_n(t) = \begin{cases}
\inf\{a(t), a(\frac{1}{n})\}, & 0 < t \leq \frac{1}{n}, \\
a(t), & \frac{1}{n} \leq t \leq 1 - \frac{1}{n}, \\
\inf\{a(t), a(1 - \frac{1}{n})\}, & 1 - \frac{1}{n} \leq t \leq 1.
\end{cases}
\]

Next, for $n \geq 2$, we define the operator $T_n : K \to K$ by
\[
T_n u(t) = \lambda \int_0^1 G(t, s)a_n(s)f(u(s)) \, ds.
\]

Obviously, $T_n$ is completely continuous on $K$ for any $n \geq 2$ by an application of Ascoli–Arzela theorem (see [26]). Denote $B_R = \{u \in K : \|u\| \leq R\}$. Then $T_n$ converges uniformly to $T_\lambda$ as $n \to \infty$. In fact, for any $t \in [0, 1]$, for each fixed $R > 0$ and $u \in B_R$, from (H2), we have that
\[
\left| T_n u(t) - T_\lambda u(t) \right| \leq \left| \lambda \int_0^1 G(t, s)[a(s) - a_n(s)]f(u(s)) \, ds \right|
\leq \int_0^{1/n} G(1, s)|a(s) - a_n(s)|f(u(s)) \, ds
+ \int_{1-n}^1 G(1, s)|a(s) - a_n(s)|f(u(s)) \, ds
\to 0 \quad (n \to \infty),
\]
where we have used the fact that $G(t, s) \leq G(1, s)$ for $t, s \in [0, 1]$. Hence $T_n$ converges uniformly to $T_\lambda$ as $n \to \infty$, and therefore $T_\lambda$ is completely continuous also. This completes the proof. □
The key tool in our approach is the following Krasnosel’skii’s fixed point theorem of cone expansion–compression type.

**Theorem 2.1.** (See [16,17].) Let $E$ be a Banach space and $K \subset E$ be a cone in $E$. Assume $\Omega_1$ and $\Omega_2$ are open subset of $E$ with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that

(A) $\|Tu\| \leq \|u\|$, $\forall u \in K \cap \partial \Omega_1$ and $\|Tu\| \geq \|u\|$, $\forall u \in K \cap \partial \Omega_2$; or

(B) $\|Tu\| \geq \|u\|$, $\forall u \in K \cap \partial \Omega_1$ and $\|Tu\| \leq \|u\|$, $\forall u \in K \cap \partial \Omega_2$.

Then $T$ has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Throughout this paper, we shall use the following notations:

\[
\begin{align*}
  f_0 &= \lim_{x \to 0^+} \frac{f(x)}{x}, & f^- &= \lim_{u \to 0^+} \frac{f(x)}{x}, & f^+ &= \lim_{u \to 0^+} \frac{f(x)}{x}, \\
  f_\infty &= \lim_{x \to \infty} \frac{f(x)}{x}, & f^- &= \lim_{x \to \infty} \frac{f(x)}{x}, & f^+ &= \lim_{x \to \infty} \frac{f(x)}{x}, \\
  A &= \max_{t \in [0,1]} \int_0^1 G(t,s)a(s) \, ds, & B &= \max_{t \in [0,1]} \int_0^\theta G(t,s)a(s) \, ds,
\end{align*}
\]

where we can choose $\theta \in (0, 1)$ such that $t_0 \in (\theta, 1)$, so $0 < B \leq A < \infty$.

### 3. Existence results

In this section, we discuss the existence of at least one positive solution for BVP (1.1). We obtain the following existence results.

**Theorem 3.1.** Suppose $(H_1)$, $(H_2)$ hold. In addition, assume that there exist two positive constants $R_1 \neq R_2$ such that

(A) $f(x) \leq \frac{R_1}{x^A}$, $\forall x \in [0, R_1]$;

(A2) $f(x) \geq \frac{R_2}{x^B}$, $\forall x \in [\frac{\theta^2}{2} R_2, R_2]$.

Then BVP (1.1) has at least one positive solution $u^* \in K$ with $\min\{R_1, R_2\} \leq \|u^*\| \leq \max\{R_1, R_2\}$.

**Proof.** Without loss of generality, we may assume that $R_1 < R_2$. Let $\Omega_1 = \{u \in C[0,1]: \|u\| < R_1\}$, $\Omega_2 = \{u \in C[0,1]: \|u\| < R_2\}$. It follows from $(A_1)$ that for any $u \in K \cap \partial \Omega_1$,

\[
\|T_{\lambda}u\| = \max_{t \in [0,1]} \lambda \int_0^1 G(t,s)a(s)f(u(s)) \, ds \leq \lambda \max_{t \in [0,1]} \int_0^1 G(t,s)a(s) \frac{R_1}{\lambda A} \, ds = R_1 = \|u\|
\]

therefore,

\[
\|T_{\lambda}u\| \leq \|u\|, \quad \text{for } u \in K \cap \partial \Omega_1.
\]

(3.1)
On the other hand, for any $u \in K \cap \partial \Omega_2$, we have that $\frac{\theta_2^2}{2} R_2 \leq u(s) \leq R_2$, $\theta \leq s \leq 1$. It follows from (A2) that for any $u \in K \cap \partial \Omega_2$,

$$
\|T_\lambda u\| = \max_{t \in [0,1]} \frac{\lambda}{\|u\|} \int_0^1 G(t,s) a(s) f\left(u(s)\right) ds
$$

$$
\geq \lambda \max_{t \in [0,1]} \int_\theta^1 G(t,s) a(s) f\left(u(s)\right) ds
$$

$$
\geq \lambda \max_{t \in [0,1]} \int_\theta^1 G(t,s) a(s) \frac{R_2}{\lambda B} ds
$$

$$
= R_2 = \|u\|,
$$

therefore,

$$
\|T_\lambda u\| \geq \|u\|, \quad \text{for} \quad u \in K \cap \partial \Omega_2.
$$

(3.2)

Applying (A) of Theorem 2.1 to (3.1) and (3.2) yields that $T_\lambda$ has a fixed point $u^* \in K \cap (\Omega_2 \setminus \Omega_1)$, and then $u^*$ is a positive solution of BVP (1.1). The proof is complete. □

**Theorem 3.2.** Suppose $\text{(H}_1\text{)}, \text{(H}_2\text{)}$ hold. Then for each $\lambda \in \left(\frac{2}{\theta^2 B f_\infty}, \frac{1}{A f_0^+}\right)$, BVP (1.1) has at least one positive solution.

**Proof.** We construct the sets $\Omega_1$ and $\Omega_2$ in order to apply Theorem 2.1. Let $\lambda \in \left(\frac{2}{\theta^2 B f_\infty}, \frac{1}{A f_0^+}\right)$, and choose $\varepsilon > 0$ such that

$$
\frac{2}{\theta^2 B (f_\infty - \varepsilon)} \leq \lambda \leq \frac{1}{A (f_0^+ + \varepsilon)}.
$$

By the definition of $f_0^+$, there exists $R_1 > 0$ such that $f(x) \leq (f_0^+ + \varepsilon)x$, for $x \in [0, R_1]$. So, choosing $u \in K$ with $\|u\| = R_1$, we have

$$
\|T_\lambda u\| = \max_{t \in [0,1]} \frac{\lambda}{\|u\|} \int_0^1 G(t,s) a(s) f\left(u(s)\right) ds
$$

$$
\leq \lambda \max_{t \in [0,1]} \int_0^1 G(t,s) a(s) \left(f_0^+ + \varepsilon\right) u(s) ds
$$

$$
\leq \lambda \max_{t \in [0,1]} \int_0^1 G(t,s) a(s) \left(f_0^+ + \varepsilon\right) \|u\| ds
$$

$$
= \lambda A \left(f_0^+ + \varepsilon\right) \|u\| \leq \|u\|.
$$

Consequently, $\|T_\lambda u\| \leq \|u\|$. So, if we set $\Omega_1 = \{u \in K: \|u\| < R_1\}$, then

$$
\|T_\lambda u\| \leq \|u\|, \quad \text{for} \quad u \in K \cap \partial \Omega_1.
$$

(3.3)
Next we construct the set $\Omega_2$. Considering the definition of $f_\infty^-$, there exists $R_2$ such that $f(x) \geq (f_\infty^- - \varepsilon)x$, for all $x \in [R_2, \infty)$. Let $R_2 = \max\{2R_1, 2\theta^{-2}R_2\}$ and set $\Omega_2 = \{u \in K: \|u\| < R_2\}$. If $u \in K$ with $\|u\| = R_2$, then $\min_{s \in [\theta, 1]} u(s) \geq \frac{1}{2}\theta^2\|u\| \geq R_2$. Thus, we have

$$
\|T_\lambda u\| = \max_{t \in [0, 1]} \int_0^1 G(t, s)\alpha(s)f(u(s))\,ds \\
\geq \lambda \max_{t \in [0, 1]} \int_\theta^1 G(t, s)\alpha(s)f(u(s))\,ds \\
\geq \lambda \max_{t \in [0, 1]} \int_\theta^1 G(t, s)\alpha(s)(f_\infty^- - \varepsilon)u(s)\,ds \\
\geq \lambda \max_{t \in [0, 1]} \int_\theta^1 G(t, s)\alpha(s)(f_\infty^- - \varepsilon)\frac{1}{2}\theta^2\|u\|\,ds \\
= \lambda \frac{1}{2}\theta^2B(f_\infty^- - \varepsilon)\|u\| \geq \|u\|.
$$

Hence,

$$
\|T_\lambda u\| \geq \|u\|, \quad \text{for } u \in K \cap \partial \Omega_2. \tag{3.4}
$$

Applying (A) of Theorem 2.1 to (3.3) and (3.4) yields that $T_\lambda$ has a fixed point $u^* \in K \cap (\Omega_2 \setminus \Omega_1)$, and then $u^*$ is a positive solution of BVP (1.1). The proof is complete. □

From the proof of Theorem 3.2, we have the following corollaries.

**Corollary 3.1.** Suppose that (H1), (H2) hold. In addition, assume that $f_0 = 0$ and $f_\infty = \infty$, i.e., $f$ is superlinear, then for any $\lambda \in (0, \infty)$, BVP (1.1) has at least one positive solution.

**Corollary 3.2.** Suppose that (H1), (H2) hold. In addition, assume that $f_\infty^- = \infty$, $0 < f_0^+ < \infty$. Then for each $\lambda \in (0, \frac{1}{Af_\infty^+})$, BVP (1.1) has at least one positive solution.

**Corollary 3.3.** Suppose that (H1), (H2) hold. In addition, assume that $f_0^+ = 0$, $0 < f_\infty^- < \infty$. Then for each $\lambda \in (\frac{2}{\theta^2Bf_0^-}, \frac{1}{Af_\infty^-})$, BVP (1.1) has at least one positive solution.

**Theorem 3.3.** Suppose (H1), (H2) hold. Then for each $\lambda \in (\frac{2}{\theta^2Bf_0^-}, \frac{1}{Af_\infty^+})$, BVP (1.1) has at least one positive solution.

**Proof.** We construct the sets $\Omega_1$ and $\Omega_2$ in order to apply Theorem 2.1. Let $\lambda \in (\frac{2}{\theta^2Bf_0^-}, \frac{1}{Af_\infty^+})$, and choose $\varepsilon > 0$ such that

$$
\frac{2}{\theta^2B(f_0^- - \varepsilon)} \leq \lambda \leq \frac{1}{A(f_\infty^- + \varepsilon)}.
$$
By the definition of $f_0^-$, there exists $R_1 > 0$ such that $f(x) \geq (f_0^- - \varepsilon)x$, for $x \in [0, R_1]$. So, choosing $u \in K$ with $\|u\| = R_1$, then $\min_{s \in [\theta, 1]} u(s) \geq \frac{1}{2}\theta^2\|u\|$. Thus, we have

$$
\|T_2 u\| = \max_{\lambda \in [0,1]} \lambda \int_0^1 G(t,s) a(s) f(u(s)) \, ds \\
\geq \lambda \max_{\theta \in [0,1]} \int_\theta^1 G(t,s) a(s) f(u(s)) \, ds \\
\geq \lambda \max_{\theta \in [0,1]} \int_\theta^1 G(t,s) a(s) (f_0^- - \varepsilon) u(s) \, ds \\
\geq \lambda \max_{\theta \in [0,1]} \int_\theta^1 G(t,s) a(s) (f_0^- - \varepsilon) \frac{1}{2}\theta^2\|u\| \, ds \\
= \lambda \frac{1}{2}\theta^2 B(f_0^- - \varepsilon)\|u\| \geq \|u\|.
$$

Consequently, $\|T_2 u\| \geq \|u\|$. So, if we set $\Omega_1 = \{u \in K : \|u\| < R_1\}$, then

$$
\|T_2 u\| \geq \|u\|, \quad \text{for } u \in K \cap \partial \Omega_1.
$$

(3.5)

Next we construct the set $\Omega_2$. Considering the definition of $f_0^+$, there exists $R_0$ such that $f(x) \leq (f_0^+ + \varepsilon)x$, for $x \in [R_0, \infty)$. We consider two cases: $f$ is bounded or $f$ is unbounded.

Case (i). Suppose that $f$ is bounded, say $f \leq M$. Set $R_2 = \max\{2R_1, \lambda MA\}$. Then if $u \in K$ with $\|u\| = R_2$, we have

$$
\|T_2 u\| = \max_{\lambda \in [0,1]} \lambda \int_0^1 G(t,s) a(s) f(u(s)) \, ds \\
\leq \lambda \max_{\theta \in [0,1]} \int_0^1 G(t,s) a(s) M \, ds \\
= \lambda AM \leq R_2 = \|u\|.
$$

Consequently, $\|T_2 u\| \leq \|u\|$. So, if we set $\Omega_2 = \{u \in K : \|u\| < R_2\}$. Then

$$
\|T_2 u\| \leq \|u\|, \quad \text{for } u \in K \cap \partial \Omega_2.
$$

(3.6)

Case (ii). When $f$ is unbounded, we let $R_2 > \max\{2R_1, R_0\}$ be such that $f(x) \leq f(R_2)$, for all $x \in [0, R_2]$. For $u \in K$ with $\|u\| = R_2$, by (2.2) and (3.5), we have

$$
\|T_2 u\| = \max_{\lambda \in [0,1]} \lambda \int_0^1 G(t,s) a(s) f(u(s)) \, ds \\
\leq \lambda \max_{\theta \in [0,1]} \int_0^1 G(t,s) a(s) f(R_2) \, ds
$$
\[ \leq \lambda \max_{t \in [0,1]} \int_{0}^{1} G(t, s) a(s) (f_{\infty}^{+} + \varepsilon) R_2 \, ds \]

\[ = \lambda \max_{t \in [0,1]} \int_{0}^{1} G(t, s) a(s) (f_{\infty}^{+} + \varepsilon) \|u\| \, ds \]

\[ = \lambda A (f_{\infty}^{+} + \varepsilon) \|u\| \leq \|u\|. \]

Thus, \( \|T_\lambda u\| \leq \|u\| \). Hence, if we set \( \Omega_2 = \{u \in K : \|u\| < R_2\} \), then

\[ \|T_\lambda u\| \leq \|u\|, \quad \text{for } u \in K \cap \partial \Omega_2. \quad (3.7) \]

Applying (B) of Theorem 2.1 to (3.5) and (3.6) to obtain \( T_\lambda \) has a fixed point \( u^* \in K \cap (\mathcal{O}_2 \setminus \Omega_1) \). Also, applying (B) of Theorem 2.1 to (3.5) and (3.7) yields that \( T_\lambda \) has a fixed point \( u^* \in K \cap (\mathcal{O}_2 \setminus \Omega_1) \). Then \( u^* \) is a positive solution of BVP (1.1). The proof is complete. \( \square \)

From the proof of Theorem 3.3 we have the following corollaries.

**Corollary 3.4.** Suppose that \( (H_1), (H_2) \) hold. In addition, assume that \( f_0 = \infty \) and \( f_{\infty} = 0 \), i.e. \( f \) is sublinear, then for any \( \lambda \in (0, \infty) \), BVP (1.1) has at least one positive solution.

**Corollary 3.5.** Suppose that \( (H_1), (H_2) \) hold. In addition, assume that \( f_{\infty}^{+} = \infty \), \( 0 < f_{\infty}^{-} < \infty \). Then for each \( \lambda \in (0, \frac{1}{A f_0}) \), BVP (1.1) has at least one positive solution.

**Corollary 3.6.** Suppose that \( (H_1), (H_2) \) hold. In addition, assume that \( f_{\infty}^{-} = 0 \), \( 0 < f_{\infty}^{+} < \infty \). Then for each \( \lambda \in \left( \frac{2}{\theta^2 B f_{\infty}}, \infty \right) \), BVP (1.1) has at least one positive solution.

### 4. Multiplicity results

In this section, we discuss the existence of multiplicity positive solutions for BVP (1.1). First discuss the existence of at least two positive solutions for BVP (1.1), we obtain the following existence results.

**Theorem 4.1.** Suppose that \( (H_1), (H_2) \) hold. In addition, assume that \( f_0 = \infty \) and \( f_{\infty} = \infty \). Then for each \( \lambda \in (0, \lambda_1) \), BVP (1.1) has at least two positive solutions, where

\[ \lambda_1 = \sup_{m > 0} \frac{m}{A \max_{u \in [0, m]} f(u)}. \]

**Proof.** Define a function \( h \) by

\[ h(m) = \frac{m}{A \max_{x \in [0, m]} f(x)}. \]

It is easy to see that \( h : (0, \infty) \to (0, \infty) \) is continuous and \( \lim_{m \to +0} h(m) = \lim_{m \to \infty} h(m) = 0 \). Thus there exists \( m_0 \in (0, \infty) \) such that \( h(m_0) = \sup_{m > 0} h(m) = \lambda^* \). For \( \lambda \in (0, \lambda^*) \), there exist constants \( c_1, c_2 \) (\( 0 < c_1 < m_0 < c_2 < \infty \)) with \( h(c_1) = h(c_2) = \lambda \). Thus

\[ f(x) \leq \frac{c_1}{\lambda A}, \quad \forall x \in [0, c_1], \quad \text{and} \quad f(x) \leq \frac{c_2}{\lambda A}, \quad \forall x \in [0, c_2]. \]
On the other hand, from \( f_0 = f_\infty = \infty \) we know there exist constants \( d_1, d_2 \) \((0 < d_1 < c_1 < c_2 < d_2 < \infty)\) with \( \frac{f(x)}{x} \geq 1/(\lambda \frac{1}{2} \theta^2 B) \) for \( x \in (0, d_1) \cup (\frac{1}{2} \theta^2 d_2, \infty) \). Thus

\[
f(x) \geq \frac{d_1}{\lambda B}, \quad \forall x \in \left[ \frac{1}{2} \theta^2 d_1, d_1 \right], \quad \text{and} \quad f(x) \geq \frac{d_2}{\lambda B}, \quad \forall x \in \left[ \frac{1}{2} \theta^2 d_2, d_2 \right].
\]

By Theorem 3.1, there exist two positive solutions \( u_1, u_2 \in K \) with \( d_1 \leq \|u_1\| \leq c_1 \) and \( c_2 \leq \|u_2\| \leq d_2 \). The proof is complete. \( \Box \)

**Theorem 4.2.** Suppose that \((H_1), (H_2)\) hold. In addition, assume that \( f_0 = f_\infty = 0 \). Then for each \( \lambda \in (\lambda_2, \infty) \), BVP (1.1) has at least two positive solutions, where

\[
\lambda_2 = \inf_{m > 0} \frac{m}{B \min_{u \in [\frac{1}{2} \theta^2 m, m]} f(u)}.
\]

**Proof.** Define a function \( p \) by

\[
p(m) = \frac{m}{B \min_{u \in [\frac{1}{2} \theta^2 m, m]} f(u)}.
\]

It is easy to see that \( p : (0, \infty) \rightarrow (0, \infty) \) is continuous and

\[
\lim_{m \rightarrow +0} p(m) = \lim_{m \rightarrow \infty} p(m) = \infty.
\]

There exists \( m_0 \in (0, \infty) \) such that \( p(m_0) = \inf_{m > 0} p(m) = \lambda_2 \). For \( \lambda \in (\lambda_2, \infty) \), there exist constants \( d_1, d_2 \) \((0 < d_1 < m_0 < d_2 < \infty)\) with \( p(d_1) = p(d_2) = \lambda \). Thus

\[
f(x) \geq \frac{d_1}{\lambda B}, \quad \forall x \in \left[ \frac{1}{2} \theta^2 d_1, d_1 \right], \quad \text{and} \quad f(x) \geq \frac{d_2}{\lambda B}, \quad \forall x \in \left[ \frac{1}{2} \theta^2 d_2, d_2 \right].
\]

On the other hand, because \( f_0 = 0 \), there exists constant \( c_1 \) \((0 < c_1 < d_1)\) with \( \frac{f(x)}{x} \leq \frac{1}{\lambda A} \) for any \( x \in (0, c_1) \). Thus

\[
f(x) \leq \frac{c_1}{\lambda A}, \quad \forall x \in [0, c_1].
\]

By \( f_\infty = 0 \), there exists constant \( c \) \((d_2 < c < \infty)\) with \( \frac{f(x)}{x} \leq \frac{1}{\lambda A} \) for any \( x \in (c, \infty) \). Let \( M = \sup_{x \in [0, c]} f(x) \) and \( c_2 \geq \max\{\lambda M A, c\} \). It is easy to see that

\[
f(x) \leq \frac{c_2}{\lambda A}, \quad \forall x \in [0, c_2].
\]

By Theorem 3.1, there exist two positive solutions \( u_1, u_2 \in K \) with \( c_1 \leq \|u_1\| \leq d_1 \) and \( d_2 \leq \|u_2\| \leq c_2 \). The proof is complete. \( \Box \)

Now we discuss the existence of infinitely many positive solutions for BVP (1.1). We obtain the following results.

**Theorem 4.3.** Suppose that \((H_1), (H_2)\) hold. In addition, assume that \( f \) is nondecreasing on \([0, \infty)\) and \( 0 < f_0^+ < f_0^- < \infty \). Then for each \( \lambda \in \left( \frac{2}{\theta^2 B f_0^+}, \frac{1}{A f_0^-} \right) \), BVP (1.1) has infinitely many positive solutions \( \{u_k\}_{k=1}^\infty \) with \( \|u_k\| \rightarrow 0 \) \((k \rightarrow \infty)\).
Proof. For $\lambda \in (\frac{2}{a^2Bf_0'}, \frac{1}{A f_0'})$ we have $f_0^- < \frac{1}{\lambda A}$ and $f_0^+ > \frac{2}{\lambda \theta^2 B}$. Therefore, there exist positive number sequences $\{a_k\}, \{b_k\}$ with $a_k \to 0$, $b_k \to 0$ ($k \to \infty$) such that

$$f(a_k) \leq \frac{a_k}{\lambda A}, \quad f(b_k) \geq \frac{2b_k}{\lambda \theta^2 B}, \quad k = 1, 2, 3, \ldots.$$  

Without loss of generality, we may assume that $a_1 > b_1 > a_2 > b_2 > \cdots > a_k > b_k > \cdots$.

For any natural number $k$, set

$$\Omega_{1k} = \{ u \in K : \|u\| < b_k \}, \quad \Omega_{2k} = \{ u \in K : \|u\| < a_k \}.$$  

Let $u \in \partial \Omega_{2k}$, then $\|u\| = a_k$, so $0 \leq u(s) \leq a_k$, $0 \leq s \leq 1$. Because of $f$ is nondecreasing we have

$$f(u(s)) \leq f(a_k) \leq \frac{a_k}{\lambda A}, \quad 0 \leq s \leq 1.$$  

Therefore,

$$\|T_\lambda u\| = \max_{t \in [0, 1]} \lambda \int_0^1 G(t, s)a(s)f(u(s))ds \leq \frac{a_k}{A} \max_{t \in [0, 1]} \int_0^1 G(t, s)a(s)ds = a_k = \|u\|.$$  

Consequently,

$$\|T_\lambda u\| \leq \|u\| \quad \text{for } u \in K \cap \partial \Omega_{2k}. \quad (4.1)$$  

If $u \in K \cap \partial \Omega_{1k}$, then $\|u\| = b_k$ and $\min_{0 \leq s \leq 1} u(s) \geq \frac{1}{2} \theta^2 b_k$, so $\frac{1}{2} \theta^2 b_k \leq u(s) \leq b_k$, $\theta \leq s \leq 1$. Because of $f$ is nondecreasing we have

$$f(u(s)) \geq f\left(\frac{1}{2} \theta^2 b_k\right) \geq \frac{b_k}{\lambda B}, \quad \theta \leq s \leq 1.$$  

Therefore,

$$\|T_\lambda u\| = \max_{t \in [0, 1]} \lambda \int_0^1 G(t, s)a(s)f(u(s))ds$$  

$$\geq \max_{t \in [0, 1]} \lambda \int_0^1 G(t, s)a(s)f(u(s))ds$$  

$$\geq \frac{b_k}{B} \max_{t \in [0, 1]} \int_0^1 G(t, s)a(s)ds = b_k.$$  

Consequently,

$$\|T_\lambda u\| \geq \|u\| \quad \text{for } u \in K \cap \partial \Omega_{1k}. \quad (4.2)$$  

Applying (A) of Theorem 2.1 to (4.1) and (4.2) yields that $T_\lambda$ has a fixed point $u_k \in K \cap (\Omega_{2k} \setminus \Omega_{1k})$, so $b_k \leq \|u_k\| \leq a_k$. From $a_k \to 0$, $b_k \to 0$ ($k \to \infty$) we obtain $\|u_k\| \to 0$ ($k \to \infty$).

The proof is complete. \qed

In the same way, we can prove the following theorem.
Theorem 4.4. Suppose that \((H_1), (H_2)\) hold. In addition, assume that \(f\) is nondecreasing on \([0, \infty)\) and \(0 < f^+_\infty, f^-\infty < \infty\). Then for each \(\lambda \in (Bf^+\infty, Af^-\infty)\), BVP (1.1) has infinitely many positive solutions \(\{u_k\}_{k=1}^\infty\) with \(\|u_k\| \to \infty (k \to \infty)\).

Acknowledgments

The author expresses her sincere gratitude to the anonymous referees for their helpful suggestions in improving the paper.

References

[25] Q. Yao, The existence and multiplicity of positive solutions for a third-order three-point boundary value problem,
[27] W. Zhao, Existence and uniqueness of solutions for third order nonlinear boundary value problems, Tohoku