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The symmetry algebras of Euclidean M-theory

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Abstract

We study the Euclidean supersymmetric $D=11$ M-algebras. We consider two such $D=11$ superalgebras: the first one is $N=(1,1)$ self-conjugate complex-Hermitean, with 32 complex supercharges and 1024 real bosonic charges, the second is $N=(1,0)$ complex-holomorphic, with 32 complex supercharges and 528 bosonic charges, which can be obtained by analytic continuation of known Minkowski M-algebra. Due to the Bott's periodicity, we study at first the generic $D=3$ Euclidean supersymmetry case. The role of complex and quaternionic structures for $D=3$ and $D=11$ Euclidean supersymmetry is elucidated. We show that the additional $1024-528=496$ Euclidean tensorial central charges are related with the quaternionic structure of Euclidean $D=11$ supercharges, which in complex notation satisfy $SU(2)$ pseudo-Majorana condition. We consider also the corresponding Osterwalder–Schrader conjugations as implying for $N=(1,0)$ case the reality of Euclidean bosonic charges. Finally, we outline some consequences of our results, in particular for $D=11$ Euclidean supergravity.

1. Introduction

The physical spacetime is Minkowskian, but there are several reasons justifying the interest in Euclidean theories. We can recall here that

(i) The functional integrals acquire a precise mathematical meaning only in the context of Euclidean quantum theory (see, e.g., [1,2]).
(ii) The topologically nontrivial field configurations (such as instantons) are solutions of Euclidean field theories (see, e.g., [3,4]).
(iii) The generating functional of an Euclidean field theory can be related to the description of statistical and stochastic systems ([5,6]).

At present the $D=11$ M-theory has been considered as the most recent proposal for a "theory of everything" (see, e.g., [7,8]). We still do not know the dynamical content of the M-theory, however, it seems that the algebraic description of its symmetries is well embraced by the so-called M-algebra[1] [9,10]

$$\{Q_A, Q_B\} = P_{AB}$$

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[1] In fact it is a superalgebra, also sometimes called M-superalgebra.

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$$= (C\Gamma_\mu)_{AB} P^\mu + (C\Gamma_{[\mu\nu]})_{AB} Z^{[\mu\nu]} + (C\Gamma_{[\mu_1\cdots\mu_5]})_{AB} Z^{[\mu_1\cdots\mu_5]}, \tag{1.1}$$

where the Q_A ($A = 1, 2, \ldots, 32$) are 32 $D = 11$ real Minkowskian supercharges, P_μ describe the 11-momenta, while the remaining 517 bosonic generators describe the tensorial central charges $Z^{[\mu\nu]}$ and $Z^{[\mu_1\cdots\mu_5]}$.

It should be stressed that the M-algebra (1.1) is the generalized $D = 11$ Poincaré algebra with maximal number of additional bosonic generators. These additional generators indicate the presence of $D = 11$ $M2$ and $M5$ branes. Indeed, it has been shown (see, e.g., [11,12]) that $D = 11$ supergravity contains the super-2-brane (supermembrane) and super-5-brane solutions.

Our aim here is to study the Euclidean counterpart of the M-algebra, described by the relation (1.1). The problem of Euclidean continuation of superalgebras is not trivial, because the dimensionality of Minkowski and Euclidean spinors may differ, as it is well known from $D = 4$ case (see, e.g., [13–16]). In a four-dimensional world the Minkowski spinors are $\mathbf{C}^2$ (two-dimensional Weyl spinors) which can be described as $\mathbf{R}^4$ Majorana spinors, but the fundamental $D = 4$ Euclidean spinors are described by $\mathbf{H}\otimes\mathbf{H} = \mathbf{H}^2$ (due to the relation $O(4) = O(3) \times O(3)$ we deal with a pair of $D = 3$ Euclidean spinors) or $\mathbf{C}^2 \otimes \mathbf{C}^2 = \mathbf{C}^4$, i.e., the number of spinor components is doubled. Further, one can describe the $D = 4$ Minkowski Dirac matrices as real four-dimensional ones (the so-called Majorana representation), but the four-dimensional $D = 4$ Euclidean Dirac matrices are necessarily complex. The doubling of spinor components is reflected in the analytic continuation procedure, and the reality condition in $D = 4$ Minkowski space is replaced by the so-called Osterwalder–Schrader reality condition [16,17]).

In $D = 11$ Minkowski case the fundamental spinors are $\mathbf{R}^{32}$, and the corresponding fundamental representation of the Minkowskian $D = 11$ Clifford algebra

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}, \quad \eta_{\mu\nu} = (-1, 1, \ldots, 1), \tag{1.2}$$

is $\mathbf{R}^{32} \times \mathbf{R}^{32}$, which allows writing the M-algebra (1.1) as a real algebra. In the $D = 11$ Euclidean case the fundamental spinors are $\mathbf{H}^{16}$, while the fundamental Hermitean Clifford algebra representation in $D = 11$ Euclidean space is $\mathbf{C}^{32} \times \mathbf{C}^{32}$. We shall describe the Euclidean M-algebra using Hermitean products of complex Hermitean $D = 11$ Euclidean gamma-matrices satisfying the Euclidean counterpart of algebraic relations (1.2):

$$\{\tilde\Gamma_\mu, \tilde\Gamma_\nu\} = 2\,\delta_{\mu\nu}, \quad \mu, \nu = (1, \ldots, 11). \tag{1.3}$$

If we introduce 32 complex supercharges $\tilde Q_A$ we get the following formula for the Euclidean $D = 11$ superalgebra

$$\begin{aligned}\{\tilde Q_A, \tilde Q_B^+\} &= \delta_{AB}\tilde Z + (\tilde\Gamma_\mu)_{AB}\tilde P_\mu + (\tilde\Gamma_{[\mu_1\cdots\mu_4]})_{AB}\tilde Z_{[\mu_1\cdots\mu_4]} \\ &\quad + (\tilde\Gamma_{[\mu_1\cdots\mu_5]})_{AB}\tilde Z_{[\mu_1\cdots\mu_5]} \\ &\quad + (\tilde\Gamma_{[\mu_1\cdots\mu_8]})_{AB}\tilde Z_{[\mu_1\cdots\mu_8]} \\ &\quad + (\tilde\Gamma_{[\mu_1\cdots\mu_9]})_{AB}\tilde Z_{[\mu_1\cdots\mu_9]},\end{aligned} \tag{1.4}$$

where on r.h.s. of (1.4) all linearly independent Hermitean antisymmetric products of Γ-matrices appear.

Since in $D = 11$ Euclidean space we get the relation

$$\tilde\Gamma_{12} = \frac{1}{11!}\epsilon_{\mu_1\cdots\mu_{11}}\tilde\Gamma_{\mu_1}\cdots\tilde\Gamma_{\mu_{11}} = i \tag{1.5}$$

we obtain the identity

$$\tilde\Gamma_{[\mu_1\cdots\mu_k]} = \frac{i}{(11-k)!}\epsilon_{\mu_1\cdots\mu_{11}}\tilde\Gamma_{[\mu_{k+1}\cdots\mu_{11}]}. \tag{1.6}$$

Applying (1.6) for $k = 2$ and 3 one can write the relation (1.4) as follows

$$\begin{aligned}\{\tilde Q_A, \tilde Q_B^+\} &= \delta_{AB}\tilde Z + (\tilde\Gamma_\mu)_{AB}\tilde P_\mu + \tilde\Gamma_{[\mu\nu]}\tilde Z_{[\mu\nu]} + \tilde\Gamma_{[\mu\nu\rho]}\tilde Z_{[\mu\nu\rho]} \\ &\quad + \tilde\Gamma_{[\mu_1\cdots\mu_4]}\tilde Z_{[\mu_1\cdots\mu_4]} + \tilde\Gamma_{[\mu_1\cdots\mu_5]}\tilde Z_{[\mu_1\cdots\mu_5]},\end{aligned} \tag{1.7}$$

where

$$\begin{aligned}\tilde Z_{[\mu\nu]} &= \frac{i}{2!}\epsilon_{\mu\nu\nu_1\cdots\nu_9}\tilde Z_{[\nu_1\cdots\nu_9]}, \\ \tilde Z_{[\mu\nu\rho]} &= \frac{i}{3!}\epsilon_{\mu\nu\rho\nu_1\cdots\nu_8}\tilde Z_{[\nu_1\cdots\nu_8]}.\end{aligned} \tag{1.8}$$

We see that in (1.7) we have two sets of Abelian bosonic charges, also called tensorial central charges

(i) the $528 = 11 + 55 + 462$ charges $\tilde P_\mu$, $\tilde Z_{[\mu_1\mu_2]}$ and $\tilde Z_{[\mu_1\cdots\mu_5]}$ corresponding to the tensorial central charges occurring in (1.1);

(ii) the $496 = 1 + 165 + 330$ additional Euclidean tensorial central charges $\tilde{Z}$, $\tilde{Z}_{[\mu_1 \cdots \mu_3]}$ and $\tilde{Z}_{[\mu_1 \cdots \mu_4]}$ which describe the maximal complex Hermitean extension of the set of real bosonic charges occurring in the Minkowski case. We shall show, however, that one can find a holomorphic subalgebra of (1.4) defining holomorphic Euclidean M-algebra with 32 complex supercharges and 528 bosonic generators.

In our Letter, in order to be more transparent, we study at first in Section 2 the lower-dimensional case of $D = 3$ Euclidean superalgebra.

It appears that to 528 bosonic charges of Minkowski M-algebra (1.1) correspond in $D = 3$ just three bosonic charges describing $D = 3$ three-momenta (there are no Minkowski central charges in $D = 3$), and to the extension in $D = 11$ from 528 Minkowskian to 1024 Euclidean bosonic charges corresponds in $D = 3$ the extension of three momentum generators by one additional central charge. We see therefore that, when passing from $D = 3$ to $D = 11$, instead of one $D = 3$ Euclidean central charge we obtain 496 central charges in $D = 11$.

In Section 3 we study in more detail the $D = 11$ case and in particular the $D = 11$ tensorial structure of 496 Euclidean central charges. We introduce in $D = 3$ and $D = 11$ the Osterwalder–Schrader conjugation which is required if we wish to obtain the holomorphic Euclidean M-theory with real bosonic charges. In Section 4 we present an outlook, considering in particular the possible applications to $D = 11$ Euclidean superbrane scan as well as the Euclidean version of the generalized AdS, dS and conformal superalgebras. We would also like to recall here that recently Euclidean symmetry and Euclidean superspace was considered as a basis for noncommutative supersymmetric field theory [18–20].

2. The $D = 3$ Euclidean superalgebra and the role of quaternionic and complex structure

The $D = 3$ Euclidean spinors are described by real quaternions ($q_0, q_r \in R;\ r = 1, 2, 3$)

$$q = q_0 + e_r q_r, \qquad e_r e_s = -\delta_{rs} + \varepsilon_{rst} e_t, \tag{2.1}$$

which carry the representations of $\overline{SO(3)} = Sp(1) = U(1; \mathbf{H})$, described by unit quaternions

$$\alpha = \alpha_0 + \alpha_r e_r, \qquad \alpha_0^2 + \alpha_r^2 = 1. \tag{2.2}$$

The quaternionic spinor (2.1) is modified under $Sp(1)$ transformation law as follows

$$q' = \alpha q. \tag{2.3}$$

One can describe the real quaternion $q \in \mathbf{H}(1)$ by a pair of complex variables (z_1, z_2). Further, introducing the 2×2 complex matrix representation $e_r = -i\sigma_r$, one can represent unit quaternions (2.2) as 2×2 unitary matrices A:

$$\alpha \quad \Leftrightarrow \quad A = \alpha_0 \mathbf{1} - i\alpha_r \sigma_r, \tag{2.4}$$

i.e., $Sp(1) \simeq SU(2)$.

We should assume that $D = 3$ Euclidean supercharges are the $SO(3)$ spinors. Unfortunately, since the Clifford algebra

$$\{\tilde{\gamma}_r, \tilde{\gamma}_s\} = 2\delta_{rs}, \quad r, s = 1, 2, 3 \tag{2.5}$$

has the fundamental $\mathbf{C}^2 \times \mathbf{C}^2$ representation, one cannot employ single quaternionic supercharges as describing the Hermitean $D = 3$, $N = 1$ Euclidean superalgebra. In fact, if we introduce the quaternionic Hermitean superalgebra with supercharges described by fundamental $SO(3)$ spinor

$$\{\bar{\mathbf{R}}, \mathbf{R}\} = \mathbf{Z}, \tag{2.6}$$

where $\mathbf{R} = R_0 + e_r R_r \to \bar{\mathbf{R}} = R_0 - e_r R_r$ describes the quaternionic conjugation, it will contain only *one* bosonic charge $\mathbf{Z} \in \mathbf{H}$ ($\mathbf{Z} = \bar{\mathbf{Z}} \to \mathbf{Z} = Z_0$) and can be successfully used rather for the description of $D = 1$, $N = 4$ supersymmetric quantum mechanics [18]. In order to obtain the "supersymmetric roots" of $D = 3$ Euclidean momenta one should introduce, however, in agreement with the representation theory of $D = 3$ Euclidean Clifford algebra (2.5), two complex supercharges ($\tilde{Q}_1, \tilde{Q}_2$). One can write the $D = 3$ Euclidean superalgebra in the following familiar complex-Hermitean form ($\alpha, \beta = 1, 2$)

$$\{\tilde{Q}^*_\alpha, \tilde{Q}_\beta\} = (\sigma_r)_{\alpha\beta} \tilde{P}_r + \delta_{\alpha\beta} Z,$$
$$\{\tilde{Q}_\alpha, \tilde{Q}_\beta\} = \{\tilde{Q}^*_\alpha, \tilde{Q}^*_\beta\} = 0. \tag{2.7}$$

We see that among the bosonic generators besides the three momenta we obtain a fourth real central charge Z. In fact (2.7) can be obtained by dimensional reduction from standard $D=4$ $N=1$ super-algebra.

In order to find the quaternionic structure in the superalgebra (2.7) one should introduce the following pair of two-component spinors R_α, R_α^H

$$R_\alpha = \frac{1}{\sqrt{2}}(\tilde{Q}_\alpha + \varepsilon_{\alpha\beta}\tilde{Q}_\beta^*),$$
$$R_\alpha^H = \frac{i}{\sqrt{2}}(\tilde{Q}_\alpha - \varepsilon_{\alpha\beta}\tilde{Q}_\beta^*), \tag{2.8}$$

satisfying the relation

$$R_\alpha^H = -i\varepsilon_{\alpha\beta}R_\beta^*. \tag{2.9}$$

The formula (2.9) implies quaternionic reality condition in complex framework [21–23] described by $SU(2)$-Majorana condition. Indeed, introducing $R_\alpha^1 = R_\alpha$, $R_\alpha^2 = R_\alpha^H$ one can rewrite (2.9) in the following way [21]

$$R_\alpha^a = i\varepsilon^{ab}\varepsilon_{\alpha\beta}\big(R_\beta^b\big)^*. \tag{2.10}$$

The self-conjugate super-algebra (2.7) can be written as follows:

$$\{R_\alpha, R_\beta\} = (\varepsilon\sigma_r)_{\alpha\beta}\tilde{P}_r, \tag{2.11a}$$
$$\big\{R_\alpha, R_\beta^H\big\} = i\varepsilon_{\alpha\beta}Z, \tag{2.11b}$$
$$\big\{R_\alpha^H, R_\beta^H\big\} = (\varepsilon\sigma_r)_{\alpha\beta}\tilde{P}_r, \tag{2.11c}$$

and describes the $N=(1,1)$ $D=3$ Euclidean supersymmetry.

It is easy to check that Eqs. (2.11a)–(2.11c) are consistent with the relation (2.9), i.e., the $SU(2)$-Majorana reality condition (2.10) can be imposed.

The real superalgebra (2.11a)–(2.11c) contains as subsuperalgebras the holomorphic $N=(1,0)$ superalgebra (2.11a) as well as the antiholomorphic $N=(0,1)$ superalgebra (2.11c). In particular we can consider separately (2.11a) as describing the basic $N=1$ $D=3$ Euclidean non-self-conjugate superalgebra, supersymmetrizing three Euclidean momenta P_r. If the superalgebras (2.11a) or (2.11c) are considered as independent algebraic structures, the Euclidean three-momentum P_r can be complex.

The superalgebra (2.11a) can be obtained by analytic continuation of the real $D=3$ $N=1$ Minkowski superalgebra ($\mu=0,1,2$; $Q_\alpha \in R$)

$$\{Q_\alpha, Q_\beta\} = (C\gamma_\mu)_{\alpha\beta}P^\mu, \tag{2.12}$$

where $\{\gamma_\mu,\gamma_\nu\} = 2\eta_{\mu\nu}$ ($\eta_{\mu\nu} = \mathrm{diag}(-1,1,1)$, $C=\gamma_0$, $\gamma_\mu \in \mathbf{R}^2\times\mathbf{R}^2$) and one can choose, e.g.,

$$\gamma_0 = -i\sigma_2, \qquad \gamma_1 = \sigma_1, \qquad \gamma_2 = \sigma_3. \tag{2.13}$$

The analytic continuation of the Minkowski superalgebra (2.12) to the Euclidean one, given by (2.11a), is obtained by complexification of the real $D=3$ Minkowski spinors and Wick rotation of Minkowski vectors into Euclidean ones. We replace $Q_\alpha \in \mathbf{R}$ by $\tilde{Q}_\alpha \in \mathbf{C}$ and

$$P^0 \quad\Rightarrow\quad \tilde{P}_3 = iP^0,$$
$$\tilde{P}_k = P_k \quad (k=1,2),$$
$$\gamma^0 \quad\Rightarrow\quad \tilde{\gamma}_3 = i\gamma_0 = \sigma_2,$$
$$\tilde{\gamma}_k = \gamma_k = \sigma_k \quad (k=1,2), \tag{2.14}$$

where $\tilde{C} = \varepsilon = i\gamma_2$ and $\tilde{C}\tilde{\gamma}_r = -\tilde{\gamma}_r^T\tilde{C}$, i.e., we keep the same $D=3$ charge conjugation matrices in Euclidean and Minkowski case. One gets ($r=1,2,3$)

$$\{R_\alpha, R_\beta\} = (\tilde{C}\tilde{\gamma}_r)_{\alpha\beta}\tilde{P}_r, \tag{2.15}$$

i.e., one can identify after putting $Q_\alpha^E = R_\alpha$ the superalgebras (2.15) and (2.11a).

In order to justify the real values of P_r in (2.15) one should introduce the Osterwalder–Schrader (OS) conjugation $A \to A^\#$, which is defined in any dimension D with complex Euclidean spinors replacing real Majorana spinors as complex conjugation supplemented by time reversal transformation, i.e.,

$$R_A^\# = T_{AB}R_B^*, \tag{2.16}$$

where (D is the Euclidean time direction; following (2.14) we choose $\tilde{k}_\mu$ ($k=1,\dots,D-1$) real and Γ_D purely imaginary)

$$T\tilde{\Gamma}_kT^{-1} = \tilde{\Gamma}_k, \qquad T\tilde{\Gamma}_DT^{-1} = \tilde{\Gamma}_D. \tag{2.17}$$

The relation (2.17) implies that

$$\tilde{P}_k^\# = \tilde{P}_k, \qquad \tilde{P}_D^\# = -\tilde{P}_D. \tag{2.18}$$

It appears that in $D=3$ the OS conjugation of supercharges can be identified with the conjugation (2.9), i.e., $T_{\alpha\beta} = -i\varepsilon_{\alpha\beta}$, $\tilde{\Gamma}_k = (\sigma_1,\sigma_3)$ and $\tilde{\Gamma}_D = \sigma_2$.

The real values of $\tilde{P}_r$ in (2.15) are required if we assume the invariance of the superalgebra (2.15) under OS conjugation, i.e., the conjugation (2.9). This reality requirement is satisfied inside the superalgebra (2.11a)–(2.11c) and is equivalent to the consistency of (2.11a)–(2.11c) with subsidiary condition (2.10). Indeed performing such a conjugation one obtains from the $N = (1,0)$ superalgebra (2.11a) the identical in its form $N = (0,1)$ superalgebra (2.11c).

One can therefore say that

(i) our $D = 3$, $N = (0,1)$ Euclidean superalgebra (2.11c) is OS-conjugate to the superalgebra (2.11a) (or equivalently (2.15)).

(ii) It is not possible to impose the OS reality conditions on single pair of supercharges R_α

$$(R_\alpha)^\# = R_\alpha \quad \Leftrightarrow \quad R_\alpha = i\varepsilon_{\alpha\beta}(R_\alpha)^*, \tag{2.19}$$

because such a condition is not consistent, i.e., the superalgebra $(1,0)$ cannot be made selfconjugate.

3. The Euclidean M-algebra and the role of quaternionic and complex structure

In this section we shall translate the Euclidean superalgebra structures from $D = 3$ to $D = 11$. Due to the Bott's periodicity conditions these algebraic structures should be analogous.

The $D = 11$ Euclidean spinors are described by 16 quaternions $R_m \in \mathbf{H}^{16}$ $(m = 1,\dots,16)$ and the fundamental representation of the $D = 11$ Euclidean Clifford algebra (1.3) is $\mathbf{C}^{32} \times \mathbf{C}^{32}$. The quaternionic $D = 11$ Hermitean superalgebra, generalizing relation (2.6), is given by the relation

$$\{\bar{\mathbf{R}}_m, \mathbf{R}_n\} = \mathbf{Z}_{mn}, \quad \mathbf{Z}_{mn} = \bar{\mathbf{Z}}_{nm}. \tag{3.1}$$

The 16×16 Hermitean-quaternionic matrix Z_{mn} is described by 496 real bosonic Abelian generators.

In order to describe the complex-Hermitean Euclidean M-algebra we should introduce 32 complex supercharges $\tilde{Q}_A \in \mathbf{C}^{32}$. The most general complex-Hermitean $D = 11$ Euclidean algebra is given by the relation

$$\{\tilde{Q}_A^+, \tilde{Q}_B\} = \tilde{Z}_{AB}, \quad \tilde{Z}_{AB} = \tilde{Z}_{AB}^+, \tag{3.2}$$

containing 1024 real bosonic charges. We introduce $D = 11$ Euclidean gamma-matrices by putting

$$\tilde{\Gamma}_\mu = \Gamma_\mu, \quad \mu = 1,2,\dots,10,$$
$$\tilde{\Gamma}_{11} = i\Gamma_0, \tag{3.3}$$

where the matrices Γ_μ, Γ_0 describe the real 32-dimensional Majorana representation of the $D = 11$ Minkowskian Clifford algebra (1.2) occurring in the M-algebra (1.1). If we introduce the antisymmetric products $\tilde{\Gamma}_{[\mu_1\cdots\mu_p]} = (1/p!)\sum_{(\mu_1\cdots\mu_p)}(-1)^{\text{perm}}\tilde{\Gamma}_{\mu_1}\cdots\tilde{\Gamma}_{\mu p}$ we can check that

$$\tilde{\Gamma}^+_{[\mu_1\cdots\mu_p]} = \tilde{\Gamma}_{[\mu_1\cdots\mu_p]} \quad \text{for } p = 0,1,4,5,8,9,$$
$$\tilde{\Gamma}^+_{[\mu_1\cdots\mu_p]} = -\tilde{\Gamma}_{[\mu_1\cdots\mu_p]} \quad \text{for } p = 2,3,6,7,10,11. \tag{3.4}$$

Following the symmetry properties (3.4) one can present the superalgebra (3.2) in the form (1.4) or (1.7).

In order to write down the Euclidean M-algebra with only 528 bosonic charges, the ones corresponding to the bosonic charges of the M-algebra (1.1), one should exhibit in $D = 11$ the quaternionic or $SU(2)$-Majorana structure in analogous way as in Eqs. (2.11a)–(2.11c) for the $D = 3$ case.

Let us introduce the following pair of 32-component complex supercharges:

$$R_A = \frac{1}{\sqrt{2}}(\tilde{Q}_A + C_{AB}\tilde{Q}_B^*),$$
$$R_A^H = \frac{i}{\sqrt{2}}(\tilde{Q}_A - C_{AB}\tilde{Q}_B^*), \tag{3.5}$$

where the $D = 11$ charge conjugation matrix satisfies the relations

$$C\tilde{\Gamma}_\mu = -\tilde{\Gamma}_\mu^T C, \qquad C^2 = -1, \qquad C^T = -C, \tag{3.6}$$

and can be chosen the same in Minkowski and Euclidean case.

One can show that

$$R_A^H = -iC_{AB}R_B^*. \tag{3.7}$$

Introducing 64-component complex spinor $R_A^a = (R_A, R_A^H)$ one can rewrite (3.7) as the $SU(2)$-Majorana condition [13]

$$R_A^a = i\varepsilon^{ab}C_{AB}\big(R_B^b\big)^*. \tag{3.8}$$

The superalgebra (1.7) can be written in the following form $(C = \Gamma_0;\ \mu = 1,\dots,11)$

$$\{R_A, R_B\} = (C\tilde{\Gamma}_\mu)_{AB}\tilde{P}_\mu + (C\tilde{\Gamma}_{[\mu\nu]})_{AB}\tilde{Z}_{[\mu\nu]} + (C\tilde{\Gamma}_{[\mu_1\cdots\mu_5]})\tilde{Z}_{[\mu_1\cdots\mu_5]}, \tag{3.9a}$$

$$\big\{R_A, R_B^H\big\} = i\big\{C_{AB}\tilde{Z} + (C\tilde{\Gamma}_{[\mu_1\mu_2\mu_3]})_{AB}\tilde{Z}_{[\mu_1\mu_2\mu_3]} \times (C\tilde{\Gamma}_{[\mu_1\cdots\mu_4]})_{AB}\tilde{Z}_{[\mu_1\cdots\mu_4]}\big\}, \tag{3.9b}$$

$$\big\{R_A^H, R_B^H\big\} = (C\tilde{\Gamma}_\mu)_{AB}\tilde{P}_\mu + (C\tilde{\Gamma}_{[\mu\nu]})_{AB}\tilde{Z}_{\mu\nu} + (C\tilde{\Gamma}_{[\mu_1\cdots\mu_5]})_{AB}\tilde{Z}_{[\mu_1\cdots\mu_5]}. \tag{3.9c}$$

We see that

(i) the relation (3.9a)–(3.9c) describe the selfconjugate $(1,1)$ Euclidean M-algebra with 1024 bosonic Abelian charges which can be written also in the form (1.4);
(ii) the relation (3.9a) describes the holomorphic $(1,0)$ Euclidean M-algebra, with 528 Abelian bosonic charges. The antiholomorphic $(0,1)$ Euclidean M-algebra obtained by conjugation (3.7), which contains also 528 Abelian bosonic charges, is given by the relation (3.9c);
(iii) the subsidiary condition (3.8) relates the bosonic generators of the $(1,0)$ and $(0,1)$ sectors of the superalgebra (3.9a)–(3.9c). It should be observed that the 496 real bosonic generators occurring in quaternionic superalgebra (3.1) can be found in the "cross anticommutator" (3.9b).

By analogy with the $D=3$ case one can treat the $(1,0)$ holomorphic Euclidean M-algebra (3.9a) as analytic continuation of the $D=11$ Minkowski M-algebra, given by (1.1). For such purpose one should in (1.1) complexify the supercharges and perform the Wick rotation of Minkowski 11-tensors, i.e., perform the change $\Gamma_\mu \to \tilde{\Gamma}_\mu$ (see (3.3)) supplemented by the redefinitions $(k,l,\ldots,=1,2,\ldots,10)$:

$$\tilde{P}_k = P_k, \qquad \tilde{Z}_{[kl]} = Z_{[kl]}, \qquad \tilde{Z}_{[k_1\cdots k_5]} = Z_{[k_1\cdots k_5]},$$
$$\tilde{P}_{11} = iP_0, \qquad \tilde{Z}_{[k\,11]} = iZ_{[k0]},$$
$$\tilde{Z}_{[11k_1\cdots k_4]} = iZ_{[0k_1\cdots k_4]}, \tag{3.10}$$

changing Minkowski tensors into Euclidean tensors.

In such a way we shall obtain from (1.1) the holomorphic Euclidean M-algebra (3.9a).

The quaternionic conjugation (3.7) describes the $D=11$ OS conjugation in Euclidean space. We would like to point out that in holomorphic and antiholomorphic Euclidean M-algebra (i.e., if we consider the relations (3.9a) and (3.9c) as separate) the bosonic generators, in particular the 11-momenta, can be complex.

In order to obtain, e.g., in (3.9a) the real Abelian bosonic generators

$$\tilde{P}_k = P_k^*, \qquad \tilde{Z}_{[kl]} = Z_{[kl]}^*, \qquad \tilde{Z}_{[k_1\cdots k_5]} = Z_{[k_1\cdots k_5]}^*.$$

One should impose the invariance of the superalgebra (3.9a) under the OS conjugation (3.7), i.e., assume that the form of holomorphic and antiholomorphic Euclidean M-algebras related by OS conjugation is identical.

4. Conclusions

In relation with Euclidean M-theories and their algebraic description presented in this Letter we would like to make the following comments:

(i) It is well known [9] that the presence of tensorial central charges in generalized D-dimensional supersymmetry algebra can be linked with the presence of p-brane solution of D-dimensional supergravity. We have two $D=11$ Euclidean M-theories described by $(1,1)$ self-dual (see (2.7)) and $(1,0)$ holomorphic (see (2.11a) or (2.15)) supersymmetry algebras. In Euclidean theory the role of p-brane solutions will be played by Euclidean instantons and space branes (S-branes). It appears that in holomorphic $N=(1,0)$ Euclidean M-theory the set of instanton solutions corresponds to p-dimensional solutions in standard Minkowskian $D=11$ M-theory (2-branes and 5-branes supplemented by six-dimensional Kaluza–Klein monopoles and nine-dimensional Horava–Witten boundaries). The Euclidean $N=(1,1)$ M-theory with symmetry algebra (2.7) will have additional instanton solutions corresponding to 3-tensor and 4-tensor central charges which do not have their Minkowski space counterparts.

(ii) The superalgebras either with Hermitean self-conjugate algebra structure or holomorphic structure can be considered in any dimension with complex fundamental spinors ($D=0,4$ modulo 8 for Minkowski metric and $D=2,6$ modulo 8 for Euclidean metric) or fundamental quaternionic spinors ($D=5,6,7$ modulo 8 for Minkowski metric and $D=3,4,5$ modulo 8 for Euclidean metric). The physical choice of the algebraic structure of supersymmetry is indicated by the presence in the bosonic sector of the vectorial momentum generators. For example, in $D=4$ one can choose either the Hermitean algebra $(\alpha,\beta=1,2)$

$$\{Q^+_{\dot\alpha}, Q_\beta\} = P_{\dot\alpha\beta}, \tag{4.1}$$

or the pair of holomorphic/antiholomorphic algebras:

$$\{Q_\alpha, Q_\beta\} = Z_{\alpha\beta},$$

$$\{Q^+_{\dot\alpha}, Q^+_{\dot\beta}\} = Z_{\dot\alpha\dot\beta} = Z^+_{\alpha\beta},$$
$$\{Q^+_{\dot\alpha}, Q_\beta\} = 0. \qquad (4.2)$$

Since the four-momentum generators are present only in the relation (4.1), these relations are the basic $D = 4$ supersymmetry relations.

In the quaternionic $D = 3$ and $D = 11$ Euclidean case the Hermitean algebras (2.11b) and (3.9b) do not contain the momentum generators; these generators occur in the superalgebras (2.11a), (2.11c) and (3.9a), (3.9c). We see therefore that in these cases the holomorphic/antiholomorphic algebra is more physical.

(iii) If the fundamental spinors are complex, from algebraic point of view one can consider the minimally extended supersymmetry algebra with either Hermitean or holomorphic complex structure. If we assume however that the Hermitean anticommutator $\{Q^+_A, Q_B\}$ as well as the holomorphic one $\{Q_A, Q_B\}$ are saturated by Abelian bosonic generators (tensorial central charges), we obtain the most general real superalgebra. In such a way in $D = 4$ Minkowski case six tensorial central charges are generated by the relations (4.2), while the Hermitean superalgebra (4.1) describes only the four-momentum generators.

In quaternionic case we have two levels of generalizations. Assuming that the fundamental spinors belong to $\mathbf{H}^n$, one can consider:

(1) generalized Hermitean superalgebra for the supercharges belonging to $\mathbf{C}^{2n}$. In such a way we generate $4n^2$ real Abelian bosonic generators.
(2) One can write down also the most general real superalgebra, with supercharges belonging to $\mathbf{R}^{4n}$. In such a way we obtain $2n(4n+1)$ real Abelian generators.

In $D = 11$ Euclidean case $n = 16$ the generalized complex-Hermitean superalgebra is given by the relation (2.7); the minimal real superalgebra containing (2.7) takes the form $(S, T = 1, \ldots, 64)$

$$\{Q_S, Q_T\} = P_{ST},$$

and can be obtained by contraction of $OSp(1|64; \mathbf{R})$.

(iv) In this Letter we discussed the analytic continuation of $D = 11$ Minkowski superalgebra with only supersymmetrized Abelian bosonic charges. It has been argued (see, e.g., [24–26]), due to the relation $Sp(32) \supset O(10, 2)$, that $OSp(1|32; \mathbf{R})$ is the generalized $D = 11$ *AdS* superalgebra, and $OSp(1|64; \mathbf{R})$ describes the generalized $D = 11$ superconformal algebra. If we consider the holomorphic version of Euclidean M-theory the corresponding generalized Euclidean *AdS* superalgebra and generalized Euclidean conformal superalgebra are obtained by holomorphic complexification and respectively given by the $OSp(1|32; \mathbf{C})$ and $OSp(1|64; \mathbf{C})$ superalgebras.

(v) It appears that the M-algebra (1.1) describes as well the symmetries of nonstandard M^*-theory with signature $(9, 2)$, and M'-theory, with signature $(6, 5)$, which are related with standard M-theory by dualities ([27,28]; see also [29]). Different choices of signature corresponds to different choices of holonomy groups embedded in $sl(32; \mathbf{R})$. If we pass to complex Hermitean holonomy structures, embedded in $sl(32; \mathbf{C})$, we can describe 12 different versions of M-theories, with any signature $(11-k, k)$ $(k = 0, 1, \ldots, 11)$. In particular, for signatures $(8, 3)$ and $(4, 7)$ we should use the holonomy groups embedded in $Sl(16; \mathbf{H}) \subset Sl(32; \mathbf{C})$.

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