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Homogenization of the two-dimensional Hall effect

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Abstract

In this paper, we study the two-dimensional Hall effect in a highly heterogeneous conducting material in the low magnetic field limit. Extending Bergman's approach in the framework of H-convergence we obtain the effective Hall coefficient which only depends on the corrector of the material resistivity in the absence of a magnetic field. A positivity property satisfied by the effective Hall coefficient is then deduced from the homogenization process. An explicit formula for the effective Hall coefficient is derived for anisotropic interchangeable two-phase composites.

Keywords: Homogenization; Hall effect; Dimension two; Two-phase composite

0. Introduction

Consider a conducting material with symmetric resistivity ρ. In electrodynamics it is well known (see e.g. [9]) that a magnetic field h induces a nonsymmetric conductivity $\rho(h)$ which corresponds to the Hall effect. In two dimensions and under the low field limit, $h \to 0$, the modified resistivity reads as

$$\rho(h) = \rho + rhJ + o(h), \tag{0.1}$$

where r is the Hall coefficient and J is the 90° rotation matrix. Now, consider a highly heterogeneous material with resistivity ρ^ε, where ε is a small parameter representing the scale of the microstructure. According to the first-order expansion (0.1), a low magnetic field h induces a perturbed resistivity $\rho^\varepsilon(h)$ satisfying

$$\rho^\varepsilon(h) = \rho^\varepsilon + r_\varepsilon hJ + o(h), \tag{0.2}$$

with a heterogeneous Hall coefficient r_ε. The problem is to compute the effective Hall coefficient r_* obtained from r_ε in the homogenization process as $\varepsilon \to 0$. Bergman [4] obtained for a periodic composite a formula for the effective Hall coefficient as an average-value only involving the local Hall coefficient and some local current fields in the absence of a magnetic field. His method is based on a small perturbation argument.

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In this paper, we extend the Bergman approach in the theoretical framework of H-convergence due to Murat and Tartar [14]. To this end, we consider the general setting of a sequence of equi-coercive and equi-bounded matrix-valued functions $A^\varepsilon(h)$ (not necessarily symmetric) in a bounded open set Ω of $\mathbb{R}^N$, $N \geqslant 1$, and which depends on a vector $h \in \mathbb{R}^n$, $n \geqslant 1$. We assume that $A^\varepsilon(h)$ satisfies the uniform Lipschitz condition (1.12) with respect to h. According to the H-convergence theory the sequence $A^\varepsilon(h)$ converges, up to a subsequence, in a suitable sense (see Definition 1.1 and the compactness Theorem 1.2) to some homogenized or effective matrix-valued $A^*(h)$. Then, if $A^\varepsilon(h)$ admits a first-order expansion of type (0.2), so does the homogenized matrix $A^*(h)$, hence

$$A^\varepsilon(h) = A^\varepsilon(0) + A_1^\varepsilon \cdot h + o(h) \underset{\varepsilon \to 0}{\overset{H}{\rightharpoonup}} A^*(h) = A^*(0) + A_1^* \cdot h + o(h). \tag{0.3}$$

Then, we prove (see Theorem 1.7) that the effective first-order term $A_1^* \cdot h$ is deduced from a weak limit only involving the first-order term $A_1^\varepsilon \cdot h$ combined with the correctors (see Definition 1.3) associated with the unperturbed matrix-valued functions $A^\varepsilon(0)$ and $A^\varepsilon(0)^T$ (and we do not necessarily assume that $A^\varepsilon(0)$ is symmetric).

We apply this homogenization process to the two-dimensional Hall effect with the conductivity $\sigma^\varepsilon(h) := \rho^\varepsilon(h)^{-1}$ satisfying the uniform Lipschitz condition (2.4) with respect to h and the first-order expansion (0.2). Therefore, the conductivity $\sigma^\varepsilon(h)$ H-converges to the homogenized conductivity $\sigma^*(h)$ so that the effective resistivity defined by $\rho^*(h) := \sigma^*(h)^{-1}$ satisfies the expansion

$$\rho^*(h) = \rho^* + r_* h J + o(h). \tag{0.4}$$

We then obtain the effective Hall coefficient r_* in (0.4) by the following process (see Theorem 2.3): the product $r_* \det(\sigma^*(0))$ is the limit in the distributions sense of the local Hall coefficient r_ε times the determinant of the unperturbed current field, i.e. the product of the conductivity $\sigma^\varepsilon(0)$ by the corrector associated with $\sigma^\varepsilon(0)$ in the absence of a magnetic field. This limit process allows us to prove the following positivity property (see Theorem 2.4): if the original Hall coefficient r_ε is bounded (from below or above) by a continuous function independent of ε, so is the effective Hall coefficient r_*.

We illustrate this homogenization approach of the two-dimensional Hall effect with two examples. The first one is based on an explicit formula (see Theorem 3.1) obtained by the third author [11] for an isotropic composite with two isotropic phases, which immediately gives the effective Hall coefficient and clearly shows the positivity property. The result of the second example seems new although it is also based on the same duality transformations due to Dykhne [7]. It consists of a periodic two-phase material the phases of which are not necessarily isotropic but interchangeable from the point of view of the homogenization process. For this geometry we obtain an explicit formula for the determinant and for the antisymmetric part of the homogenized matrix. From this we deduce (see Corollary 3.9) an explicit formula for the effective Hall coefficient when the interchangeable phases have an unperturbed conductivity matrix $\sigma^\varepsilon(0)$ in proportion to one another. As a consequence of the explicit formulas in the former two-phase examples, we also derive (see Corollaries 3.4 and 3.9) the limit value of the determinant of the corrector associated with $\sigma^\varepsilon(0)$ in each of the two phases.

The paper is organized as follows. In Section 1, we recall some results about H-convergence and the correctors, and we state a result of H-convergence with a parameter (Theorem 1.7). In Section 2 we show the homogenization process involving the Hall coefficient in a general two-dimensional microstructure, and the positivity property satisfied by the effective Hall coefficient. Section 3 is devoted to explicit formulas for the effective Hall coefficient for particular two-phase composites.

All along this article, we will use the following basic notations:

Notations.

- $N \in \mathbb{N}$, $N \geqslant 1$.
- For $x, y \in \mathbb{R}^N$, $x \cdot y := \sum_{i=1}^N x_i y_i$ where $x := (x_1, \ldots, x_N)$, $y := (y_1, \ldots, y_N)$.
- $\mathbb{R}^{M \times N}$ is the set of the $(M \times N)$ real matrices.
- For $A \in \mathbb{R}^{M \times N}$, $A = [A_{ij}]$, we denote by $A^T \in \mathbb{R}^{N \times M}$ its transpose defined by $[A^T] := [A_{ji}]$.
- I_2 is the unit matrix of $\mathbb{R}^{2\times 2}$ and J is the rotation matrix of 90°.

- $\mathcal{M}_+ \subset \mathbb{R}^{2\times 2}$ is the set of (2×2) matrices with a positive quadratic form, and $\mathcal{M}^s \subset \mathbb{R}^{2\times 2}$ is the set of (2×2) symmetric matrices (i.e. $A^T = A$, $\forall A \in \mathcal{M}^s$). Then, any matrix $A \in \mathcal{M}_+$ is uniquely decomposed into

$$A = A^s + \alpha(A)J, \quad \text{where } A^s \in \mathcal{M}^s \text{ and } \alpha(A) \in \mathbb{R}. \tag{0.5}$$

- $\mathcal{M}_+^s := \mathcal{M}_+ \cap \mathcal{M}^s$.
- $\mathcal{D}(\Omega)$ denotes the space of functions of class C^∞ on Ω with compact support in Ω, and $\mathcal{D}'(\Omega)$ denotes the space of distributions on Ω.
- We denote by $\lim_{\mathcal{D}'(\Omega)}$ the weak limit in the distributions sense.
- $\mathcal{M}(\Omega)$ denotes the space of Radon measures on Ω and we denote by $\lim_{\mathcal{M}(\Omega)}$ the weak-$*$ limit in the Radon measures sense.
- For $u:\mathbb{R}^N \to \mathbb{R}$, $\nabla u := (\frac{\partial u}{\partial x_i})_{1\leqslant i\leqslant N}$.
- For $U:\mathbb{R}^N \to \mathbb{R}^N$, $U := (u_1,\dots,u_N)$,

$$DU := \left(\frac{\partial u_j}{\partial x_i}\right)_{1\leqslant i,j\leqslant N} \quad \text{and} \quad \operatorname{div}(U) := \sum_{i=1}^N \frac{\partial u_i}{\partial x_i}. \tag{0.6}$$

- For $M:\mathbb{R}^N \to \mathbb{R}^{N\times N}$,

$$\operatorname{Div}(M) := \left(\sum_{i=1}^N \frac{\partial M_{ij}}{\partial x_i}\right)_{1\leqslant j\leqslant N} \quad \text{and} \quad \operatorname{Curl}(M) := \left(\frac{\partial M_{ij}}{\partial x_k} - \frac{\partial M_{kj}}{\partial x_i}\right)_{1\leqslant i,j,k\leqslant N}. \tag{0.7}$$

1. A few results from homogenization theory

1.1. Review of H-convergence

We recall the definition and some properties of H-convergence theory for second-order elliptic scalar equations introduced by Murat and Tartar [14] in the general case and by De Giorgi and Spagnolo [17] (under the name of G-convergence) in the symmetric case. Furthermore, we also give the definition of the correctors in homogenization.

Definition 1.1. (See Murat and Tartar [14].)

(i) Let Ω be a bounded open set of $\mathbb{R}^N$. We define the space $\mathcal{M}(\alpha,\beta;\Omega)$ as the set of measurable matrix-valued functions A defined on Ω such that

$$\forall \xi \in \mathbb{R}^N, \quad A(x)\xi\cdot\xi \geqslant \alpha|\xi|^2 \quad \text{and} \quad A^{-1}(x)\xi\cdot\xi \geqslant \beta^{-1}|\xi|^2, \quad \text{a.e. } x \in \Omega. \tag{1.1}$$

(ii) A sequence A^ε of $\mathcal{M}(\alpha,\beta;\Omega)$ is said to H-converge to A^* if $A^* \in \mathcal{M}(\alpha,\beta;\Omega)$, $f \in H^{-1}(\Omega)$ and the solution u_ε of

$$\begin{cases} -\operatorname{div}\big(A^\varepsilon \nabla u_\varepsilon\big) = f & \text{in } \Omega,\\ u_\varepsilon \in H_0^1(\Omega) \end{cases} \tag{1.2}$$

satisfies the weak convergences

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 & \text{in } H^1(\Omega)\text{-weak},\\ A^\varepsilon \nabla u_\varepsilon \rightharpoonup A^*\nabla u_0 & \text{in } L^2(\Omega)\text{-weak}, \end{cases} \tag{1.3}$$

where u_0 is the solution of

$$\begin{cases} -\operatorname{div}(A^* \nabla u_0) = f & \text{in } \Omega,\\ u_0 \in H_0^1(\Omega). \end{cases} \tag{1.4}$$

The H-convergence of A^ε to A^* is denoted by $A^\varepsilon \overset{H}{\rightharpoonup} A^*$.

An important result of H-convergence is the following “compactness theorem” due to Murat and Tartar [14]:

Theorem 1.2. *(See Murat and Tartar [14].) If A^ε is a sequence of $\mathcal{M}(\alpha,\beta;\Omega)$, then there exists a subsequence, still denoted by ε, and $A^* \in M(\alpha,\beta;\Omega)$ such that $A^\varepsilon \overset{H}{\rightharpoonup} A^*$.*

Finally, we recall the definition of correctors in homogenization and a result about the convergence of the correctors (see [14]).

Definition 1.3. Let A^ε be a sequence of $\mathcal{M}(\alpha,\beta;\Omega)$. Any matrix-valued function P^ε in $L^2(\Omega)^{N\times N}$ satisfying the properties

$$\begin{cases} P^\varepsilon \rightharpoonup I_N & \text{in } L^2(\Omega)^{N\times N}\text{-weak},\\ \operatorname{Curl}(P^\varepsilon) & \text{is compact in } H^{-1}(\Omega)^{N\times N\times N},\\ \operatorname{Div}(A^\varepsilon P^\varepsilon) & \text{is compact in } H^{-1}(\Omega)^N, \end{cases} \tag{1.5}$$

is called a corrector associated with A^ε.

Example 1.4. Let A^ε be a sequence of $\mathcal{M}(\alpha,\beta;\Omega)$ with H-limit A^* and let $U^\varepsilon \in H^1(\Omega)^N$ be the solution of

$$\begin{cases} \operatorname{Div}(A^\varepsilon DU^\varepsilon) = \operatorname{Div}(A^*) & \text{in } \Omega,\\ U^\varepsilon = I_N & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Then, the matrix-valued function defined by $P^\varepsilon := DU^\varepsilon$ is a corrector associated with A^ε.

We have the following result which is a consequence of the div-curl lemma of Murat and Tartar [13,14].

Proposition 1.5.

(i) *Assume that $A^\varepsilon \overset{H}{\rightharpoonup} A^*$. Then, any corrector P^ε associated with A^ε satisfies the weak convergences*

$$\begin{cases} A^\varepsilon P^\varepsilon \rightharpoonup A^* & \text{in } L^2(\Omega)^{N\times N}\text{-weak},\\ (P^\varepsilon)^T A^\varepsilon P^\varepsilon \rightharpoonup A^* & \text{in } \mathcal{D}'(\Omega)^{N\times N}. \end{cases} \tag{1.7}$$

(ii) *Conversely, let $A^\varepsilon \in \mathcal{M}(\alpha,\beta;\Omega)$ and let P^ε be a sequence such that*

$$\begin{cases} P^\varepsilon \rightharpoonup I_N & \text{in } L^2(\Omega)^{N\times N}\text{-weak},\\ \operatorname{Curl}(P^\varepsilon) & \text{is compact in } H^{-1}(\Omega)^{N\times N\times N},\\ \operatorname{Div}(A^\varepsilon P^\varepsilon) & \text{is compact in } H^{-1}(\Omega)^N,\\ A^\varepsilon P^\varepsilon \rightharpoonup A^* & \text{in } L^2(\Omega)^{N\times N}\text{-weak}. \end{cases} \tag{1.8}$$

Then, $A^\varepsilon \overset{H}{\rightharpoonup} A^*$.

(iii) *If P^ε and Q^ε are two correctors associated with A^ε, then $P^\varepsilon - Q^\varepsilon$ strongly converges to 0 in $L^2_{\mathrm{loc}}(\Omega)^{N\times N}$.*

1.2. H-convergence with a parameter

In the sequel, we use the following notation:

Notation 1.6. Let $n \in \mathbb{N}$, $n \geqslant 1$, and let $(E, \|\cdot\|)$ be a normed space. Let $f_0 \in E$ and $f, f_1 : \mathbb{R}^n \to E$. We set

$$f(h) = f_0 + f_1(h) + o_E(h), \tag{1.9}$$

whenever there exists $\delta : [0,+\infty) \to [0,+\infty)$ such that, for any $h \in \mathbb{R}^n$ with small enough norm, we have

$$\big\| f(h) - f_0 - f_1(h) \big\| \leqslant |h|\delta\big(|h|\big) \quad \text{with} \quad \lim_{t\to 0} \delta(t) = 0. \tag{1.10}$$

If $E := \mathbb{R}^n$, we will simply denote $o_E(h) = o(h)$. Moreover, when $f = f^\varepsilon$, $f_0 = f_0^\varepsilon$, $f_1 = f_1^\varepsilon$ depend on an additional small parameter $\varepsilon > 0$, the expansion

$$f^\varepsilon(h) = f_0^\varepsilon + f_1^\varepsilon(h) + o_E(h) \tag{1.11}$$

has the same sense as (1.9), the remainder $o_E(h)$ then being uniform with respect to ε.

Theorem 1.7. *Let $n \in \mathbb{N}$, $n \geqslant 1$, let B_κ be the open ball of $\mathbb{R}^n$ of radius κ and let $\alpha, \beta > 0$. Let $A^\varepsilon(h)$, for $h \in B_\kappa$, be a sequence in $\mathcal{M}(\alpha, \beta; \Omega)$ which satisfies the uniform Lipschitz condition*

$$\exists C > 0,\ \forall h, k \in B_\kappa, \quad \left\| A^\varepsilon(h) - A^\varepsilon(k) \right\|_{L^\infty(\Omega)^{N\times N}} \leqslant C|h-k|, \tag{1.12}$$

and the first-order expansion at $h = 0$

$$A^\varepsilon(h) = A^\varepsilon + A_1^\varepsilon \cdot h + o_{L^\infty(\Omega)^{N\times N}}(h), \tag{1.13}$$

where $A^\varepsilon = A^\varepsilon(0)$ and A_1^ε is a uniformly bounded sequence in $L^\infty(\Omega)^{n\times N\times N}$.

(i) *Then, there exists a subsequence of ε, still denoted by ε, such that $A^\varepsilon(h)$ H-converges to $A^*(h)$ in $\mathcal{M}(\alpha, \beta; \Omega)$ for any $h \in B_\kappa$, and*

$$A^*(h) = A^* + A_1^* \cdot h + o_{L^2(\Omega)^{N\times N}}(h), \tag{1.14}$$

where $A^* = A^*(0)$ and $A_1^* \in L^2(\Omega)^{n\times N\times N}$.

(ii) *Moreover, if P^ε and Q^ε are correctors associated respectively with A^ε and $(A^\varepsilon)^T$ we get, for any $h \in B_\kappa$,*

$$\left(Q^\varepsilon\right)^T \left(A_1^\varepsilon \cdot h\right) P^\varepsilon \rightharpoonup A_1^* \cdot h \quad \text{in } \mathcal{D}'(\Omega)^{N\times N}. \tag{1.15}$$

Remark 1.8. Colombini and Spagnolo proved in [6] that the homogenized matrix $A^*(h)$ is of class C^k with respect to the parameter h when all the derivatives $D_h^j A^\varepsilon(h)$, $j = 0, \ldots, k$, satisfy the uniform Lipschitz condition in h. In Theorem 1.7 we show that the Lipschitz control (1.12) of $A^\varepsilon(h)$ in h allows us to obtain the differentiability (1.14) of $A^*(h)$ at zero. The price to pay is that the remainder in (1.14) is only controlled in $L^2(\Omega)^{N\times N}$ and not in $L^\infty(\Omega)^{N\times N}$.

The proof of Theorem 1.7 which is based on classical H-convergence arguments is done in Appendix A for the reader's convenience.

1.3. About duality transformations

We recall a few results about two-dimensional duality transformation in the framework of H-convergence (see e.g. [12, Chapters 3, 4] for a general presentation and complete references).

Notation 1.9. For any $a, b, c \in \mathbb{R}$, we define for $A \in \mathcal{M}_+$,

$$f(A) := (aA + bJ)(-aI_2 + cJA)^{-1}. \tag{1.16}$$

For fixed a, b, c, we call f the duality function associated with (a, b, c).

Lemma 1.10. *For any $A \in \mathcal{M}_+$, $f(A) \in \mathcal{M}_+$ if and only if $bc > a^2$. Moreover, f is an involution on $\mathcal{M}_+$.*

The following result is due to Dykhne [7] who extended the pioneering work of Keller [8] on duality transformations. Here, the statement is written in terms of H-convergence:

Theorem 1.11. *(See Dykhne [7].) Let $a, b, c \in \mathbb{R}$ be such that $bc > a^2$ and let f be the duality function associated with (a, b, c). If $A^\varepsilon \in \mathcal{M}(\alpha, \beta; \Omega)$ H-converges to A^*, then $f(A^\varepsilon)$ H-converges to $f(A^*)$.*

Remark 1.12. The case $a = 0$, $b = c = 1$ corresponds to the following homogenization formula due to Mendelson [10]:

$$A^\varepsilon \overset{H}{\rightharpoonup} A^* \quad \Longrightarrow \quad \frac{(A^\varepsilon)^T}{\det(A^\varepsilon)} \overset{H}{\rightharpoonup} \frac{(A^*)^T}{\det(A^*)}. \tag{1.17}$$

2. Homogenization of the Hall effect in dimension 2

2.1. Definition of the Hall coefficient

In dimension N, consider a conducting material with conductivity σ. Under the effect of a constant low magnetic field h, the resulting conductivity $\sigma(h)$ depends on h and the corresponding resistivity $\rho(h) := \sigma(h)^{-1}$ satisfies the first-order expansion

$$\rho(h) = \rho + \rho_1 \cdot h + o(h), \tag{2.1}$$

where $\rho := \sigma^{-1}$. Moreover, physical considerations (see e.g. [9]) imply that $\sigma(h)^T = \sigma(-h)$, or equivalently, $\rho(h)^T = \rho(-h)$, hence ρ is a symmetric matrix-valued function of x and $\rho_1 \cdot h$ is an antisymmetric matrix-valued function of x.

In dimension $N = 2$, the magnetic field h then reduces to a scalar and the first-order expansion of $\rho(h)$ thus reads as

$$\rho(h) = \rho + rh\,J + o(h), \quad \text{where } J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.2}$$

and $\rho = \rho(0)$ is symmetric and r is a scalar function.

In (2.1), (2.2) and in the text which follows, $\rho(h), \rho, \sigma(h), \sigma, \ldots$ are matrix-valued functions and $r, s, \ldots$ are scalar functions implicitly depending on spatial coordinates x.

Definition 2.1. The function r in (2.2) is called the Hall coefficient in presence of the magnetic field h.

Now consider a heterogeneous material with conductivity σ^ε. Under a low magnetic field h in $(-\kappa, \kappa)$, $\kappa > 0$ small enough, the resulting conductivity $\sigma^\varepsilon(h)$ and resistivity $\rho^\varepsilon(h)$ satisfy the first-order expansions

$$\begin{cases} \sigma^\varepsilon(h) = \sigma^\varepsilon + s_\varepsilon h J + o_{L^\infty(\Omega)^{2\times 2}}(h), \\ \rho^\varepsilon(h) = \rho^\varepsilon + r_\varepsilon h\,J + o_{L^\infty(\Omega)^{2\times 2}}(h), \end{cases} \quad \text{where } r_\varepsilon, s_\varepsilon \in L^\infty(\Omega). \tag{2.3}$$

We also assume that there exist $\alpha, \beta > 0$ such that $\sigma^\varepsilon(h) \in \mathcal{M}(\alpha, \beta; \Omega)$, and that $\sigma^\varepsilon(h)$ satisfies the uniform Lipschitz condition

$$\exists C > 0,\ \forall h, k \in (-\kappa, \kappa), \quad \left\| \sigma^\varepsilon(h) - \sigma^\varepsilon(k) \right\|_{L^\infty(\Omega)^{2\times 2}} \leqslant C|h - k|. \tag{2.4}$$

Note that, since the remainders of (2.3) are uniform with respect to ε, estimate (2.4) implies that s_ε and r_ε are bounded sequences in $L^\infty(\Omega)$.

There is a link between the Hall coefficient r_ε and the coefficient s_ε for conductivity, given by the following result:

Proposition 2.2. *One has*

$$s_\varepsilon = -\det(\sigma^\varepsilon) r_\varepsilon. \tag{2.5}$$

Proof. Since $\rho^\varepsilon(h)\sigma^\varepsilon(h) = I_2$ and $\rho^\varepsilon \sigma^\varepsilon = I_2$, we deduce from (2.3) that

$$s_\varepsilon (\sigma^\varepsilon)^{-1} J + r_\varepsilon J \sigma^\varepsilon = 0. \tag{2.6}$$

Taking into account the symmetry of σ^ε, this leads us to

$$s_\varepsilon I_2 = -r_\varepsilon J \sigma^\varepsilon J^{-1} \sigma^\varepsilon = -\det(\sigma^\varepsilon) r_\varepsilon I_2, \tag{2.7}$$

which gives equality (2.5). $\square$

2.2. Homogenization of the Hall effect

We have the following homogenization result:

Theorem 2.3. *Let Ω be a bounded open set of $\mathbb{R}^2$. Let $\sigma^\varepsilon(h)$, for $h \in (-\kappa, \kappa)$, be a sequence in $\mathcal{M}(\alpha, \beta; \Omega)$ satisfying* (2.3) *and* (2.4) *with $s_\varepsilon, r_\varepsilon$ two bounded sequences in $L^\infty(\Omega)$. Then, there exists a subsequence of ε, still denoted by ε, such that $\sigma^\varepsilon(h)$ H-converges to $\sigma^*(h)$ for any $h \in (-\kappa, \kappa)$. The homogenized conductivity $\sigma^*(h)$ and the effective resistivity defined by $\rho^*(h) := \sigma^*(h)^{-1}$, satisfy the expansions*

$$\begin{cases} \sigma^*(h) = \sigma^* + s_* h J + o_{L^2(\Omega)^{2\times 2}}(h), \\ \rho^*(h) = \rho^* + r_* h J + o_{L^2(\Omega)^{2\times 2}}(h), \end{cases} \quad \text{with } s_* = -\det(\sigma^*) r_*, \tag{2.8}$$

where σ^* is the H-limit of σ^ε and $\rho^* := (\sigma^*)^{-1}$. Moreover, s_* and the effective Hall coefficient r_* belong to $L^\infty(\Omega)$ and are given by

$$s_* = \lim_{\mathcal{D}'(\Omega)} \left[s_\varepsilon \det(P^\varepsilon) \right] \quad \text{and} \quad \det(\sigma^*) r_* = \lim_{\mathcal{D}'(\Omega)} \left[r_\varepsilon \det(\sigma^\varepsilon P^\varepsilon) \right], \tag{2.9}$$

for any corrector P^ε associated with the matrix σ^ε.

Proof. On the one hand, by Theorem 1.7(ii) $\sigma^\varepsilon(h)$ H-converges to $\sigma^*(h)$, up to a subsequence, for any $h \in (-\kappa, \kappa)$, and

$$\sigma^*(h) = \rho^* + h\sigma_1^* + o_{L^{2\times 2}(\Omega)(h)}, \quad \text{with } \sigma_1^* = \lim_{\mathcal{D}'(\Omega)^{2\times 2}} \left[r_\varepsilon (P^\varepsilon)^T J P^\varepsilon \right], \tag{2.10}$$

where P^ε is a corrector associated with σ^ε. Since by assumption $\sigma^\varepsilon(h)^T = \sigma^\varepsilon(-h)$ and by a classical property of H-convergence $\sigma^\varepsilon(h)^T$ H-converges to $\sigma^*(h)^T$, we get $\sigma^*(h)^T = \sigma^*(-h)$. Hence, the matrix-valued function σ_1^* in (2.10) is antisymmetric. Therefore, there exists $s_* \in L^2(\Omega)$ such that $\sigma_1^* = s_* J$. This combined with (2.10) yields the first-order expansion

$$\sigma^*(h) = \sigma^* + s_* h J + o_{L^{2\times 2}(\Omega)}(h), \tag{2.11}$$

where $s_* \in L^2(\Omega)$ is given by

$$s_* I_2 = \lim_{\mathcal{D}'(\Omega)^{2\times 2}} \left[s_\varepsilon J^{-1} (P^\varepsilon)^T J P^\varepsilon \right] = \lim_{\mathcal{D}'(\Omega)^{2\times 2}} \left[s_\varepsilon \det(P^\varepsilon) I_2 \right], \tag{2.12}$$

which implies the first equality of (2.9).

On the other hand, by the uniform Lipschitz condition (2.4) combined with the estimate of the difference of two H-limits (see e.g. [5]) we have

$$\left\| \sigma^*(h) - \sigma^* \right\|_{L^\infty(\Omega)^{2\times 2}} \leqslant c|h|. \tag{2.13}$$

By the second part of Theorem 2.4 above and the boundedness of s_ε in $L^\infty(\Omega)$, the function s_* belongs to $L^\infty(\Omega)$. This combined with expansion (2.11) and estimate (2.13) implies that the effective resistivity $\rho^*(h) := \sigma^*(h)^{-1}$ satisfies the second expansion of (2.8). Similarly to (2.5) we deduce from the expansions of (2.8) the equality $s_* = -\det(\sigma^*) r_*$, which concludes the proof of (2.8).

Finally, by the first equality of (2.9) and (2.5) we obtain

$$\det(\sigma^*) r_* = -s_* = -\lim_{\mathcal{D}'(\Omega)^{2\times 2}} \left[s_\varepsilon \det(P^\varepsilon) \right] = \lim_{\mathcal{D}'(\Omega)^{2\times 2}} \left[r_\varepsilon \det(\sigma^\varepsilon P^\varepsilon) \right], \tag{2.14}$$

which yields the second equality of (2.9). □

2.3. Positivity property of the Hall effect

We have the following result:

Theorem 2.4. *Under the assumptions of Theorem* 2.3*, let r_1, r_2 be two continuous functions in Ω. Then, if the effective Hall coefficient r_ε satisfies the inequalities $r_1 \leqslant r_\varepsilon \leqslant r_2$ a.e. in Ω, so does the effective Hall coefficient r_*.*

Similarly and independently, let s_1, s_2 be two continuous functions in Ω. Then, if the coefficient s_ε satisfies the inequalities $s_1 \leqslant s_\varepsilon \leqslant s_2$ a.e. in Ω, so does the effective coefficient s_*.

Remark 2.5. Let r be a continuous function in Ω. The particular case $r_\varepsilon = r$ a.e. in Ω implies that the effective Hall coefficient also satisfies $r_* = r$ a.e. in Ω.

Proof of Theorem 2.4. The proof of Theorem 2.4 is based on the result due to Raitums [15] (see also Theorem 1.3.23 of [2, p. 60]), that any H-limit is the pointwise limit of a sequence of periodic homogenized matrices, combined with the positivity of the determinant of the periodic correctors due to Alessandrini and Nesi [1] (see also [3]).

Taking into account the continuity of the functions r_1, r_2 and using a locality argument we can assume that r_1, r_2 are two constants in the sequel. Following the approach of [2], consider for fixed $\varepsilon, t, h > 0$ and $x \in \Omega$, the periodic homogenized matrix $\sigma^*_{\varepsilon,t,x}(h)$ defined by

$$\sigma^*_{\varepsilon,t,x}(h) := \int_Y \sigma^\varepsilon(h)(x+ty)\, DW_{\varepsilon,t,x}(h,y)\, dy \tag{2.15}$$

where $Y := (0,1)^2$, $\sigma^\varepsilon(h)(x+t\cdot)$ is extended by Y-periodicity in $\mathbb{R}^2$, and $W_{\varepsilon,t,x}(h,\cdot)$ is the unique vector-valued function in $H^1_{\mathrm{loc}}(\mathbb{R}^2)^2$ solution of the cell problem

$$\begin{cases} \operatorname{div}\big(\sigma^\varepsilon(h)(x+ty)\, DW_{\varepsilon,t,x}(h,y)\big) = 0 & \text{in } \mathcal{D}'\big(\mathbb{R}^2\big) \\ y \mapsto W_{\varepsilon,t,x}(h,y) - y & \text{is } Y\text{-periodic with zero } Y\text{-average.} \end{cases} \tag{2.16}$$

Consider, for fixed ε, t, x, the oscillating sequence $\rho^\varepsilon(h)(x + t\frac{y}{\delta})$ as δ tends to zero. For this resistivity the second expansion of (2.3) reads as

$$\rho^\varepsilon(h)(x+t\cdot) = \rho^\varepsilon(x+t\cdot) + r_\varepsilon(x+t\cdot)hJ + o_{L^\infty(\Omega)^{2\times 2}}(h), \tag{2.17}$$

where $r_\varepsilon(x+t\cdot)$ is Y-periodic. Then, by (2.8) the expansion of the effective resistivity is given by

$$\rho^*_{\varepsilon,t,x}(h) = \rho^*_{\varepsilon,t,x}(0) + r^*_{\varepsilon,t,x} hJ + o(h), \tag{2.18}$$

where the effective resistivity $\rho^*_{\varepsilon,t,x}(h)$ is the inverse of the constant homogenized matrix $\sigma^*_{\varepsilon,t,x}(h)$ defined by (2.15). Moreover, by (2.16) the sequence of gradients $DW_{\varepsilon,t,x}(0,\frac{y}{\delta})$ is a corrector associated with the sequence $\sigma^\varepsilon(x+t\frac{y}{\delta})$ in the sense of Definition 1.3. Therefore, by the second limit of (2.9) where the scale δ replaces ε, the product of the effective Hall coefficient $r^*_{\varepsilon,t,x}$ by $\det(\sigma^*_{\varepsilon,t,x}(h))$ is the limit in the distributions sense of the sequence $r_\varepsilon(x+t\frac{y}{\delta})\det(\sigma^\varepsilon(0)(x+t\frac{y}{\delta})\, DW_{\varepsilon,t,x}(0,\frac{y}{\delta}))$ as δ tends to zero. Hence, again by periodicity we get

$$r^*_{\varepsilon,t,x} \det\big(\sigma^*_{\varepsilon,t,x}(0)\big) = \int_Y r_\varepsilon(x+ty) \det\big(\sigma^\varepsilon(0)(x+ty)\, DW_{\varepsilon,t,x}(0,y)\big)\, dy. \tag{2.19}$$

On the other hand, since det is a null Lagrangian and $\sigma^\varepsilon(0)(x+t\cdot)\, DW_{\varepsilon,t,x}(0,\cdot)$ is Y-periodic and divergence free, by definition (2.15) we have

$$\int_Y \det\big(\sigma^\varepsilon(0)(x+ty)\, DW_{\varepsilon,t,x}(0,y)\big)\, dy = \det\big(\sigma^*_{\varepsilon,t,x}(0)\big). \tag{2.20}$$

Furthermore, thanks to the positivity result of [1] we have $\det(DW_{\varepsilon,t,x}(0,y)) > 0$ a.e. $y \in Y$. Then, from (2.19) and (2.20) we deduce that $r_1 \leqslant r^*_{\varepsilon,t,x} \leqslant r_2$. Therefore, considering the scalar product of the expansion (2.18) with the matrix J, we obtain

$$2r_1 h \leqslant \rho^*_{\varepsilon,t,x}(h) : J - \rho^*_{\varepsilon,t,x}(0) : J + o(h) \leqslant 2r_2 h. \tag{2.21}$$

Moreover, using for example Theorem 1.3.23 of [2] there exist two sequences $t, h_n > 0$ going to zero, such that

$$\lim_{t\to 0}\lim_{\varepsilon\to 0} \sigma^*_{\varepsilon,t,x}(0) = \sigma^*(x) \quad \text{and} \quad \lim_{t\to 0}\lim_{\varepsilon\to 0} \sigma^*_{\varepsilon,t,x}(h_n) = \sigma^*(h_n)(x), \quad \forall n \in \mathbb{N} \text{ and a.e. } x \in \Omega, \tag{2.22}$$

hence, by the continuity of the inverse the following similar limits hold for the resistivities:

$$\lim_{t\to 0}\lim_{\varepsilon\to 0} \rho^*_{\varepsilon,t,x}(0) = \rho^*(x) \quad \text{and} \quad \lim_{t\to 0}\lim_{\varepsilon\to 0} \rho^*_{\varepsilon,t,x}(h_n) = \rho^*(h_n)(x), \quad \forall n \in \mathbb{N} \text{ and a.e. } x \in \Omega. \tag{2.23}$$

Then, passing to the double limit $\varepsilon \to 0$, $t \to 0$ in (2.21) it follows

$$2r_1 h_n \leqslant \rho^*(h_n)(x) : J - \rho^*(x) : J + o(h_n) \leqslant 2r_2 h_n, \quad \forall n \in \mathbb{N} \text{ and a.e. } x \in \Omega. \tag{2.24}$$

On the other hand, consider a Lebesgue point $x_0 \in \Omega$ of the function r_* and let $B(x_0, \delta)$ be the ball of center x_0 and of radius $\delta > 0$. The limit expansion (2.8) satisfied by $\rho^*(h)$ yields

$$\left(\fint_{B(x_0,\delta)} \rho^*(h_n)(x)\,dx \right) : J = \left(\fint_{B(x_0,\delta)} \rho^*(x)\,dx \right) : J + 2\left(\fint_{B(x_0,\delta)} r_*(x)\,dx \right) h_n + o_\delta(h_n), \tag{2.25}$$

where, by the Cauchy–Schwarz inequality in $L^2(B(x_0, \delta))$, $|o_\delta(h_n)| \leqslant \frac{1}{\sqrt{\pi\delta^2}}\, o(h_n)$. The former estimate combined with (2.24) implies that for any $n \in \mathbb{N}$,

$$r_1 \leqslant \fint_{B(x_0,\delta)} r_*(x)\,dx + \frac{o_\delta(h_n)}{h_n} \leqslant r_2. \tag{2.26}$$

Therefore, passing successively to the limits $h_n \to 0$ and $\delta \to 0$ in (2.26), we get the desired inequalities $r_1 \leqslant r_*(x_0) \leqslant r_2$.

The proof of the inequalities for the coefficient s_* is quite similar, replacing in the previous proof the current field $\sigma^\varepsilon(0)(x + t\cdot)\, DW_{\varepsilon,t,x}(0, \cdot)$ with the electric field $DW_{\varepsilon,t,x}(0, \cdot)$. $\square$

3. Computation of the effective Hall coefficient and applications

We will consider particular cases of two-phase composites where, under some assumptions, explicit formulas of the Hall coefficient can be derived without the use of formula (2.9). These results combined with formula (2.9) then allow us to obtain the weak limit of the corrector determinant associated with the resistivity matrix in each of the two phases.

First, we recall the formula for the effective Hall coefficient for isotropic two-phase composites, obtained by the third author in [11]. Then, we prove a new (up to our knowledge) formula for anisotropic interchangeable two-phase composites, like those depicted in Figs. 1 and 2. The two results are based on the duality transformations (1.16).

3.1. The isotropic two-phase case

Let ρ_1, ρ_2, r_1, r_2 be four continuous even functions on $\mathbb{R}$, ρ_1, ρ_2 being positive. We consider a two-phase material with resistivity

$$\rho^\varepsilon(h) := \rho_\varepsilon(h)\, I_2 + r_\varepsilon(h) h J, \quad \text{where } \begin{cases} \rho_\varepsilon(h) := \chi_\varepsilon \rho_1(h) + (1-\chi_\varepsilon)\rho_2(h), \\ r_\varepsilon(h) := \chi_\varepsilon r_1(h) + (1-\chi_\varepsilon) r_2(h). \end{cases} \tag{3.1}$$

We assume that the symmetric part $\sigma^\varepsilon(h)^s$ of the conductivity $\sigma^\varepsilon(h) := \rho^\varepsilon(h)^{-1}$, H-converges to the isotropic matrix $\sigma_*(h)\, I_2$, where $\sigma_*(h)$ is a positive function in $L^\infty(\Omega)$, which is continuous and even with respect to h. Then, the third author proved the following homogenization result:

Theorem 3.1. *(See Milton [11].) Up to a subsequence, $\sigma^\varepsilon(h)$ H-converges to $\sigma^*(h) = \rho^*(h)^{-1}$, where the effective resistivity satisfies $\rho^*(h) = \rho_*(h) I_2 + r_*(h) h J$, and the effective Hall coefficient $r_*(h)$ is given by*

$$\frac{r_2(h) - r_*(h)}{r_2(h) - r_1(h)} = \frac{\rho_2(h)^2 - \rho_*(h)^2 + (r_2(h) - r_*(h))^2 h^2}{\rho_2(h)^2 - \rho_1(h)^2 + (r_2(h) - r_1(h))^2 h^2}. \tag{3.2}$$

In the low-field limit $h \to 0$, formula (3.2) *reduces to the Shklovskii's formula* [16]

$$\frac{r_2(0) - r_*(0)}{r_2(0) - r_1(0)} = \frac{\rho_2(0)^2 - \rho_*(0)^2}{\rho_2(0)^2 - \rho_1(0)^2}, \tag{3.3}$$

and $r_* = r_*(0)$ in the expansion (2.8).

Remark 3.2. In the isotropic case of Theorem 3.1 the conductivity $\sigma^\varepsilon(h)$ H-converges, up to a subsequence, to $\sigma^*(h)$ with $\sigma^*(h)^s = \sigma_*(h)\, I_2$. Then, thanks to the isotropy of the symmetric parts $\sigma^\varepsilon(h)^s, \sigma^*(h)^s$, and the duality transformation (1.17) we have

$$\rho^\varepsilon(h) = \sigma^\varepsilon(h)^{-1} = \frac{\sigma^\varepsilon(h)^T}{\det(\sigma^\varepsilon(h))} \xrightharpoonup{H} \frac{\sigma^*(h)^T}{\det(\sigma^*(h))} = \sigma^*(h)^{-1} = \rho^*(h). \tag{3.4}$$

Therefore, the resistivity $\rho^\varepsilon(h)$ H-converges to the effective resistivity $\rho^*(h)$. Moreover, a relation like (3.2) also holds for the homogenized conductivity matrix $\sigma^*(h)$.

Remark 3.3. In Section 4.3 of [12, p. 65], the third author also gives an explicit formula for the skew part of the effective matrix for ordinary checkerboards (or isotropic interchangeable two-phase composites). This leads us easily to an explicit formula for the effective Hall coefficient $r_*(h)$. We will extend this formula to anisotropic interchangeable two-phase composites in the next section.

By the classical bounds on the effective matrix $\rho_*(0)\, I_2$ we have

$$\min\bigl(\rho_1(0), \rho_2(0)\bigr) \leqslant \rho_*(0) \leqslant \max\bigl(\rho_1(0), \rho_2(0)\bigr) \quad \text{a.e. in } \Omega, \tag{3.5}$$

which implies that the right-hand side of (3.3) is nonnegative, hence

$$\min\bigl(r_1(0), r_2(0)\bigr) \leqslant r_*(0) \leqslant \max\bigl(r_1(0), r_2(0)\bigr) \quad \text{a.e. in } \Omega. \tag{3.6}$$

These bounds on the effective Hall coefficient illustrate the positivity property of Theorem 2.4 since the Hall coefficient $r_\varepsilon(0)$ of the heterogeneous material clearly satisfies

$$\min\bigl(r_1(0), r_2(0)\bigr) \leqslant r_\varepsilon(0) \leqslant \max\bigl(r_1(0), r_2(0)\bigr) \quad \text{a.e. in } \Omega. \tag{3.7}$$

Corollary 3.4. *Let $\rho_1, \rho_2 \in (0, +\infty)$, with $\rho_1 \neq \rho_2$. Consider the two-phase material with isotropic resistivity*

$$\rho^\varepsilon := \bigl(\chi_\varepsilon \rho_1 + (1-\chi_\varepsilon)\rho_2\bigr) I_2. \tag{3.8}$$

Assume that the conductivity $\sigma^\varepsilon := (\rho^\varepsilon)^{-1}$ H-converges to the isotropic matrix $(\rho_*)^{-1} I_2$. Then, any corrector P^ε associated with σ^ε satisfies the formula

$$\lim_{\mathcal{D}'(\Omega)} \bigl[\chi_\varepsilon \det\bigl(P^\varepsilon\bigr)\bigr] = \frac{\rho_*^{-2} - \rho_2^{-2}}{\rho_1^{-2} - \rho_2^{-2}}. \tag{3.9}$$

Proof. Take $r_1(0) := \rho_1^2$ and $r_2(0) := 0$ in Theorem 3.1, which yields the equality

$$r_\varepsilon(0) \det\bigl(\sigma^\varepsilon P^\varepsilon\bigr) = \chi_\varepsilon \det\bigl(P^\varepsilon\bigr). \tag{3.10}$$

Then, the second formula of (2.9) and formula (3.3) imply the desired result. □

3.2. The anisotropic interchangeable two-phase case

Definition 3.5. Consider a two-phase material with phases A and B, the conductivity matrix A^ε of which is given by

$$A^\varepsilon := \chi_\varepsilon A + (1-\chi_\varepsilon) B. \tag{3.11}$$

Also consider the two-phase material obtained by exchanging the two phases A and B, the conductivity matrix of which is thus

$$B^\varepsilon := \chi_\varepsilon B + (1-\chi_\varepsilon) A. \tag{3.12}$$

The material is said to be interchangeable if B^ε and A^ε have the same H-limit.

Example 3.6.

1. A checkerboard is a periodic microstructure whose period cell is a parallelogram shared in four equal 1/2-homothetic parallelograms (see Fig. 1). Consider a checkerboard with clockwise phases A and B ($A, B \in \mathcal{M}_+$). Then, the checkerboard of phases B and A must have the same effective matrix. Thus, the two-phase periodic checkerboard represented in Fig. 1 is a periodic interchangeable material.
2. The periodic material represented in Fig. 2 by two of its period cells, is also an interchangeable two-phase material but not of checkerboard type.

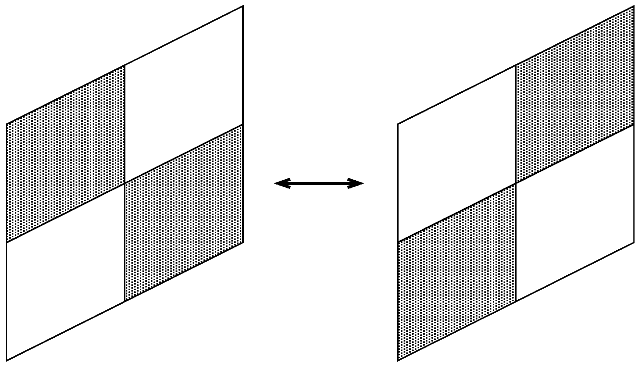

Fig. 1. Two period cells of a generalized checkerboard.

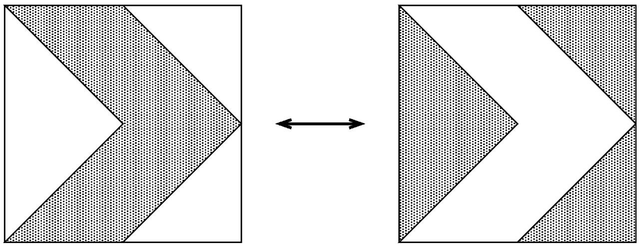

Fig. 2. Two period cells of an interchangeable material with a herring-bone pattern.

We have the following result for interchangeable two-phase composites:

Theorem 3.7. *Consider an interchangeable two-phase material with phases A and $\lambda A + \mu J$, $\lambda, \mu \in \mathbb{R}$. Assume that $\lambda > 0$ and*

$$\lambda \det(A) + \frac{\mu(\mu + 2\lambda\alpha(A))}{\lambda + 1} > \left(\frac{\mu + 2\lambda\,\alpha(A)}{\lambda + 1}\right)^2, \quad \text{where } A - A^T = 2\alpha(A)J. \tag{3.13}$$

Then, the matrix-valued function A^ε associated with this two-phase material H-converges to the constant matrix A^* such that

$$\det\big(A^*\big) = \lambda \det(A) + \frac{\mu(\mu + 2\lambda\alpha(A))}{\lambda + 1} \quad \text{and} \quad \alpha\big(A^*\big) = \frac{\mu + 2\lambda\alpha(A)}{\lambda + 1}. \tag{3.14}$$

Remark 3.8. The determinant and the antisymmetric part of A^* are explicit but not the whole matrix in general.

Applying this result to the conductivity of a two-phase microstructure with interchangeable, symmetric and proportional phases, and using Theorem 2.3, we get the following result:

Corollary 3.9. *Consider an interchangeable two-phase material with conductivity*

$$\sigma^\varepsilon := \chi_\varepsilon \sigma^1 + (1 - \chi_\varepsilon)\lambda\,\sigma^1, \quad \text{with } \sigma^1 \in \mathcal{M}_+^s \text{ and } \lambda > 0, \tag{3.15}$$

and consider the conductivity $\rho^\varepsilon(h)$ under the low magnetic field h

$$\sigma^\varepsilon(h) = \sigma^\varepsilon + s_\varepsilon h J, \quad \text{where } s_\varepsilon := \chi_\varepsilon s_1 + (1 - \chi_\varepsilon)s_2,\ s_1, s_2 \in \mathbb{R}. \tag{3.16}$$

Then, the resistivity $\rho^\varepsilon(h) := \sigma^\varepsilon(h)^{-1}$ satisfies the expansion

$$\rho^\varepsilon(h) = \rho^\varepsilon + r_\varepsilon h J + o_{L^\infty(\Omega)^{2\times 2}}, \quad \text{where } \begin{cases} \rho^\varepsilon := \chi_\varepsilon(\sigma_1)^{-1} + (1 - \chi_\varepsilon)(\lambda\sigma_1)^{-1}, \\ r_\varepsilon := \chi_\varepsilon r_1 + (1 - \chi_\varepsilon)r_2, \end{cases} \tag{3.17}$$

where the constants r_1, r_2 are defined by

$$r_1 := -\frac{s_1}{\det(\sigma_1)} \quad \textit{and} \quad r_2 := -\frac{s_2}{\det(\lambda\sigma_1)}. \tag{3.18}$$

The coefficient s_* and the effective Hall coefficient r_* in expansion (2.8) *are given by the following formulas*:

$$s_* = \frac{s_2 + \lambda s_1}{1+\lambda} \quad \textit{and} \quad r_* = \frac{r_1 + \lambda r_2}{1+\lambda}. \tag{3.19}$$

Moreover, for any corrector P^ε associated with σ^ε, we have

$$\lim_{\mathcal{D}'(\Omega)} \left[\chi_\varepsilon \det\left(P^\varepsilon\right)\right] = \frac{\lambda}{\lambda+1}. \tag{3.20}$$

3.3. Proof of the results

3.3.1. Proof of Theorem 3.7

First we prove the following result:

Lemma 3.10. *Let $A \in \mathcal{M}_+$. Then, the following equivalence holds true for any $\lambda, \mu \in \mathbb{R}$:*

$$AJA = \lambda A + \mu J \quad \Leftrightarrow \quad \lambda = -2\alpha(A) \quad \textit{and} \quad \mu = \det(A). \tag{3.21}$$

Proof. On the one hand, from $A = A^s + \alpha(A)J$ we deduce that

$$AJA = A^s J A^s - \alpha(A)A^s - \alpha(A)A^s - \alpha(A)^2 J = -2\alpha(A)A^s + \left(\det\left(A^s\right) - \alpha(A)^2\right)J, \tag{3.22}$$

taking into account that $A^s J A^s = \det(A^s)\, J$. Furthermore, it is easy to check that

$$\det(A) = \det\left(A^s\right) + \alpha(A)^2, \tag{3.23}$$

hence

$$AJA = -2\alpha(A)A^s + \left(\det(A) - 2\alpha(A)^2\right)J. \tag{3.24}$$

On the other hand, $AJA = \lambda A + \mu J$ is equivalent to

$$AJA = \lambda A^s + \left(\lambda\alpha(A) + \mu\right)J. \tag{3.25}$$

From the uniqueness of the decompositions (3.24) and (3.25) we deduce the desired result. □

Now, let us prove Theorem 3.7. First, let us show there exist $a, b, c \in \mathbb{R}$ with $bc > a^2$, such that $f(A) = \lambda A + \mu J$, where $f(A)$ is given by (1.16) and $\lambda > 0$. We have

$$\lambda c AJA = (a + a\lambda + c\mu)A + (b + a\mu)J. \tag{3.26}$$

Since $\lambda c \neq 0$ by assumption, we deduce from Lemma 3.10 that

$$a + a\lambda + c\mu = -2\lambda c\alpha(A) \quad \text{and} \quad b + a\mu = \lambda c \det(A), \tag{3.27}$$

which implies that

$$\frac{a}{c} = -\frac{\mu + 2\lambda\alpha(A)}{1+\lambda} \quad \text{and} \quad \frac{b}{c} = \lambda\det(A) + \frac{\mu(\mu + 2\lambda\alpha(A))}{1+\lambda}. \tag{3.28}$$

Then, the condition $bc > a^2$ is equivalent to condition (3.13).

On the other hand, we have $A^\varepsilon := \chi_\varepsilon A + (1-\chi_\varepsilon)f(A)$. Set $B^\varepsilon := \chi_\varepsilon f(A) + (1-\chi_\varepsilon)A$. Since the phases are interchangeable, B^ε H-converges to A^*. Furthermore, by Lemma 1.10 we clearly have $f(A^\varepsilon) = B^\varepsilon$, hence $f(A^\varepsilon)$ H-converges to A^*. The condition $bc > a^2$ being satisfied, we deduce from Theorem 1.11 and the uniqueness of the H-limit, that $f(A^*) = A^*$. This equality also reads as

$$aA^* + bJ = -aA^* + cA^*JA^*, \tag{3.29}$$

or equivalently

$$cA^*JA^* = 2aA^* + bJ. \tag{3.30}$$

Therefore, Lemma 3.10 and formulas (3.28) imply that

$$\alpha\big(A^*\big) = \frac{\mu + 2\lambda\,\alpha(A)}{1+\lambda} \quad \text{and} \quad \det\big(A^*\big) = \lambda\det(A) + \frac{\mu(\mu + 2\lambda\alpha(A))}{1+\lambda}, \tag{3.31}$$

which concludes the proof.

3.3.2. Proof of Corollary 3.9

We apply Theorem 3.7 to the interchangeable two-phase material with conductivity $A^\varepsilon := \sigma^\varepsilon(h)$. In this case $A := \sigma^1 + s_1 hJ$ and $\mu := (s_2 - \lambda s_1)h$. Hence, condition (3.13) reads as

$$\lambda\big(\det\big(\sigma^1\big) + s_1^2h^2\big) + \frac{(s_2^2 - \lambda^2 s_1^2)h^2}{1+\lambda} > \left(\frac{s_2 + \lambda s_1}{1+\lambda}\right)^2 h^2, \tag{3.32}$$

which is equivalent to

$$\lambda\det\big(\sigma^1\big) > O\big(h^2\big). \tag{3.33}$$

This holds true for small enough $|h|$, since $\lambda > 0$ and $\sigma^1 \in \mathcal{M}_+^s$. Then, condition (3.13) holds true without additional assumption for small enough $|h|$. Therefore, by the formula (3.14) of Theorem 3.7 we obtain

$$\alpha\big(\sigma^*(h)\big) = \frac{(s_2 - \lambda s_1)h + 2\lambda\alpha(\sigma^1 + s_1hJ)}{1+\lambda} = \left(\frac{s_2 + \lambda s_1}{1+\lambda}\right)h. \tag{3.34}$$

Furthermore, by Theorem 2.3 and for any h, $\sigma^\varepsilon(h)$ H-converges to

$$\sigma^*(h) = \sigma^* + s_*hJ + o_{L^2(\Omega)^{2\times 2}}(h), \quad \text{with } s_* = \lim_{\mathcal{D}'(\Omega)}\big[s_\varepsilon\det\big(P^\varepsilon\big)\big], \tag{3.35}$$

which implies that the antisymmetric part of $\sigma^*(h)$ satisfies

$$\alpha\big(\sigma^*(h)\big) = s_*h + o(h). \tag{3.36}$$

Hence, by (3.34) we get

$$s_* = \frac{s_2 + \lambda\,s_1}{1+\lambda}. \tag{3.37}$$

This combined with (3.35) yields in the case $s_1 := 1$ and $s_2 := 0$,

$$\lim_{\mathcal{D}'(\Omega)}\big[\chi_\varepsilon\det\big(P^\varepsilon\big)\big] = \frac{\lambda}{\lambda+1}, \tag{3.38}$$

which yields (3.20). On the other hand, by the formula (3.14) applied with $A := \sigma^1$ and $\mu := 0$, we obtain

$$\det\big(\sigma^*\big) = \det\big(\sigma^*(0)\big) = \lambda\det\big(\sigma^1\big). \tag{3.39}$$

Hence, by the third equality of (2.8), formulas (3.37) and (3.18) it follows that

$$r_* = -\frac{s_*}{\det(\sigma^*)} = \frac{1}{\lambda\det(\sigma^1)}\left(\frac{\det(\lambda\sigma^1)r_2 + \lambda\det(\sigma^1)r_1}{1+\lambda}\right) = \frac{r_1 + \lambda r_2}{1+\lambda}, \tag{3.40}$$

which gives (3.19) and concludes the proof.

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Appendix A. Proof of Theorem 1.7

A.1. Proof of part (i)

We follow the construction of the H-limit used by Murat and Tartar (see [14]) which depends on the vector parameter h.

Let $\tilde{\Omega}$ be a bounded open set of $\mathbb{R}^N$ such that $\Omega \subset \tilde{\Omega}$. We extend $A^\varepsilon(h)$ in $\tilde{\Omega} \backslash \Omega$ by αI_N (in order to have $A^\varepsilon \in \mathcal{M}(\alpha, \beta; \tilde{\Omega})$). We define $\mathcal{A}^\varepsilon(h) \in \mathcal{L}(H_0^1(\tilde{\Omega}); H^{-1}(\tilde{\Omega}))$ by

$$\forall u \in H_0^1(\tilde{\Omega}), \quad \mathcal{A}^\varepsilon(h)u := -\operatorname{div}\big(A^\varepsilon(h)\nabla u\big). \tag{A.1}$$

We proceed in two steps.

First step. For any $h \in B_\kappa$, $\mathcal{A}^\varepsilon(h)$ is bounded by β and equi-coercive, i.e.

$$\forall u \in H_0^1(\tilde{\Omega}), \quad \big\langle \mathcal{A}^\varepsilon(h)u, u\big\rangle_{H^{-1}(\tilde{\Omega}), H_0^1(\tilde{\Omega})} \geqslant \alpha \|u\|^2_{H_0^1(\Omega)}. \tag{A.2}$$

So, from the Lax–Milgram theorem, $\mathcal{A}^\varepsilon(h)$ is invertible and, since $A^\varepsilon(h)$ admits a first-order expansion, so does $\mathcal{A}^\varepsilon(h)$ and $\mathcal{B}^\varepsilon(h) := \mathcal{A}^\varepsilon(h)^{-1}$. Furthermore, $\mathcal{B}^\varepsilon(h)$ is bounded by α^{-1}, hence there exist a subsequence, still denoted by ε, and a linear operator $\mathcal{B}^*(h)$ from $H^{-1}(\tilde{\Omega})$ to $H_0^1(\tilde{\Omega})$ such that, for any $f \in H^{-1}(\tilde{\Omega})$,

$$\mathcal{B}^\varepsilon(h)f \rightharpoonup \mathcal{B}^*(h)f \quad \text{in } H_0^1(\tilde{\Omega})\text{-weak}, \tag{A.3}$$

for any countable dense set of h. Due to the condition (1.12) satisfied by $A^\varepsilon(h)$, $\mathcal{A}^\varepsilon(h)$ and $\mathcal{B}^\varepsilon(h)$ satisfy a uniform Lipschitz condition

$$\exists C > 0,\ \forall h, k \in B_\kappa, \quad \big\|\mathcal{B}^\varepsilon(h) - \mathcal{B}^\varepsilon(k)\big\|_{\mathcal{L}(H^{-1}(\tilde{\Omega}); H_0^1(\tilde{\Omega}))} \leqslant C|h-k|. \tag{A.4}$$

Therefore, convergence (A.3) holds true for any $h \in B_\kappa$. Moreover, there exists a linear operator $\mathcal{B}_1^\varepsilon \in \mathcal{L}(\mathbb{R}^n; \mathcal{L}(H^{-1}(\tilde{\Omega}); H_0^1(\tilde{\Omega})))$ such that

$$\mathcal{B}^\varepsilon(h)f = \mathcal{B}^\varepsilon(0)f + \big(\mathcal{B}_1^\varepsilon \cdot h\big)f + o_{H_0^1(\tilde{\Omega})}(h), \quad \forall f \in H^{-1}(\tilde{\Omega}),\ \|f\|_{H^{-1}(\tilde{\Omega})} \leqslant 1. \tag{A.5}$$

Since

$$\big\|\big(\mathcal{B}_1^\varepsilon \cdot h\big)f\big\|_{H_0^1(\tilde{\Omega})} = \big\|\mathcal{B}^\varepsilon(h)f - \mathcal{B}^\varepsilon(0)f\big\|_{H_0^1(\tilde{\Omega})} + o(h) = O(h), \quad \forall f \in H^{-1}(\tilde{\Omega}),\ \|f\|_{H^{-1}(\tilde{\Omega})} \leqslant 1, \tag{A.6}$$

there exist a subsequence of ε, still denoted by ε, and a linear operator $\mathcal{B}_1^* \in \mathcal{L}(\mathbb{R}^n; \mathcal{L}(H^{-1}(\tilde{\Omega}); H_0^1(\tilde{\Omega})))$ such that, for any $h \in \mathbb{R}^n$ and any $f \in H^{-1}(\tilde{\Omega})$,

$$\big(\mathcal{B}_1^\varepsilon \cdot h\big)f \rightharpoonup \big(\mathcal{B}_1^* \cdot h\big)f \quad \text{in } H_0^1(\tilde{\Omega})\text{-weak}. \tag{A.7}$$

Then, passing to the weak limit in (A.5) and using the semicontinuity of the $H_0^1(\tilde{\Omega})$-norm, we get

$$\mathcal{B}^*(h)f = \mathcal{B}^*(0)f + \big(\mathcal{B}_1^* \cdot h\big)f + o_{H_0^1(\tilde{\Omega})}(h), \quad \forall f \in H^{-1}(\tilde{\Omega}),\ \|f\|_{H^{-1}(\tilde{\Omega})} \leqslant 1. \tag{A.8}$$

Since $\mathcal{B}^\varepsilon(h)$ is β^{-1}-coercive so is $\mathcal{B}^*(h)$ and $\mathcal{B}^*(h)$ is thus invertible, which allows us to define

$$\mathcal{A}^*(h) := \mathcal{B}^*(h)^{-1} : H_0^1(\Omega) \to H^{-1}(\tilde{\Omega}). \tag{A.9}$$

Then, $\mathcal{A}^*(h)$ satisfies

$$\mathcal{A}^*(h)u = \mathcal{A}^*u + \big(\mathcal{A}_1^* \cdot h\big)u + o_{H^{-1}(\tilde{\Omega})}(h), \quad \forall u \in H_0^1(\tilde{\Omega}). \tag{A.10}$$

Moreover, thanks to (A.4) we have

$$\exists C > 0,\ \forall h, k \in B_\kappa, \quad \big\|\mathcal{A}^\varepsilon(h) - \mathcal{A}^\varepsilon(k)\big\|_{\mathcal{L}(H_0^1(\tilde{\Omega}); H^{-1}(\tilde{\Omega}))} \leqslant C\,|h-k|. \tag{A.11}$$

Second step. To obtain an expansion of the H-limit of $A^\varepsilon(h)$, we construct a corrector $P^\varepsilon(h)$ associated with $A^\varepsilon(h)$. Let $\psi \in \mathcal{D}(\tilde{\Omega})$ such that $\psi \equiv 1$ on Ω and $\lambda \in \mathbb{R}^N$. We set $u^\lambda(x) := \psi(x)\lambda \cdot x$ and we define $u_\varepsilon^\lambda(h) \in H_0^1(\tilde{\Omega})$ by

$$u_\varepsilon^\lambda(h) := \mathcal{B}^\varepsilon(h)\big(\mathcal{A}^*(h)u^\lambda\big). \tag{A.12}$$

Then, we define

$$P^\varepsilon(h)\lambda := \nabla u_\varepsilon^\lambda(h) = \nabla\big[\mathcal{B}^\varepsilon(h)\big(\mathcal{A}^*(h)u^\lambda\big)\big]. \tag{A.13}$$

The uniform Lipschitz assumptions (A.4), (A.11) satisfied by $\mathcal{B}^\varepsilon$ and $\mathcal{A}^\varepsilon$ and the first-order expansions (A.5) and (A.10) satisfied by $\mathcal{B}^\varepsilon(h)$ and $\mathcal{A}^*(h)$ yield

$$\exists C > 0,\ \forall h, k \in B_\kappa, \quad \big\| P^\varepsilon(h) - P^\varepsilon(k)\big\|_{L^2(\Omega)^{N\times N}} \leqslant C|h-k|, \tag{A.14}$$

and

$$P^\varepsilon(h) = P^\varepsilon(0) + P_1^\varepsilon\cdot h + o_{L^2(\tilde\Omega)^{N\times N}}(h), \tag{A.15}$$

with $\|P_1^\varepsilon\cdot h\|_{L^2(\tilde\Omega)^{N\times N}} = O(h)$.

Since $A^\varepsilon \in \mathcal{M}(\alpha,\beta;\tilde\Omega)$ we have (up to a subsequence) $A^\varepsilon \overset{H}{\rightharpoonup} A^*$. From the definition (A.13) of $P^\varepsilon(h)$, it is clear that $P^\varepsilon := P^\varepsilon(0)$ is a corrector associated with A^ε in Ω, hence by Proposition 1.5 we have

$$A^\varepsilon P^\varepsilon \rightharpoonup A^* \quad \text{in } L^2(\Omega)^{N\times N}\text{-weak}. \tag{A.16}$$

Since

$$u_\varepsilon^\lambda(h) \rightharpoonup \mathcal{B}^*\big(\mathcal{A}^*(h)u^\lambda\big) = u^\lambda \quad \text{in } L^2(\tilde\Omega)\text{-weak}, \tag{A.17}$$

we obtain, for any $h \in B_\kappa$,

$$P^\varepsilon(h) \rightharpoonup I_N \quad \text{in } L^2(\Omega)^{N\times N}\text{-weak}. \tag{A.18}$$

Moreover, by (1.12) and (A.14) $A^\varepsilon(h)P^\varepsilon(h)$ satisfies the uniform Lipschitz condition in $L^2(\Omega)^{N\times N}$ for $h \in B_\kappa$. Hence, there exist a new subsequence of ε, still denoted by ε, and $A^* \in L^2(\Omega)^{N\times N}$ such that

$$\forall h \in B_\kappa,\ A^\varepsilon(h)P^\varepsilon(h) \rightharpoonup A^*(h) \quad \text{in } L^2(\Omega)^{N\times N}\text{-weak}. \tag{A.19}$$

By Proposition 1.5(ii), the previous convergence combined with (A.14) and (A.18) implies that $A^\varepsilon(h)$ H-converges to $A^*(h)$. Finally, by (1.12) and (A.15) we have

$$A^\varepsilon(h)P^\varepsilon(h) = A^\varepsilon P^\varepsilon + Q_1^\varepsilon\cdot h + o_{L^2(\tilde\Omega)^{N\times N}}(h), \tag{A.20}$$

with $\|Q_1^\varepsilon\cdot h\|_{L^2(\tilde\Omega)^{N\times N}} = O(h)$. Therefore, passing to the limit in the previous equality, we get

$$A^*(h) = A^* + A_1^*\cdot h + o_{L^2(\Omega)^{N\times N}}(h). \tag{A.21}$$

The proof of the part (i) of Theorem 1.7 is done.

Remark A.11. From (A.15) we deduce that if $P^\varepsilon(h)$, $Q^\varepsilon(h)$ are the correctors associated with $A^\varepsilon(h)$ and $A^\varepsilon(h)^T$ respectively, then $P^\varepsilon(h)$ and $Q^\varepsilon(h)$ admit the first-order expansions

$$P^\varepsilon(h) = P^\varepsilon + P_1^\varepsilon\cdot h + o_{L^2(\Omega)^{N\times N}}(h) \quad\text{and}\quad Q^\varepsilon(h) = Q^\varepsilon + Q_1^\varepsilon\cdot h + o_{L^2(\Omega)^{N\times N}}(h), \tag{A.22}$$

where P^ε and Q^ε are the correctors associated with A^ε and $(A^\varepsilon)^T$ respectively. Since $P^\varepsilon(h)$ and P^ε are curl-free, we have

$$\big\|\mathrm{Curl}\big(P_1^\varepsilon\cdot h\big)\big\|_{H^{-1}(\Omega)^{N\times N\times N}} = o(h), \tag{A.23}$$

hence $P_1^\varepsilon\cdot h$ is also curl-free for any $h \in B_\kappa$. Moreover, since $P^\varepsilon(h)$ and P^ε weakly converge to I_N in $L^2(\Omega)^{N\times N}$, for any weakly convergent subsequence $P_1^{\varepsilon'}\cdot h$ in $L^2(\Omega)^{N\times N}$, the lower semicontinuity of the $L^2(\Omega)^{N\times N}$-norm implies that

$$\Big\|\lim_{\varepsilon'\to 0}\big(P_1^{\varepsilon'}\cdot h\big)\Big\|_{L^2(\Omega)^{N\times N}} = o(h), \tag{A.24}$$

hence, for the whole sequence ε and for any $h \in B_\kappa$, we have

$$P_1^\varepsilon\cdot h \rightharpoonup 0 \quad \text{in } L^2(\Omega)^{N\times N}\text{-weak}, \tag{A.25}$$

A.2. Proof of part (ii)

By the part (i) we obtain that for any $h \in B_\kappa$, $A^\varepsilon(h)$ H-converges to $A^*(h)$ where

$$A^*(h) = A^* + A_1^* \cdot h + o_{L^2(\Omega)^{N\times N}}(h). \tag{A.26}$$

Since, for any $\lambda, \mu \in \mathbb{R}^N$, we have $Q^\varepsilon(h)^T A^\varepsilon(h) P^\varepsilon(h)\lambda \cdot \mu = A^\varepsilon(h) P^\varepsilon(h)\lambda \cdot Q^\varepsilon(h)\mu$, we obtain by Proposition 1.5(i) and the div-curl lemma

$$A^*(h) = \lim_{\mathcal{D}'(\Omega)^{N\times N}} \big[Q^\varepsilon(h)^T A^\varepsilon(h) P^\varepsilon(h)\big]. \tag{A.27}$$

On the other hand, the expansion (1.13) of $A^\varepsilon(h)$ and Remark A.11 lead us to

$$\begin{aligned} Q^\varepsilon(h)^T A^\varepsilon(h) P^\varepsilon(h) = {} & \big(Q^\varepsilon\big)^T A^\varepsilon P^\varepsilon + \big(Q^\varepsilon\big)^T \big(A_1^\varepsilon \cdot h\big) P^\varepsilon \\ & + \big(Q^\varepsilon\big)^T A^\varepsilon \big(P_1^\varepsilon \cdot h\big) + \big(Q_1^\varepsilon \cdot h\big)^T A^\varepsilon P^\varepsilon + o_{L^1(\Omega)^{N\times N}}(h). \end{aligned} \tag{A.28}$$

Let $\lambda, \mu \in \mathbb{R}^N$, we have

$$\big(Q^\varepsilon\big)^T A^\varepsilon \big(P_1^\varepsilon \cdot h\big)\lambda \cdot \mu = \big(A^\varepsilon\big)^T Q^\varepsilon \mu \cdot \big(P_1^\varepsilon \cdot h\big)\lambda \quad \text{and} \quad \big(Q_1^\varepsilon \cdot h\big)^T A^\varepsilon P^\varepsilon \lambda \cdot \mu = A^\varepsilon P^\varepsilon \lambda \cdot \big(Q_1^\varepsilon \cdot h\big)\mu. \tag{A.29}$$

Hence, by the div-curl lemma and convergences (1.5) and (A.25) we get

$$\lim_{\mathcal{D}'(\Omega)^{N\times N}} \big[\big(Q^\varepsilon\big)^T A^\varepsilon \big(P_1^\varepsilon \cdot h\big)\big] = \lim_{\mathcal{D}'(\Omega)^{N\times N}} \big[\big(Q_1^\varepsilon \cdot h\big)^T A^\varepsilon P^\varepsilon\big] = 0. \tag{A.30}$$

There exists a subsequence ε', which is actually independent of h (by linearity), such that the sequence $(Q^{\varepsilon'})^T (A_1^{\varepsilon'} \cdot h) P^{\varepsilon'}$ converges in the weak-$*$ sense of the Radon measures. Hence, by Proposition 1.5 combined with (A.27) and (A.28) we get

$$A^*(h) = A^* + \lim_{\mathcal{M}(\Omega)^{N\times N}} \big[\big(Q^{\varepsilon'}\big)^T \big(A_1^{\varepsilon'} \cdot h\big) P^{\varepsilon'}\big] + o_{\mathcal{M}(\Omega)^{N\times N}}(h). \tag{A.31}$$

Therefore, equating (A.31) to (A.26) it follows that

$$A_1^* \cdot h = \lim_{\mathcal{M}(\Omega)^{N\times N}} \big[\big(Q^{\varepsilon'}\big)^T \big(A_1^{\varepsilon'} \cdot h\big) P^{\varepsilon'}\big]. \tag{A.32}$$

Since the limit is independent of the subsequence ε', the whole sequence $(Q^\varepsilon)^T (A_1^\varepsilon) P^\varepsilon$ thus converges to $A_1^* \cdot h$ in $\mathcal{D}'(\Omega)^{N\times N}$.

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